ELECTROMAGNETISM AND RELATIVITY

with particular reference to moving media and electromagnetic induction

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BY

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PREFACE

A STUDY of the contents of texts on electricity and magnetism, as a branch of physics, reveals the almost complete absence of a treatment of the electromagnetics of moving media. Since the experimental and theoretical investigation of the electromagnetic effects of moving bodies played a vital part in the development of modern physical theory, from a broad educational viewpoint such an omission is unfortunate. Moreover, not only does this neglect result in unbalanced presentations of electrical science but it has led to the persistence of erroneous views about certain cases of electromagnetic induction.

It is now evident that teachers are becoming more aware of the inadequacy of the over-simplified statements of the laws of induction so often found in older books, but nevertheless elementary theory has not yet been brought completely into accord even with the first-order form of relativisitic electromagnetism. This inconsistency is obscured when moving bodies are conductors, and is easily overlooked since it does not lead to a wrong result for any problem of practical importance. The simplest example of the elusive flaw is that of the old puzzle of a rotating cylindrical magnet: according to the form of theory which is widely accepted, the electromagnetic forces in the moving iron depend solely on the motion of the material through its own magnetic field, and therefore the consequent displacement of free charges, or conduction electrons, is exactly the same as that which would occur in a non-magnetic conducting cylinder rotating in an identical magnetic field. In other words, it is concluded that when such a rotating magnet is made to produce unipolar induction as in one of Faraday's original experiments, the e.m.f. is induced entirely in the magnet itself. This conclusion, although consistent with the original non-relativistic electron theory of H. A. Lorentz, is incompatible with the first-order form of relativistic theory.

The prolonged controversy about this particular problem is a symptom of a basic weakness, namely that relativity theory has not yet been properly assimilated into physics. Although relativistic electromagnetism is now some fifty years old, it has remained, at least in this country, largely an advanced subject in the education of mathematicians,* and in texts on electricity and magnetism is either omitted or, in a few cases, confined to a superficial introduction. For the mathematician, the transformations of relativity theory provide the standard method of obtaining expressions for electromagnetic phenomena which arise from the motion of material bodies, and all that is required is a theory of electromagnetism for bodies at rest, together with the formal mathematical application of the principles of relativity. If correctly used, this method provides an elegant specification of correct results, but at the expense of a consideration and understanding of basic electromagnetic *physics*. Hence it has come about that, except in treatises on relativity, the electromagnetics of moving bodies are rarely considered at all.

There appears to be a need, therefore, for a treatment which stresses the fundamentals of electromagnetic physics within the relativistic framework, and which includes a more thorough examination of the theory of electromagnetic induction. Standard books on relativity usually contain no more than a general outline of Minkowski's theory of moving media, with little concern for its practical application and interpretation in actual experiments. Further, a study of the original literature shows that the subject is by no means free from confusion and inconsistency. Hence a treatment from a physical rather than a mathematical viewpoint may be of interest even to those who are familiar with relativity theory as usually presented, while providing the basis for a first-order and elementary theory which can be taught to undergraduate students of physics and electrical engineering without recourse to the full Theory of Relativity.

In Part I of this book I have attempted such a task. The

* Cf. Sir James Jeans, *The Mysterious Universe* (Cambridge U.P., 1930) p. 98 : " The signal for the revolution was a short paper which Einstein published in June 1905. And with its publication, the study of the inner workings of nature passed from the engineer-scientist to the mathematician " ; p. 126 : " no one except a mathematician need ever hope fully to understand . . . the theory of relativity . . . " ; p. 134 : " the Great Architect of the Universe now begins to appear as a pure mathematician." It seems scarcely necessary to add that Jeans was a mathematician.

theoretical method adopted is based on the recognition, on the one hand, of the components of all the electromagnetic field vectors, **E**, **D**, **H** and **B**, which specifically arise from sources within a moving body, and on the other hand the components of these same quantities which arise from sources outside it. This leads to a consistent first-order theory which is remarkably simple in its application to specific cases. Further, it provides a form of relativistic theory which, in contrast to the usual "transformations", gives a detailed account of the effect of each moving body, thus facilitating the exact solution of practical problems.

This emphasis on field *components* reflects a change in physical viewpoint. In Maxwell's theory the electromagnetic field is postulated as a physical state in a material medium or aether, the field being the primary concept. Its sources—charges and currents—are secondary in importance and their individual contributions to the resultant field, which alone enters into Maxwell's equations, are of no physical significance. Now although Maxwell's equations have survived to the present day, the discovery of the electron and the development of relativity theory have removed the physical props upon which they were built. If an aether exists, it is impossible to measure the velocity of a body relative to it, and the primary particles of matter have attained a status of their own. An electron may reasonably be regarded as the source of its field, rather than the other way about. Indeed the field itself may be regarded as no more than a mathematical construct which simplifies the description of the effect of the behaviour of charged particles, and it is clearly in keeping with this viewpoint to describe the electromagnetic effect of a moving body in terms of its contribution to the resultant field.

Nearly all the work on the basic physics of relativistic electromagnetism has been based on the electron theory, in which electric charges are taken as fundamental and magnetic poles have no explicit part. We shall see, however, that an intelligible interpretation of some of the basic relations, such as those between **H** and **D**, is possible only in terms of poles. The concept still exists, rather like the black sheep of the family who is never mentioned, even in Lorentz's own equations. A theory of the macroscopic field of magnetized bodies based on magnetic dipoles is perfectly valid and well founded in the historical development of electromagnetism, and the neglect of the concept in relativistic theory tends to suggest a discontinuity with the established theory of magnetostatics which does not, in fact, exist. I have therefore developed the theory of moving magnets from both points of view, with frequent examples of their equivalence. The use of the pole concept, moreover, makes possible a valid first-order form of theory which can be presented without the aid of the Theory of Relativity, whereas this is not true of the electron theory.

Part II of the book deals with the laws of electromagnetic induction, and various problems including those of magnetic-field sources rotating about an axis of symmetry. Of considerable interest is the case of induction in a rotating cylinder which is both an insulator and magnetic, for in this case the prevalent theory gives an incorrect result. The experiment of M. and H. A. Wilson on this phenomenon does not seem to have received the attention it deserves, for it is fundamental. Another problem for which prerelativistic theory is inadequate is that of the velocity of propagation of an electromagnetic wave in a moving dielectric which is magnetic. Apart from these fundamental cases, the component-field theory simplifies the solution of more practical problems of electromagnetic induction, but as an aid to its use it is necessary to clarify the meaning of the terms *induced e.m.f.* and *potential difference.* In particular, it is found essential to distinguish between the potentials of "true" and "apparent" charges, for unless this is done it is impossible, in some cases, to disentangle the relativistic and the non-relativistic theories.

A short chapter deals in an introductory way with the interesting but complex subject of electromagnetic induction in conducting fluids, a matter which has assumed importance in geophysics since the theoretical deduction of magneto-hydrodynamic waves by Alfvén in 1942 and their experimental production by Lundquist in 1949. Sir Edward Bullard's theory of the Earth's magnetic field is a most interesting example of electromagnetic induction.

Part III deals with the relativistic theory of electromagnetic induction in moving circuits, and its application to cases of e.m.f.'s of the second order of magnitude. The concept of electromotive force in a closed circuit is of practical value chiefly when it produces a single-valued current, given by $I = e/R$. This is generally true for direct and alternating currents in stationary non-radiating circuits provided the length of the circuit is very small compared with the wave-length corresponding to the frequency, and so the law has a very wide application. The relativistic transformation of time shows, however, that if the e.m.f. in a moving circuit is calculated *instantaneously* in the usual way by Faraday's law, then the relation $I = e/R$ is strictly no more than an approximation. The consequent error is quite negligible in all ordinary practical cases, but cannot be ignored, even at low velocities, when a magnetic field is due to individual moving charges as distinct from complete current-circuits or magnets. Such problems are of interest since they clearly demonstrate the necessity of incorporating the relativistic transformation of space and time in the Maxwell-Lorentz theory of electromagnetism.

The fourth and final part of the book is concerned with a study of electromagnetic energy and momentum. Maxwell's aether was assumed to be the seat of momentum but in Lorentz's theory this property disappeared. Since the resultant of the electromagnetic forces acting on the material bodies in an electromagnetic system need not be zero, Lorentz supposed that Newton's third law of motion, specifying equality of action and reaction, may not be universally valid. If, however, we abandon the aether concept altogether and accept the motion, in empty space, of electromagnetic energy possessing mass, then Newton's law is reinstated.

Although electricity is an experimental science, conventional theory is still based on an assumption of Maxwell which has since been proved by experiment to be incorrect, namely that the carriers of a conduction current have no inertia or momentum. This inertia may be brought within the scope of electromagnetic theory if we suppose that the magnetic energy of a current is actually the kinetic energy of the conduction electrons, and Chapter 18 is devoted to the development of this hypothesis. It is then found to lead to a theory of the induction of currents in a superconductor, a phenomenon previously supposed to be outside the scope of classical electromagnetism. Chapters 18 and 19 are based on two published monographs, and my thanks are due to the Council of the Institution of Electrical Engineers for permission to make use of them in this book.

The book is written for advanced students and teachers rather than as an elementary textbook. It is assumed, therefore, that readers will be familiar with basic electromagnetic theory, at least to the standard of my *Fundamentals of Electromagnetism*,* and will have some knowledge of vector analysis, but for reasons which should be clear no use has been made of the four-dimensional analysis so attractive to mathematicians. This is, after all, no more than an alternative method of describing the physical theory expressed in the reciprocal Lorentz transformation, and hence cannot legitimately be made to produce any physical results unobtainable from a correct use of this transformation. Its great danger lies in the ease with which errors in physical interpretation can be obscured by mathematical elegance, and a serious example of this is the subject of Chapter 5, Sections 7–8.

This argument about the life of space travellers, in which one side finds cogent support in Einstein's principles and the other in Einstein's equations, suggests however that some basic inconsistency has been overlooked. We find, in fact, that in the accepted interpretation of the theory there is an unjustifiable assumption about velocity. The velocity of a body can be measured either by one observer, with radar, or by two observers on the path with synchronized clocks like timekeepers at the start and finish of a race. The position of a distant moving body cannot be measured by one observer with measuring rods. Strict acceptance of the velocity c on each leg of a return path, when light is reflected by a moving body, requires identical results by the two possible methods, but this is incompatible with the reciprocity of reference systems. If, then, Einstein's two principles are to be retained the two methods, unless the velocity is small compared with c, must be accepted as giving different measures. The "observed" velocity, measured by radar, is that which must be used in the Lorentz transformation and in relativistic kinematics, and must be less than c. The "clocked" velocity, on the other hand, may exceed c. This distinction, which is not incompatible with

* Cambridge University Press, Second edition, 1949.

experiment, requires a new interpretation of relativistic results, such as that for the variation of mass, and the Lorentz transformation is necessarily limited to measures obtained by two observers who must be at the origins of their respective systems.

For those who have no previous knowledge of relativity theory it is suggested that, in a first reading, attention may be confined to Chapters 1 to 4 and 8 to 10. These provide the basis for the presentation of a first-order form of theory, not requiring relativity, in undergraduate courses, with as much or as little of the detail as may suit the individual teacher.

The rationalized m.k.s. system of units is used, for this is now widely used by electrical engineers and its merits are gaining recognition by physicists. It is ideal for the purpose of this work, since it prevents confusion, so prevalent in c.g.s. systems, between **H** and **B** or between **E** and **D**. Since most of the classical papers on the relativistic electromagnetism of moving bodies are written in rationalized Gaussian units, a note on the relation between the two systems has been included, on p. xxii.

A word of explanation about certain symbols, and their names, is desirable. The International Electrotechnical Commission, when it adopted the m.k.s. system of units in 1935, retained the names *permittivity* and *permeability* for the measures **D**/**E** and **B**/**H**, thus necessitating the introduction of the terms *permittivity and permeability of free space*. In my *Fundamentals of Electromagnetism* I suggested, instead of the latter, the names *primary electric and magnetic constants* as being less suggestive of a material aether, but I otherwise followed the nomenclature, now adopted by the British Standards Institution,* used by the I.E.C. In the present book, however, I have adopted Professor Carter's more logical system,† and have retained the old meanings of permittivity and permeability as non-dimensional numerical ratios, equal to unity in free space and denoted by κ and μ, using different symbols, ϵ_0 and η_0, for the dimensional primary electric and magnetic constants whose unit values

* *British Standard*, No. 1991, Part 1, 1954, p. 21. The system has been adopted in *The Teaching of Electricity, with special reference to the use of m.k.s. units*, (Report of a Sub-Committee of the Science Masters' Association, Murray, 1954).

† G. W. Carter, *The Electromagnetic Field in its Engineering Aspects* (Longmans, 1954) p. viii.

define the c.g.s. systems. Before the introduction of the m.k.s. system the symbols κ and μ were authorized standards * for what everybody used to understand by permittivity and permeability, and I consider that it would have been far better, on the introduction of the m.k.s. system, to leave them, with their physical meanings, unchanged and to adopt new symbols for the primary constants. The notation adopted enables us, if we wish, to put $\kappa\epsilon_0 = \epsilon$ and $\mu\eta_0 = \eta$, but a clear distinction between free space and a material medium is essential in this book and is facilitated by the retention of the separate symbols in the relations $\mathbf{D} = \kappa\epsilon_0\mathbf{E}$ and $\mathbf{B} = \mu\eta_0\mathbf{H}$. Perhaps the various official committees whose deliberations lead to standards may see fit to reconsider this matter. As Professor Carter has said, it is earnestly to be hoped that the nomenclature and notation of the m.k.s. system are not immovably fixed in a bad usage.

I acknowledge with gratitude my debt to all those who, in one way or another, have helped me to clear up particular points, including my colleague Dr. D. E. Rutherford and Professor G. W. O. Howe to whose early articles on electromagnetic problems I largely owe the awakening of my interest. Professor G. W. Carter of Leeds most kindly read the manuscript when near its final stage and made some very helpful suggestions. My thanks are also due to the Editor of the *Wireless Engineer* for his permission to reproduce Figs. 16.1 and 16.2 which first appeared in that journal (1949, p. 344), and to the Cambridge University Press and Messrs. Macmillan and Co. for permission to quote short passages from certain of their publications.

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Finally, I wish to express my gratitude for the advice and encouragement, concerning publication, which I have received from the Institution of Electrical Engineers.

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* *British Standard*, No. 423, 1931, p. 6.

LIST OF CHIEF ELECTROMAGNETIC SYMBOLS

Rationalized M.K.S. Units

ϵ_0	primary electric constant $= 8{\cdot}854 \times 10^{-12}$ coulomb/volt-metre, or farad/metre.
η_0	primary magnetic constant $= 4\pi \times 10^{-7}$ weber/ampere-metre, or henry/metre.
c	velocity of electromagnetic waves *in vacuo* $= 2{\cdot}998 \times 10^8$ m/sec ($\epsilon_0\eta_0 = 1/c^2$).
$\mathbf{A}$, A	magnetic vector potential, webers/m.
$\mathbf{B}$, B	magnetic flux-density or induction, webers/m^2.
C	capacitance, farads.
$\mathbf{D}$, D	electric flux-density or displacement, coulombs/m^2.
$\mathbf{E}$, E	electric field intensity, volts/m.
e, E	electromotive force, volts.
e	electronic charge (a negative number), $= -1{\cdot}602 \times 10^{-19}$ coulomb.
$\mathbf{H}$, H	magnetic field intensity, ampere-turns/m.
i, I	current, amp.
$\mathbf{J}$, J	current density, amp/m^2.
j_s,	surface current density, amp/m.
K	rate of ohmic heating, per unit volume, watts/m^3.
L	coefficient of self inductance, henrys.
L_m	coefficient of mutual inductance, henrys.
$\mathbf{M}$, M	intensity of magnetization, webers/m^2.
$\mathcal{M}$	magnetic pole strength, webers.
$\mathbf{n}$, n	magnetic moment, weber-metres.
$\mathbf{N}$, N	electric dipole moment, coulomb-metres.
$\mathbf{P}$, P	electric polarization, coulombs/m^2.
$\mathbf{P}$	ponderomotive force, per unit volume, newtons/m^3.
q, Q	electric charge, coulombs.
R	resistance, ohms.
r	resistivity, ohm-metres.

T kinetic energy, joules.
U volume density of field energy, joules/m³.
V potential difference, volts.
γ conductivity, mhos/m.
κ permittivity or dielectric constant (numeric).
μ permeability (numeric).
Φ magnetic flux, webers.
ϕ, ψ electric scalar potential, volts.
Ψ electric flux, coulombs.
ρ charge density (volume), coulombs/m³.
σ surface charge density, coulombs/m².

Components of the electromagnetic field measures

external components $\mathbf{E}_0, \mathbf{D}_0;\quad \mathbf{H}_0, \mathbf{B}_0$
self components $\mathbf{E}_s = \mathbf{E}_p + \mathbf{E}_q,\quad \mathbf{D}_s = \mathbf{D}_p + \mathbf{D}_q;\quad \mathbf{H}_p, \mathbf{B}_i$

Relativistic factor $\beta = (1 - v^2/c^2)^{-1/2}$

List of Greek Letters used

α	alpha	μ	mu
β	beta	π	pi
γ	gamma	ρ	rho
δ	delta	Σ, σ	sigma
ϵ	epsilon	τ	tau
η	eta	Φ, ϕ	phi
θ	theta	χ	chi
κ	kappa	Ψ, ψ	psi
Λ, λ	lambda	Ω, ω	omega

ELECTROMAGNETIC EQUATIONS IN GAUSSIAN AND M.K.S. UNITS

Electromagnetic systems of units involve three constants, ϵ_0, η_0, and A, connected by the relation $A^2/\epsilon_0\eta_0 = c^2$. In the *rationalized c.g.s. symmetric* (Gaussian) system, used by H. A. Lorentz, H. Minkowski, M. Born, Einstein, Laub and many others, ϵ_0 and η_0 are each taken equal to unity and $A = c$. In the rationalized m.k.s. (practical) system, now extensively used, $A = 1$, $\eta_0 = 4\pi \times 10^{-7}$, and $\epsilon_0\eta_0 = 1/c^2$. Maxwell's equations may be written in a form applicable to both systems :

$$\begin{aligned} \text{curl } \mathbf{H} &= \frac{1}{A}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \\ \text{div } \mathbf{D} &= \rho \\ \text{curl } \mathbf{E} &= -\frac{1}{A}\frac{\partial \mathbf{B}}{\partial t} \\ \text{div } \mathbf{B} &= 0, \end{aligned}$$

together with $\mathbf{B} = \eta_0\mathbf{H} + \mathbf{M}$, $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, $\mathbf{F}_e = q\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{A}\right)$, $\mathbf{F}_m = \mathscr{M}\left(\mathbf{H} - \frac{\mathbf{v}\times\mathbf{D}}{A}\right)$.

The relativistic transformations for the electromagnetic measures (p. 87) then become

$$\mathbf{E} = \beta\left[\frac{\mathbf{E}'}{\alpha} - \frac{\mathbf{v}\times\mathbf{B}'}{A}\right] \quad 6(24)$$

$$\mathbf{D} = \beta\left[\frac{\mathbf{D}'}{\alpha} - \frac{A}{c^2}(\mathbf{v}\times\mathbf{H}')\right] \quad 6(25)$$

$$\mathbf{H} = \beta\left[\frac{\mathbf{H}'}{\alpha} + \frac{\mathbf{v}\times\mathbf{D}'}{A}\right] \quad 6(26)$$

$$\mathbf{B} = \beta\left[\frac{\mathbf{B}'}{\alpha} + \frac{A}{c^2}(\mathbf{v}\times\mathbf{E}')\right] \quad 6(27)$$

$$\rho = \beta\left[\rho' + \frac{A}{c^2}(\mathbf{v}\cdot\mathbf{J}')\right] \quad 6(32)$$

$$\mathbf{J} = \alpha\mathbf{J}' + \beta\frac{\rho'\mathbf{v}}{A} \ . \quad 6(34)$$

The apparent polarization of a moving magnet is

$$\mathbf{P}_a = \beta \frac{\epsilon_0}{A}(\mathbf{v} \times \mathbf{M}') \quad , \qquad 6(66), 6(70)$$

and the apparent magnetization of a moving polarized dielectric is

$$\mathbf{M}_a = \beta \frac{\eta_0}{A}(\mathbf{P}' \times \mathbf{v}) \quad . \qquad 6(65)$$

CONTENTS

PART I

THE ELECTROMAGNETIC FIELD OF BODIES IN UNIFORM RECTILINEAR MOTION