Methods and Techniques of Mathematical Physics

Proceedings of a Conference held in Oberwolfach, February 3-9, 1980

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INHALTSVERZEICHNIS

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

TAGUNGSBERICHT

METHODEN UND VERFAHREN DER MATHEMATISCHEN PHYSIK

3.2. - 9.2.1980

Die nunmehr 9.Tagung über "Methoden und Verfahren der mathematischen Physik" fand unter der Leitung von B.Brosowski (Frankfurt) und E.Martensen (Karlsruhe) statt. Es kamen 41 Teilnehmer, darunter 11 aus dem Ausland. Das Ziel der Tagung war es, möglichst vielfältige Methoden der mathematischen Physik heranzuziehen und auf weitgefächerte, konkrete Fragestellungen anzuwenden. Außerdem sollte durch Beteiligung von Vertretern der Anwendungsgebiete selbst (hier der Mechanik und der Physik) die Zusammenarbeit und die gegenseitige Anregung gefördert werden.

Die insgesamt 33 Vorträge haben dieser Zielsetzung entsprochen. Im Vordergrund standen funktionalanalytische und numerische Methoden für partielle Differentialgleichungen und Integrodifferentialgleichungen. Zu den angesprochenen Gebieten gehörten u.a. die Strömungsdynamik, die Streutheorie, die spezielle Relativitätstheorie und die Ausbreitung von Wellen. Aufmerksamkeit wurde auch den Fragen der Modellbildung gewidmet. Einige Übersichtsvorträge ergänzten das Programm. Das Interesse an Fragestellungen aus den behandelten Anwendungsbereichen führte zu lebhaften Diskussionen und fruchtbarem wissenschaftlichen Austausch, nicht zuletzt auch mit den anwesenden Vertretern anderer Fachrichtungen.

E.Halter

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A SINGULAR PERTURBATION PROBLEM FOR LINEAR OPERATORS WITH AN APPLICATION TO THE LIMITING BEHAVIOUR OF STATIONARY ELECTROMAGNETIC WAVE FIELDS FOR SMALL FREQUENCIES.

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ABSTRACT.

In this paper we shall give sufficient conditions for the continuous dependence of the solution of a parameter-dependent linear operator equation of the second kind in a Banach space at a critical parameter where the operator changes from the regular to the singular case. The results are applied to the investigation of the limiting behaviour of solutions to boundary-value problems from the mathematical theory of stationary electromagnetic reflection at perfect conductors for small frequencies.

1. INTRODUCTION.

The mathematical treatment of the reflection of stationary electromagnetic fields at perfect electrical conductors leads to the following exterior boundary-value problem [5], [12], [14].

Let D be an unbounded connected domain in $\mathbb{R}^3$. The boundary of D, denoted by S, is assumed to consist of a finite number m of disjoint, closed, bounded surface $S = \bigcup_{j=1}^{m} S_j$. The complement of the closure $\bar{D}$ in $\mathbb{R}^3$ is designated by D_i. By n we denote the unit normal to S directed into D.

Problem $E_\kappa(D)$ *Find a vector field* $E_\kappa \in C^2(D) \cap C(\bar{D})$ *such that* div E_κ, rot $E_\kappa \in C(\bar{D})$ *satisfying the vector Helmholtz equation*

$$(1.1) \qquad \Delta E_\kappa + \kappa^2 E_\kappa = 0 \text{ in } D\,, \ \kappa \neq 0\,, \ \mathrm{Im}(\kappa) \geq 0\,,$$

the electric boundary conditions

$$(1.2) \qquad [n, E_\kappa] = c_\kappa\,, \ \mathrm{div}\, E_\kappa = \gamma_\kappa \quad \text{on } S$$

and the radiation condition

$$(1.3) \qquad \left[\mathrm{rot}\, E_\kappa, \frac{x}{|x|}\right] + \frac{x}{|x|}\,\mathrm{div}\, E_\kappa - i\kappa E_\kappa = o\left(\frac{1}{|x|}\right), |x| \to \infty\,,$$

uniformly for all directions $x/|x|$. *Here,* $\gamma_\kappa \in C^{0,\alpha}(S)$, $0<\alpha<1$, *is a given function and* $c_\kappa \in C^{0,\alpha}(S)$ *is a given tangential field with the additional property that its surface divergence* Div c_κ *exists and is of class* $C^{0,\alpha}(S)$.

Problem $E_\kappa(D)$ is a slight generalization of the boundary-value problem of stationary electromagnetic reflection at perfect conductors

Problem $M_\kappa(D)$ *Find two vector fields* E_κ, $H_\kappa \in C^1(D) \cap C(\bar{D})$ *satisfying the time-harmonic Maxwell equations*

$$(1.4) \qquad \mathrm{rot}\, E_\kappa - i\kappa H_\kappa = 0\,, \ \mathrm{rot}\, H_\kappa + i\kappa E = 0 \quad \text{in } D\,,$$

the boundary condition

$$(1.5) \qquad [n, E_\kappa] = c_\kappa \quad \text{on } S$$

and the radiation condition

(1.6) $$[H_\kappa, \frac{x}{|x|}] - E_\kappa = o\left(\frac{1}{|x|}\right), \ |x| \to \infty,$$

uniformly for all directions $x/|x|$.

Provided E_κ and H_κ solve Problem $M_\kappa(D)$, then E_κ solves the special case of Problem $E_\kappa(D)$ where $\gamma_\kappa = 0$. Conversely, let E_κ be a solution to Problem $E_\kappa(D)$ with $\gamma_\kappa = 0$. Then E_κ and $H_\kappa := (1/i\kappa)\text{rot } E_\kappa$ solve Problem $M_\kappa(D)$.

For the boundary-value problem $E_\kappa(D)$ it is known that for every frequency $\kappa \neq 0$, $\text{Im}(\kappa) \geq 0$ a uniquely determined solution exists [5], [14]. The aim of this paper is to study the dependence of the solution E_κ as $\kappa \to 0$. In the limiting static case $\kappa = 0$ we obtain

<u>Problem $E_0(D)$</u> *Find a vector field* E_0 *with the regularity properties required for problem* $E_\kappa(D)$ *satisfying the vector Laplace equation*

(1.7) $$\Delta E_0 = 0 \quad \textit{in } D,$$

the electric boundary conditions

(1.8) $$[n, E_0] = c_0, \ \text{div } E_0 = \gamma_0 \quad \textit{on } S$$

and at infinity

(1.9) $$E_0(x) = o(1), \ |x| \to \infty,$$

uniformly for all directions $x/|x|$.

Since the homogeneous problem $E_0(D)$ admits nontrivial solutions, the mathematical character of the limiting static problem $\kappa = 0$ essentially differs from the stationary problem $\kappa \neq 0$.

A Dirichlet field [11] in D is a vector field $Y \in C^1(D) \cap C(\bar{D})$

satisfying the differential equations

(1.10) $\operatorname{div} Y = 0$, $\operatorname{rot} Y = 0$ in D ,

the boundary condition

(1.11) $[n,Y] = 0$ on S

and at infinity

(1.12) $Y(x) = o(1)$, $|x| \to \infty$.

By DF(D) we denote the set of all Dirichlet fields in D. Note that Dirichlet fields automatically satisfy $Y(x) = O\left(\frac{1}{|x|^2}\right)$, $x \to \infty$. This linear space DF(D) has dimension dim DF(D) = m and a basis $Y^1,\ldots,Y^m$ of DF(D) can be chosen such that

(1.13) $\int_{S_j} (n,Y^k)\,ds = \delta_j^k$, $j,k = 1,\ldots,m$.

As is readily verified the space of solutions to the homogeneous problem $E_0(D)$ is given by the Dirichlet fields DF(D) [7].

Therefore, studying the limiting behaviour of the solutions E_κ to the problem $E_\kappa(D)$ as the frequency κ tends to zero faces the difficulty of investigating a singular perturbation problem. Our treatment of this problem will be by transforming the boundary-value problem into an equivalent integral equation using the following representation theorem from [5].

<u>Theorem 1.1.</u> *Let* $E_\kappa \in C^2(D) \cap C(\bar{D})$ *be a solution to the vector Helmholtz equation (1.1) satisfying the radiation condition (1.3) and let* $\operatorname{div} E_\kappa$, $\operatorname{rot} E_\kappa \in C(\bar{D})$. *Then*

...

$$(1.14)\qquad E_\kappa(x) = \operatorname{rot} \int_S [n(y), E_\kappa(y)]\Phi_\kappa(x,y)\,ds(y)$$
$$- \int_S [\operatorname{rot} E_\kappa(y), n(y)]\Phi_\kappa(x,y)\,ds(y)$$
$$- \int_S n(y)\operatorname{div} E_\kappa(y)\Phi_\kappa(x,y)\,ds(y)$$
$$- \operatorname{grad} \int_S (n(y), E_\kappa(y))\Phi_\kappa(x,y)\,ds(y),\quad x \in D .$$

Here,

$$\Phi_\kappa(x,y) := \frac{1}{4\pi}\,\frac{e^{i\kappa|x-y|}}{|x-y|}$$

denotes the fundamental solution to the scalar Helmholtz equation in $\mathbb{R}^3$.

Define integral operators by

$$(K_{11,\kappa} b)(x) := 2 \int_S [n(x), \operatorname{rot}_x\{b(y)\Phi_\kappa(x,y)\}]\,ds(y),$$

$$(1.15)\qquad (K_{21,\kappa} b)(x) := -2 \int_S (n(x), b(y))\Phi_\kappa(x,y)\,ds(y),$$

$$(K_{22,\kappa}\mu)(x) := -2 \int_S \mu(y)\,\frac{\partial\Phi_\kappa(x,y)}{\partial n(x)}\,ds(y),\quad x \in S .$$

Then, from Theorem 1.1 by letting x tend to the boundary S with the help of the jump relations we obtain (see also [9])

<u>Theorem 1.2.</u> *Let* E_κ *be a solution to problem* $E_\kappa(D)$. *Then*

$$(1.16)\qquad b_\kappa := [\operatorname{rot} E_\kappa, n] \quad , \quad \mu_\kappa := (n, E_\kappa) \qquad on\ S$$

solve the system of integral equations

$$(1.17)\qquad b_\kappa - K_{11,\kappa} b_\kappa = 2[\operatorname{rot} G_\kappa, n]$$

$$(1.18)\qquad \mu_\kappa - K_{21,\kappa} b_\kappa - K_{22,\kappa}\mu_\kappa = 2(n, G_\kappa) \qquad on\ S$$

.

where

$$G_\kappa(x) := \text{rot} \int_S c_\kappa(y)\Phi_\kappa(x,y)ds(y) - \int_S \gamma_\kappa(y)n(y)\Phi_\kappa(x,y)ds(y)\ , \quad x \in D\ . \tag{1.19}$$

Since we have existence and uniqueness for Problem $E_\kappa(D)$, $\kappa \neq 0$, the system of integral equations (1.17) and (1.18) is equivalent to Problem $E_\kappa(D)$ for those κ for which we have uniqueness for the system (1.17) and (1.18). In this case, let b_κ and μ_κ be the solution of (1.17) and (1.18). Then

$$(1.20)\quad E_\kappa(x) := \text{rot} \int_S c_\kappa(y)\Phi_\kappa(x,y)ds(y) - \int_S b_\kappa(y)\Phi_\kappa(x,y)ds(y)$$
$$- \int_S n(y)\gamma_\kappa(y)\Phi_\kappa(x,y)ds(y) - \text{grad} \int_S \mu_\kappa(y)\Phi_\kappa(x,y)ds(y),\quad x \in D,$$

is the solution of Problem $E_\kappa(D)$. To discuss the integral equations, first we solve (1.17) for b_κ and then (1.18) for μ_κ.

The homogeneous form of equation (1.17) has nontrivial solution if and only if the interior boundary-value problem

$$\text{rot}\, E_\kappa - i\kappa H_\kappa = 0,\ \text{rot}\, H_\kappa + i\kappa E_\kappa = 0 \quad \text{in } D_i\ ,$$
$$[n,H_\kappa] = 0 \quad \text{on } S$$

has nontrivial solutions [12], [14]. The homogeneous form of equation (1.18) has nontrivial solutions if and only if the interior Neumann problem

$$\Delta\varphi_\kappa + \kappa^2\varphi_\kappa = 0 \quad \text{in } D_i\ ,$$
$$\frac{\partial\varphi_\kappa}{\partial n} = 0 \quad \text{on } S$$

has nontrivial solutions [10]. As is well known, both of these homogeneous problems have nontrivial solutions only for a countable set of positive eigenvalues κ accumulating at infinity. Hence, for sufficiently small frequencies $\kappa \neq 0$ the system of integral equations (1.17) and (1.18) is uniquely solvable.

In the limiting case $\kappa = 0$ both of the homogeneous equations (1.17) and (1.18) possess nontrivial nullspaces. Firstly, corresponding to the limiting behaviour of the boundary-value problem $E_\kappa(D)$ we have the nullspace [6], [8]

(1.21) $$N(I-K_{22,0}) = \{ \mu = (n,Y) \text{ on } S \text{ , } Y \in DF(D) \} .$$

Here, I denotes the identity operator. In particular, $\dim N(I-K_{22,0}) = m$. The nullspace for equation (1.17) is given in terms of Neumann fields [11] in D, e.g. solutions $Z \in C^1(D) \cap C(\bar{D})$ of the differential equations

(1.22) $$\operatorname{div} Z = 0 \text{ , } \operatorname{rot} Z = 0 \quad \text{in } D$$

satisfying the boundary conditions

(1.23) $$(n,Z) = 0 \quad \text{on } S$$

and

(1.24) $$Z(x) = o(1) \text{ , } |x| \to \infty .$$

By NF(D) we denote the linear space of all Neumann fields in D. Note that Neumann fields automatically satisfy $Z(x) = O\left(\frac{1}{|x|^2}\right)$, $|x| \to \infty$. Let p denote the topological genus of D. Then $\dim NF(D) = p$. If $p > 0$ we choose a basis $Z^1,\dots,Z^p$ of NF(D). For the nullspace of $I - K_{11,0}$ there holds [6], [8]

(1.25) $$N(I-K_{11,0}) = \{ b = [n,Z] \text{ on } S \text{ , } Z \in NF(D) \} .$$

In particular, $\dim N(I-K_{11,0}) = p$.

Thus, like the boundary-value problem itself the integral equations change at $\kappa = 0$ their nature from unique solvability to non uniqueness. Obviously, an additional complication arises from the fact that for multiply connected domains D, this means $p > 0$, the nullspace of the system of integral equations has a higher dimension than the nullspace of the boundary-value problem.

The limiting behaviour of solutions to the problem $E_\kappa(D)$ as κ tends to zero was extensively studied by Werner. In [14] and [15] he proved convergence in the case $p = 0$ by using integral equation techniques. Recently Werner [16] extended his work to multiply connected domains but without using the integral equation approach.

In this paper we shall demonstrate how to establish convergence for multiply connected domains $p \geq 0$ in a straighforward manner by the integral equation method. This will be done by first considering a singular perturbation problem in a general setting and then applying the results to the special case of the integral equations in Theorem 1.2.

2. A SINGULAR PERTURBATION PROBLEM.

Let X be a complex Banach space and let $K \subset \mathbb{C}$ be a subset such that $0 \in K$ is an accumulation point of K. Consider a family $\{K_\kappa : X \to X, \kappa \in K\}$ of compact linear operators and define $L_\kappa := I - K_\kappa$ where I denotes the identity operator. Assume that L_κ is regular for all $\kappa \neq 0$ and that L_0 is not regular. Then for all $\kappa \neq 0$ and all $f_\kappa \in X$ the equation

$$(2.1) \qquad L_\kappa \varphi_\kappa = f_\kappa$$

has a unique solution $\varphi_\kappa = L_\kappa^{-1} f_\kappa$. We are interested in finding sufficient conditions for convergence $\varphi_\kappa \to \varphi_0$, $\kappa \to 0$, to a solution φ_0 of the limiting equation

(2.2) $$L_o\varphi_o = f_o$$

under the assumptions

(2.3) $$\| K_\kappa - K_o \| \to 0, \ \| f_\kappa - f_o \| \to 0, \ \kappa \to 0 .$$

In addition, we require that the Riesz number of K_o is one which by the Riesz-Schauder theory [2] implies

(2.4) $$X = N(L_o) \oplus R(L_o)$$

where $N(L_o)$ and $R(L_o)$ denote the nullspace and range of L_o, respectively. Because $N(L_o)$ and $R(L_o)$ are closed subspaces the canonic projector $P : X \to N(L_o)$ introduced by the composition (2.4) is bounded. Hence, since $N(L_o)$ is finite dimensional P is compact.

Let $\varphi \in N(L_o+P)$. Then $L_o\varphi \in N(L_o) \cap R(L_o) = \{0\}$. This implies $\varphi \in N(L_o)$ and $P\varphi = 0$ and therefore $\varphi = 0$. From this, by the Riesz-Schauder theory, we conclude that $L_o + P$ is regular. Now define

(2.5) $$L_o^+ := (L_o+P)^{-1}(I-P) .$$

From $P(L_o+P) = P = (L_o+P)P$ we obtain $P(L_o+P)^{-1} = P = (L_o+P)^{-1}P$. Then $PL_o^+ = L_o^+P = 0$ and therefore

(2.6) $$L_oL_o^+ = L_o^+L_o = I - P .$$

Actually, (2.6) implies that L_o^+ is a generalized inverse of L_o [1].

Define

(2.7) $$M_\kappa := L_o^+(L_o-L_\kappa) .$$

Because of (2.3), for sufficiently small κ the operator $I - M_\kappa$