A REVIEW OF UNDERGRADUATE PHYSICS

BENJAMIN F. BAYMAN

MORTON HAMERMESH

A REVIEW OF UNDERGRADUATE PHYSICS

BENJAMIN F. BAYMAN
University of Minnesota

MORTON HAMERMESH
University of Minnesota

JOHN WILEY & SONS
New York / Chichester / Brisbane / Toronto / Singapore

Library of Congress Cataloging in Publication Data:

Bayman, Benjamin F., 1930–
A rcview of undergraduate physics.

Includes indexes.
1. Physics. I. Hamermesh, M. (Morton), 1915–
II. Title.
QC21.2.B38 1986 530 85-26577
ISBN 0-471-81684-1 (pbk)

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)$$

$$j_l(\rho) \underset{\rho\to 0}{\to} \frac{\rho^l}{(2l+1)!!} \qquad j_l(\rho) \underset{\rho\to\infty}{\to} \frac{\sin(\rho - l\pi/2)}{\rho}$$

$$n_l(\rho) \underset{\rho\to 0}{\to} -\frac{(2l-1)!!}{\rho^{l+1}} \qquad n_l(\rho) \underset{\rho\to\infty}{\to} -\frac{\cos(\rho - l\pi/2)}{\rho}$$

$$e^{ik\rho\cos\theta} = J_0(k\rho) + 2\sum_{m=1}^{\infty} i^m J_m(k\rho)\cos(m\theta)$$

$$J_m(\rho) \underset{\rho\to 0}{\to} \frac{\left(\frac{1}{2}\rho\right)^m}{m!} \qquad J_m(\rho) \underset{\rho\to\infty}{\to} \sqrt{\frac{2}{\pi\rho}}\cos\left(\rho - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(\rho) \underset{\rho\to 0}{\to} -\frac{(m-1)!}{\pi\left(\frac{1}{2}\rho\right)^m} \qquad N_m(\rho) \underset{\rho\to\infty}{\to} \sqrt{\frac{2}{\pi\rho}}\sin\left(\rho - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{l+\frac{1}{2}}(\rho)$$

$$[\nabla_r^2 + k^2]\frac{e^{ik|r-\vec{r}'|}}{|r-\vec{r}'|} = -4\pi\,\delta(r-\vec{r}')$$

$$\left(\frac{d^2}{dx^2} + k^2\right) e^{ik|x-x'|} = 2ik\,\delta(x-x')$$

$$f(x) = \sum_{m=0}^{\infty}\left[a_m\cos\left(\frac{2m\pi x}{L}\right) + b_m\sin\left(\frac{2m\pi x}{L}\right)\right] \qquad \text{if} \quad f(x) = f(x+L)$$

$$\text{where } a_m = \frac{2}{L(1+\delta_{m,0})}\int_0^L \cos\left(\frac{2m\pi x}{L}\right) f(x)\,dx$$

$$b_m = \frac{2}{L}\int_0^L \sin\left(\frac{2m\pi x}{L}\right) f(x)\,dx$$

$$\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk = 2\pi\,\delta(x-x')$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x} g(\omega)\,d\omega, \qquad g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$$

$$f(x+\varepsilon) = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\frac{d^n f(x)}{dx^n} = \exp\left(\varepsilon\frac{d}{dx}\right) f(x)$$

$$f(r+\boldsymbol{\varepsilon}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\vec{\varepsilon}\cdot\vec{\nabla}_{\vec{r}})^n f(\vec{r}) = \exp(\vec{\varepsilon}\cdot\vec{\nabla}_r) f(\vec{r})$$

$$\int f(x_1, x_2, \ldots)\,dx_1\,dx_2\ldots = \int f(x_1(y_1, y_2, \ldots), x_2(y_1, \ldots), \ldots)\left|\frac{\partial(x_1 x_2 \ldots)}{\partial(y_1 y_2 \ldots)}\right| dy_1\,dy_2\ldots.$$

$$\int_0^{\infty} x^n e^{-\lambda x}\,dx = \frac{\Gamma(n+1)}{\lambda^{n+1}} \qquad \Gamma(n+1) = n\Gamma(n)$$

$$\int_0^{\infty} x^n e^{-\lambda x^2}\,dx = \frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\lambda^{\left(\frac{n+1}{2}\right)}} \qquad \Gamma(1) = 1,\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

PREFACE

In this book we have tried to present a concise summary of most of the material covered in an undergraduate program in physics. Each topic is developed from fundamental principles and then applied to the solution of illustrative problems. These problems are of the type used by American graduate schools in their comprehensive physics examinations and in the Graduate Record Examination. This book should therefore be especially useful to someone who is preparing for such a comprehensive examination. We hope it will also be useful to students who are currently in an undergraduate physics program, and to engineers and scientists who are interested in more advanced treatments of subjects they encountered in their introductory physics courses.

We have tried to make our presentation as self-contained as possible. Of course, each of our chapters is too brief to be considered as a replacement for a monograph or textbook on its subject. However, if the goal is a review of a wide variety of physical ideas and applications in a reasonable amount of time, then brevity is necessary. Furthermore, by treating different subject areas of physics within the same volume, we have emphasized the important basic ideas that are common to these different areas. This makes the review process more efficient and deepens our appreciation of the unity of physics.

We believe that a book of this sort is most useful if its size and cost are both kept reasonably small. We have therefore included very little factual material of the kind that would be covered in introductory courses in atomic, nuclear, or solid-state physics. This factual material is an important component of a physics education, but it is not easily summarized. Moreover, in the interest of brevity we have assumed that the reader has a good understanding of vector algebra and calculus, and of the elementary properties of differential equations.

Most of this text uses the cgs Gaussian system of units. This is the system used in most graduate-level work in physics and in most of the research literature. Of course, the physical description of any system should be independent of units. Thus, a student who prefers to work in a different system of units should be able to transcribe all our expressions into his or her units with no change in essential physical content.

The idea for this book developed out of an informal seminar offered during the past ten years to help first-year physics graduate students at the University of Minnesota prepare for our Graduate Written Examination. Most of our illustrative examples are taken from previous University of Minnesota examinations. We have also included problems from the comprehensive examinations given at several other universities.

B. F. B.
M. H.

CONTENTS

CHAPTER 3 ELECTRICITY AND MAGNETISM 106

CHAPTER 4 OPTICS 174

CHAPTER 5 QUANTUM MECHANICS 199

CHAPTER 1

CLASSICAL MECHANICS

Books that attempt to survey all of physics traditionally begin with classical mechanics. There are several good reasons for following this tradition. Most of our physical intuition is based on mechanical models, most of the important concepts of physics have their simplest realization in mechanical systems, and the newer ideas of relativity and quantum mechanics are perhaps best appreciated in terms of their contrast with the views of classical mechanics.

1.1 NEWTON'S SECOND LAW OF MOTION

If the force $\mathbf{f}$ acts on a point particle of mass m, then

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2} \tag{1.1a}$$

Here $\mathbf{p}$ and $\mathbf{v}$ are the momentum and velocity of the particle relative to an inertial frame of reference, and $\mathbf{r}$ is a vector from a fixed point 0 in that frame of reference to the location of the particle. From (1.1a) we can derive

$$\begin{aligned}\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f} = m\mathbf{r} \times \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}(m\mathbf{r} \times \mathbf{v}) \\ = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) &= \frac{d\mathbf{l}}{dt}\end{aligned} \tag{1.1b}$$

$\boldsymbol{\tau}$ is the torque on the particle and $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is its angular momentum, both defined relative to the point 0.

Now consider a system of particles. The force $\mathbf{f}_i$ on particle i can be written as

$$\mathbf{f}_i = \mathbf{f}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{f}(j \text{ on } i) \tag{1.2}$$

$\mathbf{f}_i^{\text{ext}}$ is the external force on particle i, and $\mathbf{f}(j \text{ on } i)$ is the force on particle i due to particle j. Newton's third law of motion asserts that

$$\mathbf{f}(j \text{ on } i) = -\mathbf{f}(i \text{ on } j) \tag{1.3}$$

If we now sum (1.1a) and (1.1b) over all the particles and use (1.2) and (1.3), we find that

$$\mathbf{F} \equiv \sum_i \mathbf{f}_i^{\text{ext}} = \frac{d}{dt}\sum_i \mathbf{p}_i = \frac{d}{dt}\mathbf{P}_{\text{tot}} \qquad (1.4a)$$

$$\boldsymbol{\tau} \equiv \sum_i \boldsymbol{\tau}_i = \frac{d}{dt}\sum_i \mathbf{l}_i = \frac{d}{dt}\mathbf{L}_{\text{tot}} \qquad (1.4b)$$

The derivation of (1.4b) also requires that we assume that $\mathbf{f}(j \text{ on } i)$ is directed along the line joining particles i and j. Both the total torque $\boldsymbol{\tau}$ and the total angular momentum $\mathbf{L}_{\text{total}}$ in (1.4b) must be defined with respect to the same point. This point may be any point fixed in an inertial frame of reference (i.e., at rest in such a frame or moving with uniform velocity relative to it), or it may be the point that moves with the mass center of the system,[1] located at

$$\mathbf{R}_{\text{CM}} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M_{\text{tot}}} \qquad (1.5)$$

Usually we need to relate $\mathbf{P}_{\text{tot}}$ and $\mathbf{L}_{\text{tot}}$ to the motion of the system. For $\mathbf{P}_{\text{tot}}$ we have

$$\mathbf{P}_{\text{tot}} = \sum_i \mathbf{p}_i = \sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{d}{dt}\sum_i m_i \mathbf{r}_i = \frac{d}{dt}(M_{\text{tot}}\mathbf{R}_{\text{CM}}) = M_{\text{tot}}\frac{d}{dt}\mathbf{R}_{\text{CM}} \qquad (1.6a)$$

The relationship between angular momentum and angular velocity is more complicated. It is discussed in Section 1.8 below. Our present considerations will be limited to uniform rigid bodies rotating about an axis about which the body has rotational symmetry, or an axis which is perpendicular to a plane of reflection symmetry. In these cases we can write

$$L = I\omega \qquad (1.6b)$$

Here ω is the angular speed of the body (in radians per unit time), and I is the moment of inertia of the body about the rotation axis, defined by

$$I = \int dm\, s^2 \qquad (1.7)$$

The integration goes over every mass element dm of the body, and s is the perpendicular distance of the mass element dm from the rotation axis. Equations (1.4) and (1.6) can be combined to yield

$$\mathbf{F} = M_{\text{tot}}\frac{d\mathbf{V}_{\text{CM}}}{dt} = M_{\text{tot}}\frac{d^2\mathbf{R}_{\text{CM}}}{dt^2} \qquad (1.8a)$$

$$\tau = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2} \qquad (1.8b)$$

where $\omega = d\theta/dt$. We purposely avoid writing (1.8b) as a vector equation since it applies only to the case of rotation about a principal axis (see Section 1.8).

[1]Another valid (but less useful) choice is any point accelerating toward or away from the mass center.

1.2 SOME COMMONLY ENCOUNTERED FORCES

1.2.1 Friction

Suppose that an object is in contact with a surface. The force that the surface exerts on the object can be resolved into a perpendicular component **N** and a tangential component **f**. If the object slides along the surface, it is often a good approximation to assume that the magnitudes of **f** and **N** are related by

$$f = \mu_k N \tag{1.9a}$$

μ_k is called the coefficient of kinetic or sliding friction. The direction of **f** is usually assumed to be opposite to the velocity of the object relative to the surface. If the object is at rest on the surface the value of f depends on the other forces acting, but cannot exceed a critical value given by

$$f \leq \mu_s N \tag{1.9b}$$

μ_s is called the coefficient of static friction. The values of μ_s and μ_k are usually assumed to depend only on the nature of the surfaces in contact, and to be independent of the area of contact and the magnitude of **N**. If a problem refers to a "smooth" surface, this implies that $\mu_s = \mu_k = 0 = f$, so that the force that such a surface exerts on an object is exactly perpendicular to the surface.

1.2.2 Gravitation

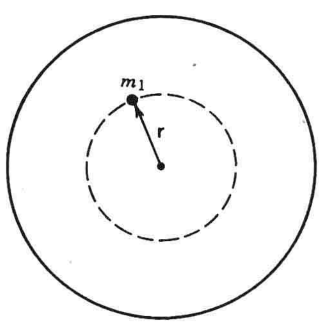
The gravitational force on a point mass m_1 due to another point mass m_2 is

$$\mathbf{F}_{(\text{on 1 due to 2})} = Gm_1m_2\frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = -Gm_1m_2\frac{\mathbf{r}}{r^3} = -Gm_1m_2\frac{\hat{r}}{r^2} \tag{1.10}$$

where G is the fundamental gravitational constant and **r** is the vector from mass m_2 to mass m_1. We can also use (1.10) to find the force on a point mass m_1 due to a *spherically symmetric* mass distribution. In this case **r** is the vector to m_1 from the center of the continuous mass distribution, and m_2 is the total mass within a distance r from the center (see Figure 1.1). In particular, if m_1 is wholly outside the continuous distribution, m_2 is the total spherical mass.

Now suppose the continuous mass distribution is the earth (assumed spherical), and we want the gravitational force on a point particle m_1 slightly above its surface. Then **r** points from the center of the earth, so an observer near the particle would say that the force on m_1 is vertically downward. If the height of

FIGURE 1.1 The gravitational force on m_1 is that of a point particle of mass m_2 located at the center of the sphere. The total mass within the dotted sphere of radius r is m_2.



the particle above the earth is small compared to the radius of the earth, (1.10) becomes

$$\mathbf{F} = -m_1\left(\frac{Gm_2}{R_e^2}\right)\hat{\mathbf{r}} = -m_1 g\hat{\mathbf{r}} \equiv m_1\mathbf{g} \tag{1.11}$$

Thus, we can describe the gravitational field near the surface of the earth as uniform, of magnitude g ($= 32.2\ \mathrm{ft/s^2} = 9.8\ \mathrm{m/s^2}$) and directed vertically downward. The total gravitational force and torque on a finite object are given by

$$\mathbf{F}_{\text{grav}} = \int dm\,\mathbf{g} = \mathbf{g}\int dm = M_{\text{tot}}\mathbf{g} \tag{1.12a}$$

$$\begin{aligned}\tau_{\text{grav}} &= \int \mathbf{r} \times dm\,\mathbf{g} = \left(\int dm\,\mathbf{r}\right) \times \mathbf{g} = M_{\text{tot}}\mathbf{R}_{\text{CM}} \times \mathbf{g} \\ &= \mathbf{R}_{\text{CM}} \times (M_{\text{tot}}\mathbf{g}) = \mathbf{R}_{\text{CM}} \times \mathbf{F}_{\text{grav}}\end{aligned} \tag{1.12b}$$

Equation (1.12b) shows that we get the correct value of the gravitational torque on an object if we assume that the entire gravitational force (weight) acts at a single point of the object, its mass center. This implies that the gravitational torque on an object, defined with respect to its mass center, is zero. This is true in general only for a uniform gravitational field.

1.2.3 Hooke's Law Springs

Suppose that a Hooke's law (or ideal) spring has an unstretched length l_0 and spring constant k. If the spring is stretched or compressed to length l it exerts a restoring force of magnitude

$$F = k|l - l_0| \tag{1.13}$$

Now consider the situation shown in Figure 1.2. Since the mass in Figure 1.2*b* is in equilibrium, the upward force due to the spring must equal the downward

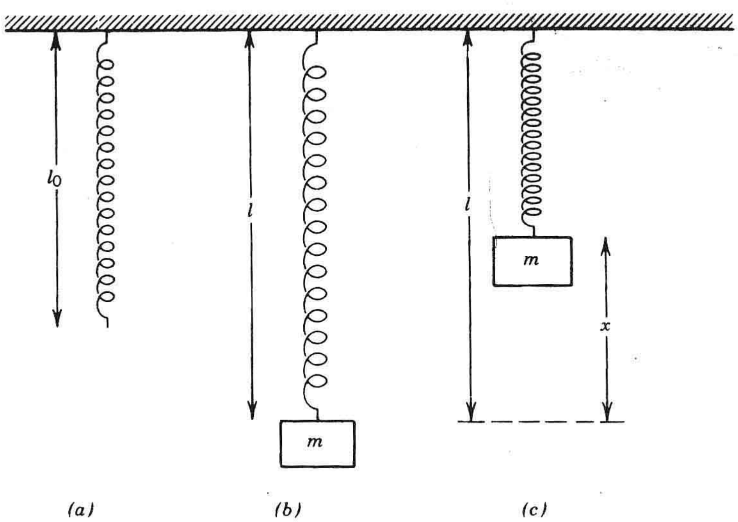


FIGURE 1.2 Three springs suspended from a ceiling: (a) the unstretched spring; (b) mass m in equilibrium under the combined forces of gravity and the spring; and (c) the mass displaced a distance x from equilibrium.

force due to gravity. Thus,

$$k(l - l_0) - mg = 0$$

In Figure 1.2*c* the mass m has been given an additional upward displacement x, so that the length of the spring is now $l - x$. The upward force due to the spring is now $k(l - x - l_0)$ while the downward gravitational force is still mg. Thus, the net upward force on the mass is

$$F = k(l - x - l_0) - mg = -kx \tag{1.14}$$

The minus sign in $-kx$ implies that an upward displacement of the mass results in a downward net force on the mass, and vice versa. We see that k governs the restoring force for oscillations about equilibrium. The equation of motion of the mass is

$$F = m\ddot{x} = -kx \tag{1.15a}$$

whose general solution

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t + \phi\right), \qquad (A, \phi, \text{constants})$$

describes oscillations about equilibrium ($x = 0$), with constant amplitude A and initial phase ϕ. A and ϕ depend on the initial conditions under which the mass is set into oscillation. The circular frequency

$$\omega = \sqrt{\frac{k}{m}} \tag{1.15b}$$

depends only on the materials of which the system is made. In particular, it is independent of the amplitude of the oscillations.

PROBLEM 1.2.1 A heavy object, when placed on a rubber pad that is to be used as a shock absorber, compresses the pad by 1 cm. If the object is given a vertical tap, it will oscillate. Ignoring the damping, estimate the oscillation frequency.

Let k be the spring constant of the rubber, and let x_0 $(= 1\text{ cm})$ be the equilibrium displacement. At equilibrium the upward force on the object is kx_0, and the downward force is mg. Thus, $kx_0 = mg$, $k = mg/x_0$. The circular frequency of small oscillations about equilibrium is $\omega = \sqrt{k/m} = \sqrt{g/x_0} = \sqrt{980/1}$ rad/s. Thus, the frequency is $(1/2\pi)\sqrt{980}$ cycles/s $= 4.98$ Hz.

PROBLEM 1.2.2 An automobile, with nobody inside, has a mass of 1000 kg, and has ground clearance 18 cm. After four persons with total mass 300 kg get into the car, the ground clearance is only 12 cm. They drive off. At what speed will the car, with its four passengers, bounce in resonance while moving along a road that is straight, level, and smooth, except for a transverse tar patch every 15 m? For simplicity assume that the shock absorbers are ineffective, and also that the fore and aft suspensions have the same bouncing frequency.

Adding passenger weight of 300g Newtons causes a 6-cm deflection. Thus, $k = 300\text{ kg} \times 9.8\text{ m/s}^2/.06\text{ m} = 5000 \times 9.8\text{ kg/s}^2$. Since the total mass of the loaded car is 1300 kg, the circular frequency is

$$\omega = \sqrt{k/M} = \sqrt{5000 \times 9.8\text{ kg/s}^2/1300\text{ kg}} = \sqrt{5000 \times 9.8/1300}\text{ s}^{-1}.$$

Thus, the period of the oscillations is

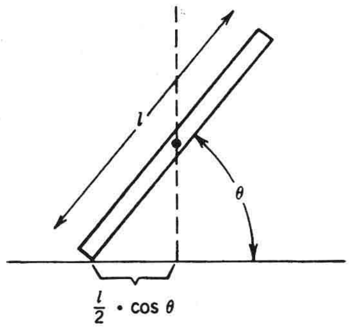
$$\tau = \frac{1}{\nu} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1300}{5000 \times 9.8}} \text{ s.}$$

If the car has a speed of

$$v = \frac{15 \text{ m}}{\tau} = \frac{15}{2\pi}\sqrt{\frac{5000 \times 9.8}{1300}} \text{ m/s} = 14.7 \text{ m/s} = 52.8 \text{ km/h}$$

the impulses due to the tar patch will be at the resonant frequency.

PROBLEM 1.2.3 A stick of length l is held so that one end rests on a smooth plane, making an angle θ with the plane. The stick is then released. How far will the left end of the stick have moved by the time the stick hits the plane?



The external forces acting on the stick (gravity and the surface contact force) are both vertical. Thus, $\mathbf{F}_{ext}$ has no horizontal component, and the acceleration of the mass center of the stick is vertical. Since the horizontal component of the velocity of the mass center is initially zero, it remains zero as the stick falls. This implies that the mass center of the stick falls vertically, so that by the time the stick is horizontal the left-hand end will have moved by $(l/2)[1 - \cos\theta]$.

PROBLEM 1.2.4 A thin stick of length L and mass m is supported at its ends by vertical strings so as to be in a horizontal position. One of the strings is cut at time t.

(a) Find the downward acceleration of the center of the stick at time $t + \delta$ (where $\delta \rightarrow 0$).

At time $t + \delta$, the external forces acting on the stick are shown in the free-body diagram (Figure 1.3b). The total external torque about the left end is $mgL/2$. Thus, the angular acceleration, α, of the stick about its left end is

$$\alpha = \frac{\tau}{I} = \frac{mgL/2}{(1/3)mL^2} = \frac{3}{2}\frac{g}{L}$$

The linear acceleration, a, of the center of the stick is then

$$a = \frac{\alpha L}{2} = \frac{3}{2}\frac{g}{L}\frac{L}{2} = \frac{3}{4}g$$

vertically downward.

(b) Find the sideward acceleration of the center of the stick. At time $t + \delta$, all the external forces acting on the stick are vertical. Thus, the total external force has no horizontal component, and the sideward acceleration of the stick is zero.

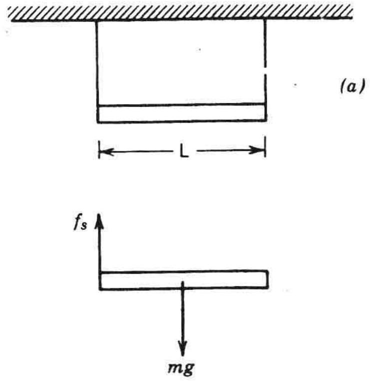


FIGURE 1.3 (a) The stick immediately after the right-hand string has been cut. (b) The forces acting on the string at that instant.

(c) Find the tension f_s in the remaining string.
Newton's second law applied to the stick gives

$$mg - f_s = ma = m \cdot \tfrac{3}{4}g$$
$$f_s = \tfrac{1}{4}mg$$

PROBLEM 1.2.5 An hourglass with vertical sides is placed on a critically damped balance, the sand trickling through the hole. What does the balance read? Discuss the direction of deflection of the balance during all stages of the flow.

Let the mass of the hourglass plus sand be M, and let F_b be the upward force that the balance pan exerts on the hourglass. According to Newton's third law, F_b is also the force that the hourglass exerts on the balance pan and thus F_b determines the reading on the balance scale. If y is the height of the center of mass of the hourglass plus sand, then

$$F_b - Mg = M\ddot{y}$$

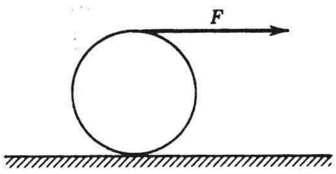
Thus, if $\ddot{y} = 0$, $F_b = Mg$, but if $\ddot{y}$ is positive (negative), F_b will be greater (less) than Mg.

Suppose that all the sand is at rest in the upper portion of the hourglass for $t < t_0$. At $t = t_0$ the sand starts to fall, and reaches a steady stream at $t = t_1$. The steady stream continues until $t = t_2$, when the flow starts to wane and comes to a stop at $t = t_3$. Thus, for $t < t_0$ and $t > t_3$ the sand is at rest, $\dot{y} = 0 = \ddot{y}$, and $F_b = Mg$. Between $t = t_1$ and $t = t_2$, $\dot{y} < 0$, but $\ddot{y} = 0$ (since the sand is falling at a constant rate) so that F_b still equals Mg. Between $t = t_0$ and $t = t_1$ we are going from a situation in which $\dot{y} = 0$ to one in which $\dot{y} < 0$. Thus, $\ddot{y} < 0$ between t_0 and t_1, so that $F_b < Mg$. Conversely, between $t = t_2$ and $t = t_3$ we are going from a situation in which $\dot{y} < 0$ to one in which $\dot{y} = 0$. Thus, $\ddot{y} > 0$ between t_2 and t_3, so that $F_b > Mg$. To summarize: the balance reads

$$\begin{aligned} Mg &\quad \text{for } t \le t_0 \\ < Mg &\quad \text{for } t_0 < t < t_1 \\ Mg &\quad \text{for } t_1 \le t \le t_2 \\ > Mg &\quad \text{for } t_2 < t < t_3 \\ Mg &\quad \text{for } t \ge t_3 \end{aligned}$$

The transitions between the different flow conditions described above will be smooth, since the momentum flux will not change discontinuously.

PROBLEM 1.2.6 A right circular cylinder has a density that is a function of distance from the symmetry axis. It rests on a frictionless surface. A string is wrapped around the periphery of the cylinder and a constant force F is applied to the string for a time T, in the horizontal direction.



(a) Describe qualitatively the translational and rotational motion of the object.

Since the surface is frictionless, the force that it exerts on the cylinder has no horizontal component. Thus, the horizontal component of the total external force is F when $0 \leq t \leq T$, and zero when $T \leq t$. The axis of the cylinder, therefore, has acceleration F/M when $0 \leq t \leq T$, and zero when $T \leq t$, so that its speed, v, is given by

$$v = \frac{F}{M}t, \qquad \text{for } 0 \leq t \leq T$$

$$= \frac{F}{M}T, \qquad \text{for } T \leq t$$

In the interval $0 \leq t \leq T$, the external torque about the axis is FR, so the angular acceleration is $\alpha = FR/I$ for $0 \leq t \leq T$, and zero for $T \leq t$. Thus, the angular speed of rotation of the cylinder about its axis is

$$\omega = \frac{FR}{I}t, \qquad \text{for } 0 \leq t \leq T$$

$$= \frac{FR}{I}T, \qquad \text{for } T \leq t$$

(b) Find a specific geometry for the object so that the kinetic energy is equally divided between translational and rotational motion.

The translational part of the kinetic energy is

$$\frac{1}{2}Mv^2 = \frac{1}{2}M\left(\frac{F}{M}t\right)^2$$

The rotational part is

$$\frac{1}{2}I\omega^2 = \frac{1}{2}I\left(\frac{FR}{I}t\right)^2$$

If these are to be equal, we must have

$$\frac{1}{M} = \frac{R^2}{I}, \qquad I = MR^2$$

The only way this can occur is if all the mass of the cylinder is at distance R from the axis. Thus, the cylinder must consist of a thin layer of material around an empty core.

PROBLEM 1.2.7 A bowling ball of mass M and radius R is thrown onto a surface with speed v_0. The coefficient of kinetic friction between the ball and the surface is μ. Initially, the ball is sliding without rolling. What will be its speed when it rolls without sliding?

The friction force f slows the speed of the mass center of the ball and increases the angular speed of the ball around its mass center:

$$M\frac{dv}{dt} = -Mg\mu \qquad \text{(horizontal component of external force)}$$

$$\frac{d}{dt}I\omega = Mg\mu R \qquad \text{(torque of external force about mass center)}$$

Thus,

$$v = v_0 - g\mu t$$

$$\omega = \omega_0 + \frac{Mg\mu Rt}{I} = \frac{Mg\mu Rt}{I}$$

Pure rolling will occur when $v = \omega R$, because then the point of contact of the ball with the surface will have zero speed relative to the surface. This occurs at a time t satisfying

$$v_0 - g\mu t = \frac{Mg\mu R^2}{I}t$$

$$t = \frac{v_0}{g\mu\left[1 + \frac{MR^2}{I}\right]}$$

The speed of the mass center at this time is

$$v = v_0 - g\mu\frac{v_0}{g\mu\left[1 + \frac{MR^2}{I}\right]} = v_0\frac{MR^2}{I + MR^2}$$

To complete the solution we need the moment of inertia of a uniform sphere about a diameter. Suppose that the center of the sphere is at the origin of a rectangular coordinate system. The moment of inertia of the sphere about the z axis is

$$I = \int dm\,(x^2 + y^2)$$

The symmetry of the sphere implies that

$$\int dm\,x^2 = \int dm\,y^2 = \int dm\,z^2 = \frac{1}{3}\int dm\,(x^2 + y^2 + z^2)$$

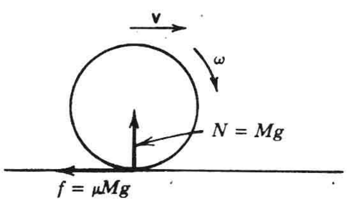


FIGURE 1.4 The center of the ball moves with velocity **v**, while the ball rotates about its center with angular speed ω. **N** and **f** are components of the force that the surface exerts on the ball.