# INTEGRATION

By

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PRINCETON
PRINCETON UNIVERSITY PRESS
1944

Second Printing 1947
Third Printing 1950
Fourth Printing 1957
Fifth Printing 1961

Printed in the United States of America

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# *Preface*

The swift development of analysis in the twentieth century, beginning with the theory of the Lebesgue integral, has been of tremendous mathematical importance. No mathematician today can afford to be ignorant of the modern theories of integration, and it is to the profit of a student of mathematics that he become acquainted with these ideas early in his graduate studies. On the other hand, most of the writings on integration are written by mature mathematicians for mature mathematicians, often in an admirably concise form which is not appreciated by a beginner. This book is written with the hope that it will open a path to the Lebesgue theory which can be travelled by students of little maturity.

It is for the sake of such readers that details are explicitly presented which could ordinarily be regarded as obvious. An experienced mathematician may regard many proofs as verbose. Probably some of them are unnecessarily wordy, even for the veriest beginners; equally probably there are details omitted as obvious which will not be obvious to all readers. In view of the audience to whom this is addressed, the latter must be considered the graver fault.

The scheme of introducing the Lebesgue and Lebesgue-Stieltjes integral here adopted is a modification of that of Daniell, the integral appearing as the result of a two-stage generalization of the Cauchy (or Stieltjes) integral. Perhaps this manifestation of a connection between continuous functions and summable functions may help the beginner to feel at home in the newer theory.

There are few historical remarks on the theorems and methods here used and there is practically no bibliography. These are not usually of great interest to a beginner, and a student who wishes to continue further into the subject will necessarily read treatises—above all, Saks' *Theory of the Integral*—which will furnish bibliographical and historical references.

In only a few features can this book make claims to novelty. An expert will usually recognize known proofs used in assorted combinations and modifications. One acknowledgement must however be made. The latter part of the chapter on differential equations owes much to a mimeographed set of lecture notes on differential equations by Professor G. A. Bliss.

Part of the material in this book has been used in teaching graduate classes at the University of Virginia, and in several respects the choice of subject matter and of forms of proof has been guided by the comments of the students, especially by those of Dr. B. J. Pettis.

Shortly after the manuscript reached the Editors of the Princeton Mathematical Series I was called to the Aberdeen Proving Ground to help with the work in exterior ballistics. As a result, I lacked the time to perform the usual final tasks. I am most grateful to the Editors for their kindness in taking over duties which properly should have devolved upon the author, and thereby advancing the date of publication by many months. In particular, I owe thanks to Dr. Paco Lagerstrom, who worked long and efficiently over the manuscript.

In the correction of proof I have been greatly assisted by Miss Mary Jane Cox, who not only read all proof-sheets but pointed out a number of places in which rewording was needed for the sake of clarity.

Finally, I wish to thank Princeton University Press for its cooperativeness and efficiency.

E. J. McShane.

Charlottesville, Virginia,
*November* 21, 1943.

# Contents

CHAPTER I

# *Some Theorems on Real-valued Functions*

The entire subject-matter of this book rests upon the properties of real numbers, with which we assume the reader to be familiar. No appeal is made to geometric intuition. Nevertheless, it is often convenient to use the language of geometry, and this is permissible if we define all our geometric expressions in terms of number.

If $q$ is a positive integer, we shall say that each ordered $q$-tuple $(x^{(1)}, \cdots, x^{(q)})$ of real numbers is a "point in $q$-dimensional space," or a "point in $R_q$." For convenience in notation we prefer to put the indices $^{(1)}, \cdots, ^{(q)}$ up instead of down; the lower position will be reserved for subscripts distinguishing different points from each other. Usually we abbreviate by writing $x$ for $(x^{(1)}, \cdots, x^{(q)})$. A standing notational convention will be the following. If any letter, with or without affixes, is used to denote a point in $q$-dimensional space, the $q$ numbers defining the point will be denoted by the same symbol (with the same affixes if any) with superscripts $^{(1)}, \cdots, ^{(q)}$. Thus if we speak of a point $y_0$ in $q$-dimensional space $R_q$ we mean the ordered $q$-tuple

$$(y_0^{(1)}, y_0^{(2)}, \cdots, y_0^{(q)}).$$

Two points of a space $R_q$ are identical if and only if corresponding numbers in the two $q$-tuples are equal; that is, if $x$ and $y$ are both in $R_q$ the equation $x = y$ has the same meaning as the $q$ equations

$$x^{(1)} = y^{(1)}, \qquad \cdots, \qquad x^{(q)} = y^{(q)}.$$

An ordered $q$-tuple and an ordered $p$-tuple $(p \neq q)$ will never be regarded as identical.

Having this system of abbreviation, it is reasonable to proceed a step further and define sums $x + y$ and products $cx$, where $x$ and $y$ are points in $R_q$ and $c$ is a real number. The definitions are

$$\begin{aligned} x + y &= (x^{(1)} + y^{(1)}, \cdots, x^{(q)} + y^{(q)}), \\ cx &= (cx^{(1)}, \cdots, cx^{(q)}). \end{aligned}$$

We have little use for these symbols until the later chapters.

In order that a collection $E$ of points of $R_q$ shall be called a *point set* in $R_q$ we require only that, given any point $x$ of $R_q$, it must be possible to determine whether or not $x$ belongs to the collection $E$.

However, there is a great deal of trouble concealed in this statement. The difficulty lies in giving a precise meaning to the word "determine." Clearly it is not possible to list all the infinitely many points of $R_q$ one by one, marking each as belonging to $E$ or not belonging to $E$. Some rule must be given. This leads to a further question. What is a rule? Now we have begun to enter the domain of foundations of mathematics; and, without denying the importance of such studies, we shall turn back again to the narrower study of the points of our spaces $R_q$. We shall assume that the reader has some reasonably adequate concept of a rule; if he has doubts of this, as we all may well have, we can only refer him to the various publications on mathematical logic and the foundations of mathematics.

A simple example of a point-set in $R_q$ is $R_q$ itself; for, given any $x$ in $R_q$, we know at once that it belongs to $R_q$. The "empty" set $\Lambda$, which contains no points whatever, is also a point-set in $R_q$; for, given any $x$ in $R_q$, we know that it does *not* belong to $\Lambda$. Two point-sets $E_1$, $E_2$ in $R_q$ are identical if and only if each point $x$ which belongs to $E_1$ belongs also to $E_2$ and each point $x$ which belongs to $E_2$ belongs to $E_1$.

The set of all $x$ for which the statement $S$ holds will sometimes be denoted by $\{x \mid S\}$. E.g. $\{x \mid 0 \leqq x \leqq 1\}$ would be the closed interval consisting of all real numbers between 0 and 1. (Cf. also §4.)

Given any set $E$ in $R_q$, the set of all points of $R_q$ which do not belong to $E$ is called the *complement* of $E$, and is denoted by $CE$. Thus the complement of the whole space $R_q$ is the empty space $\Lambda$, and conversely; in symbols, $CR_q = \Lambda$ and $C\Lambda = R_q$. It is easy to see that for every point-set $E$ in $R_q$ the equation $C(CE) = E$ holds. For if $x$ is in $E$, it is not in $CE$, and is therefore in $C(CE)$; and if $x$ is in $C(CE)$, it is not in $CE$, and is therefore in $E$.

If $E_1$ and $E_2$ are point sets in $R_q$, we say that $E_1$ *is contained in* $E_2$ (in symbols, $E_1 \subset E_2$) or that $E_2$ *contains* $E_1$ (in symbols, $E_2 \supset E_1$) in case every point $x$ which belongs to $E_1$ also belongs to $E_2$. Thus in particular $E \subset E$ and $\Lambda \subset E$ for every set $E$. Further, we define $E_1 \cup E_2$ to be the set of all points $x$ belonging to one or both of $E_1$, $E_2$; we define* $E_1 \cap E_2$ or $E_1E_2$ to be the set of all points $x$ belonging to both $E_1$ and $E_2$; and we define $E_1 - E_2$ to be the set of all points $x$ which belong to $E_1$ but not to $E_2$.†

* $E_1 \cap E_2$ is sometimes called the *product*, sometimes the *intersection* of $E_1$ and $E_2$, $E_1 \cup E_2$ is called the sum or the union of $E_1$ and $E_2$.

† In defining the set theoretical operations we might of course have considered any collection of elements instead of $R_q$.

EXAMPLE. In $R_1$, put $E_1 = \{x \mid 0 \leqq x \leqq 2\}$ and $E_2 = \{x \mid 1 \leqq x \leqq 3\}$. Then $E_1 \cup E_2$ is the set $\{x \mid 0 \leqq x \leqq 3\}$, $E_1E_2$ is the set $\{x \mid 1 \leqq x \leqq 2\}$, $E_1 - E_2$ is $\{x \mid 0 \leqq x < 1\}$, $E_2 - E_1$ is $\{x \mid 2 < x \leqq 3\}$.

If $\{E_\alpha\}$ is any (finite or infinite) collection of sets in $R_q$, we define the sum (union) $\bigcup E_\alpha$ to be the set of all $x$ contained in at least one of the sets $E_\alpha$, and we define the product (intersection) $\bigcap E_\alpha$ to be the set of all $x$ belonging to all the sets $E_\alpha$.*

EXAMPLE. In one dimensional space $R_1$, let $E_n$ be $\{x \mid 0 \leqq x \leqq 1/n\}$ $(n = 1, 2, 3, \cdots)$. Then $\bigcup E_n$ is $\{x \mid 0 \leqq x \leqq 1\}$, $\bigcap E_n$ is the single point 0. If $E_n$ is $\{x \mid 0 < x < 1/n\}$, then $\bigcup E_n$ is $\{x \mid 0 < x < 1\}$, while $\bigcap E_n = \Lambda$.

The following relationships are easily verified:

$$C(E_1 \cup E_2) = CE_1 \cap CE_2, \qquad C(\bigcup E_\alpha) = \bigcap (CE_\alpha),$$
$$C(E_1 \cap E_2) = CE_1 \cup CE_2, \qquad C(\bigcap E_\alpha) = \bigcup (CE_\alpha),$$
$$E_1 - E_2 = E_1 \cap CE_2,$$
$$C(E_1 - E_2) = CE_1 \cup E_2.$$

For instance, a point $x$ is in $C(\bigcup E_\alpha)$ if and only if it is not in $\bigcup E_\alpha$, which is true if and only if it is in no one of the sets $E_\alpha$, which is true if and only if it is in every set $CE_\alpha$ and therefore in $\bigcap (CE_\alpha)$. Again, applying this equality to the sets $CE_\alpha$, we have

$$C(\bigcup CE_\alpha) = \bigcap (CCE_\alpha),$$

whence by taking complements

$$\bigcup CE_\alpha = C(\bigcap E_\alpha).$$

If $\{E\}$ is a collection of sets, the sets of the collection $\{E\}$ are *disjoint* if no point $x$ belongs to more than one of the sets of the collection.

A useful tool in studying properties of point sets $E$ is the characteristic function.

**1.1.** *The characteristic function $K_E(x)$ of the set $E$ is that function whose value is* 1 *if $x$ is in $E$ and whose value is* 0 *if $x$ is not in $E$.*

In the next theorem we assemble some simple properties of characteristic functions. However, it is desirable first to define sums and products of characteristic functions. This is trivial for finite sums and products. Given an infinite aggregate of symbols $\alpha$, and

* Some authors write $E_1 + E_2$, $E_1 \cdot E_2$, $\Sigma E_\alpha$, $\Pi E_\alpha$ for $E_1 \cup E_2$, $E_1 \cap E_2$, $\bigcup E_\alpha$, $\bigcap E_\alpha$.

corresponding to each $\alpha$ a number $t_\alpha$ which is either 0 or 1, we define the product of all $t_\alpha$ to be 0 if any one of them is 0 and to be 1 if all the $t_\alpha$ are 1. The sum of all $t_\alpha$ is the number $n$ if exactly $n$ of the $t_\alpha$ have the value 1, and is $\infty$ $(>1)$ if an infinite number of the $t_\alpha$ have the value 1. Concerning this symbol $\infty$ we shall have more to say shortly.

**1.2.** (*a*) *For any collection* $\{E_\alpha\}$ *of sets,* $K_{\bigcap E_\alpha}(x) = \Pi K_{E_\alpha}(x)$.

(*b*) *For any collection* $\{E_\alpha\}$ *of sets,* $K_{\bigcup E_\alpha}(x)$ *is the smaller of the numbers* 1 *and* $\Sigma K_{E_\alpha}(x)$.

(*c*) *The sets* $E_\alpha$ *are disjoint if and only if* $\Sigma K_{E_\alpha}(x) \leqq 1$.

(*d*) *If the sets* $E_\alpha$ *are disjoint, then* $K_{\bigcup E_\alpha} = \Sigma K_{E_\alpha}$.

(*e*) $K_{CE}(x) = 1 - K_E(x)$.

(*f*) *If* $E_1 \subset E_2$, *then* $K_{E_1}(x) \leqq K_{E_2}(x)$.

To prove (*a*), we observe that if the left member has the value 1, then $x$ is in $\bigcap E_\alpha$, so it is in every $E_\alpha$, so $K_{E_\alpha}(x) = 1$ for every $\alpha$, and the product of the characteristic functions is 1. Otherwise the left member has the value zero, $x$ is not in $\bigcap E_\alpha$, it is therefore lacking from some $E_\alpha$; for this $E_\alpha$ we have $K_{E_\alpha}(x) = 0$, and the product of the characteristic functions is 0.

To prove (*b*), if $x$ is in $\bigcup E_\alpha$ it is in at least one $E_\alpha$, so at least one term of the sum $\Sigma K_{E_\alpha}(x)$ is 1. Hence $K_{\bigcup E_\alpha}(x) = 1$, and the smaller of 1 and $\Sigma K_{E_\alpha}(x)$ is also 1. If $x$ is not in $\bigcup E_\alpha$, then $K_{\bigcup E_\alpha}(x)$ is 0 and so is every term of the sum $\Sigma K_{\bigcup E_\alpha}(x)$. So the sum is 0, which is the smaller of 0 and 1.

In (*c*), if the sets $E_\alpha$ are disjoint, each $x$ belongs to at most one set $E_\alpha$, so at most one term in the sum is 1, the others all being 0. So the sum is 0 or 1. Conversely, if the sum is never more than 1, there is no $x$ for which two or more of the characteristic functions have the value 1. That is, no $x$ belongs to more than one of the sets $E_\alpha$, and the $E_\alpha$ are disjoint.

Statement (*d*) follows at once from (*b*) and (*c*). Or we can prove it directly. If $x$ is in $\bigcup E_\alpha$, it is in exactly one of the sets $E_\alpha$, so both members of the equation have the value 1. If $x$ is not in $\bigcup E_\alpha$, it is not in any $E_\alpha$, so both members of the equation have the value 0.

If $x$ is in $CE$, it is not in $E$, so $K_E(x) = 0$ and $1 = K_{CE}(x) = 1 - K_E(x)$. If $x$ is not in $CE$ it is in $E$, so $K_E(x) = 1$ and $0 = K_{CE}(x) = 1 - K_E(x)$. This proves (*e*).

For (*f*) we observe that if $x$ is not in $E_1$, then

$$0 = K_{E_1}(x) \leqq K_{E_2}(x),$$

while if $x$ is in $E_2$ it is also in $E_1$, so

$$1 = K_{E_1}(x) = K_{E_2}(x).$$

**2.** Next we proceed to investigate some properties of sets in $R_q$ which depend, at least in part, on the concept of distance. If $x$ is a point in $R_q$ (or, as an alternative name, a vector in $R_q$) we define its distance from the origin (or, alternatively, the length of the vector) to be the quantity $||\,x\,||$ defined by the equation

$$||\,x\,|| = \left[\sum_{i=1}^{q} (x^{(i)})^2\right]^{\frac{1}{2}}.$$

If $x$ and $y$ are points of $R_q$, we define their distance $||\,x, y\,||$ by the equation

$$||\,x, y\,|| = ||\,x - y\,|| = \left[\sum_{i=1}^{q} (x^{(i)} - y^{(i)})^2\right]^{\frac{1}{2}}.$$

We now establish the four fundamental properties of this distance, which are the following.

(1) For all points $x, y$ of $R_q$,

$$||\,x, y\,|| \geqq 0.$$

(2) If $x$ and $y$ are points of $R_q$, $||\,x, y\,|| = 0$ if and only if $x = y$.
(3) For all points $x, y$ of $R_q$,

$$||\,x, y\,|| = ||\,y, x\,||.$$

(4) For all points $x, y, z$ of $R_q$,

$$||\,x, y\,|| + ||\,y, z\,|| \geqq ||\,x, z\,||.$$

Properties (1) and (3) are evident from the definition. Also, $||\,x, y\,|| = 0$ if and only if each difference $x^{(i)} - y^{(i)}$ has the value zero, which establishes (2). Property (4) is called the "triangle inequality"; in geometric language, it states that the sum of two sides of a triangle is at least equal to the third side. In order to prove it, it is convenient first to establish the highly useful Cauchy inequality.

*If $a_1, \cdots, a_q, b_1, \cdots, b_q$ are real numbers, then*

$$\left[\sum_{i=1}^{q} a_i^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{q} b_i^2\right]^{\frac{1}{2}} \geqq \sum_{i=1}^{q} a_i b_i.$$

It is evident that

$$\sum_{i,j=1}^{q} [a_i b_j - a_j b_i]^2 \geqq 0;$$

that is,

$$\sum_{i,j=1}^{q} a_i^2 b_j^2 - 2 \sum_{i,j=1}^{q} a_i b_j a_j b_i + \sum_{i,j=1}^{q} a_j^2 b_i^2 \geqq 0.$$

In the first double sum we first collect all the terms containing $a_1$, then those containing $a_2$, and so on. We find

$$\begin{aligned} \sum_{i,j=1}^{q} a_i^2 b_j^2 &= a_1^2(b_1^2 + \cdots + b_q^2) + a_2^2(b_1^2 + \cdots + b_q^2) + \cdots \\ &\qquad + a_q^2(b_1^2 + \cdots + b_q^2) \\ &= (a_1^2 + \cdots + a_q^2)(b_1^2 + \cdots + b_q^2) \\ &= \left(\sum_{i=1}^{q} a_i^2\right)\left(\sum_{i=1}^{q} b_i^2\right). \end{aligned}$$

A similar process can be applied to each of the other two double sums; we thus find

$$\left(\sum_{i=1}^{q} a_i^2\right)\left(\sum_{i=1}^{q} b_i^2\right) - 2\left(\sum_{i=1}^{q} a_i b_i\right)\left(\sum_{i=1}^{q} a_i b_i\right) + \left(\sum_{i=1}^{q} a_i^2\right)\left(\sum_{i=1}^{q} b_i^2\right) \geqq 0.$$

If we transpose the middle term and divide by 2, we obtain

$$\left(\sum_{i=1}^{q} a_i^2\right)\left(\sum_{i=1}^{q} b_i^2\right) \geqq \left(\sum_{i=1}^{q} a_i b_i\right)^2.$$

The left member of the Cauchy inequality is non-negative. If the right member is also non-negative, the Cauchy inequality follows from the preceding inequality by taking the square roots of both members; if the right member is negative, the inequality is evidently satisfied.

EXERCISE. Let us say that the $q$-tuples $(a_1, \cdots, a_q)$ and $(b_1, \cdots, b_q)$ are proportional if there are numbers $h$, $k$ not both zero such that $ha_i = kb_i$, $i = 1, \cdots, q$. Show that the absolute values of the two members of the Cauchy inequality are equal if and only if the $q$-tuples are proportional. (If they are proportional and, say, $h \neq 0$, we can substitute $kb_i/h$ for $a_i$ and verify equality. If equality holds, show that it also holds in the first inequality in the proof. From this deduce proportionality of the $q$-tuples.)

Returning to the proof of the triangle inequality, we first observe that by the Cauchy inequality

$$\left[\sum_{i=1}^{q}(x^{(i)}-y^{(i)})^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{q}(y^{(i)}-z^{(i)})^2\right]^{\frac{1}{2}} \geqq \sum_{i=1}^{q}(x^{(i)}-y^{(i)})(y^{(i)}-z^{(i)}).$$

We add the same quantity to both members of this inequality to obtain

$$\tfrac{1}{2}\sum_{i=1}^{q}(x^{(i)}-y^{(i)})^2+\left[\sum_{i=1}^{q}(x^{(i)}-y^{(i)})^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{q}(y^{(i)}-z^{(i)})^2\right]^{\frac{1}{2}} + \tfrac{1}{2}\sum_{i=1}^{q}(y^{(i)}-z^{(i)})^2$$

$$\geqq \tfrac{1}{2}\sum_{i=1}^{q}(x^{(i)}-y^{(i)})^2+\sum_{i=1}^{q}(x^{(i)}-y^{(i)})(y^{(i)}-z^{(i)})+\tfrac{1}{2}\sum_{i=1}^{q}(y^{(i)}-z^{(i)})^2,$$

or

$$\tfrac{1}{2}\left\{\left[\sum_{i=1}^{q}(x^{(i)}-y^{(i)})^2\right]^{\frac{1}{2}}+\left[\sum_{i=1}^{q}(y^{(i)}-z^{(i)})^2\right]^{\frac{1}{2}}\right\}^2 \geqq \tfrac{1}{2}\left[\sum_{i=1}^{q}[(x^{(i)}-y^{(i)})+(y^{(i)}-z^{(i)})]^2\right].$$

Multiplying both members by 2 and changing notation, we find

$$[\,|\,|x,y|\,|+|\,|y,z|\,|\,]^2 \geqq |\,|x,z|\,|^2,$$

whence the triangle inequality follows at once.

If $x_0$ is any point of the space $R_q$, and $\epsilon$ is any positive number, we define the *$\epsilon$-neighborhood* $N_\epsilon(x_0)$ of the point $x_0$ to be $\{x \mid \,|\,|x, x_0|\,| < \epsilon\}$. Thus in space of one dimension the $\epsilon$-neighborhood of $x_0$ consists of $\{x \mid x_0 - \epsilon < x < x_0 + \epsilon\}$; in three-space, $N_\epsilon(x_0)$ consists of the points inside of the sphere of radius $\epsilon$ with center at $x_0$.

EXERCISE. If $x$ is in $R$, and $h$ and $k$ are both in $N_\epsilon(x)$, so is every number $y$ between $h$ and $k$.

EXERCISE. Given two points $x_1$, $x_2$ of $R_q$, we say that a point $x$ is on the line-segment joining $x_1$ and $x_2$ if there is a number $t$ between 0 and 1 such that $x^{(i)} = tx_1^{(i)} + (1-t)x_2^{(i)}$. Show that if $x_1$ and $x_2$ are both in $N_\epsilon(y)$, so is every point on the line-segment joining $x_1$ and $x_2$. (Use the equation $|\,|cx|\,| = |c| \cdot |\,|x|\,|$ and the triangle inequality.)

A point $x$ is *interior* to a set $E$ if it is possible to find a neighborhood* $N_\epsilon(x)$ every point of which belongs to $E$. A point-set $E$ is *open* if every point $x$ which belongs to $E$ is interior to $E$. For example,

* In such a case as this, in which we merely wish to state that there is *some* neighborhood $N_\epsilon(x)$ with a given property, and the size of $\epsilon$ is of no importance, we shall sometimes write merely $N(x)$ instead of $N_\epsilon(x)$.

let us give the name "open interval" to a point set consisting of $\{x \mid a^{(1)} < x^{(1)} < b^{(1)}, \cdots, a^{(q)} < x^{(q)} < b^{(q)}\}$ where the $a^{(i)}$ and $b^{(i)}$ are finite constants for which $a^{(i)} < b^{(i)}$. Then every open interval is an open set. For if $x$ belongs to the interval, each of the numbers $x^{(i)} - a^{(i)}$ and $b^{(i)} - x^{(i)}$ is positive. Denote by $2\epsilon$ the smallest of them, and consider the neighborhood $N_\epsilon(x)$. If $x_0$ is in this neighborhood, then

$$a^{(i)} < x^{(i)} - \epsilon < x^{(i)} - \| x, x_0 \| \leqq x^{(i)} - | x^{(i)} - x_0^{(i)} | \leqq x_0^{(i)} \\ \leqq x^{(i)} + | x^{(i)} - x_0^{(i)} | \leqq x^{(i)} + \| x, x_0 \| < x^{(i)} + \epsilon < b^{(i)},$$

so $x_0$ is also in the open interval. Hence every open interval is an open set.

If we give the name "closed interval" to $\{x \mid a^{(i)} \leqq x^{(i)} \leqq b^{(i)}\}$, where the $a^{(i)}$ and $b^{(i)}$ are finite constants such that $a^{(i)} \leqq b^{(i)}$, we see that a closed interval is *not* an open set. For the point $a$ belongs to the closed interval; but every neighborhood of $a$ contains points $x_0$ with $x_0^{(i)} < a^{(i)}$, so that $x_0$ can not belong to the closed interval.

A point $x$ is called an *accumulation point* of a set $E$ if every neighborhood of $x$ contains infinitely many points of $E$. The point $x$ itself may or may not belong to $E$. For example, every point of an open interval is an accumulation point of the interval. If in one-space we take $E$ to be the set of points $1, \frac{1}{2}, \frac{1}{3}, \cdots, 1/n, \cdots$, then 0 is an accumulation point of $E$; and in fact we easily verify that it is the only accumulation point of $E$.

The set of all points $x$ which are accumulation points of a set $E$ is called the *derived set* of $E$, and is denoted by $E'$. The set $E \cup E'$ is the *closure* of $E$, and is denoted by $\bar{E}$.

A set $E$ is *closed* in case every accumulation point of $E$ is itself a point of $E$; in symbols, if $E' \subset E$. Thus the open interval

$$\{x \mid a^{(i)} < x^{(i)} < b^{(i)}\}$$

is *not* a closed set; for the point $a$ is an accumulation point of the interval, but does not belong to the interval. The closed interval $\{x \mid a^{(i)} \leqq x^{(i)} \leqq b^{(i)}\}$ is a closed set. For suppose that a point $x$ is not in the interval. Then the above inequalities do not all hold. Suppose, to be specific, that $x^{(1)} < a^{(1)}$. Define $\epsilon = \frac{1}{2}(a^{(1)} - x^{(1)})$. For every point $x_0$ in $N_\epsilon(x)$ we have $x_0^{(1)} \leqq x^{(1)} + | x_0^{(1)} - x^{(1)} | < x^{(1)} + \epsilon < a^{(1)}$, so $x_0$ is not in the interval. Hence $x$ can not be an accumulation point of the interval; and since no point outside of the interval is an accumulation point of the interval, the interval is a closed set.