Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann
Subseries: Fondazione C.I.M.E., Firenze
Adviser: Roberto Conti

1365

M. Giaquinta (Ed.)

Topics in Calculus of Variations

Montecatini Terme 1987

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann
Subseries: Fondazione C.I.M.E., Firenze
Adviser: Roberto Conti

1365

M. Giaquinta (Ed.)

Topics in Calculus of Variations

Lectures given at the 2nd 1987 Session of the
Centro Internazionale Matematico Estivo (C.I.M.E.)
held at Montecatini Terme, Italy, July 20–28, 1987

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

Editor

Mariano Giaquinta
Istituto di Matematica Applicata, Università di Firenze
Via di S. Marta, 3, 50139 Firenze, Italy

Mathematics Subject Classification (1980): 49, 35, 53

ISBN 3-540-50727-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-50727-2 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

INTRODUCTION

The international course on "Topics in Calculus of Variations" was held at Montecatini, Italy, July 20-28, 1987, organized by the Fondazione CIME.

These proceedings contain the texts of the lectures presented by L. Caffarelli, J. Moser, L. Nirenberg, R. Schoen, A. Tromba. They also contain the lectures H. Brezis had originally planned and kindly agreed to provide, though he was prevented from coming.

I wish to express my gratitude to the lecturers and to all participants for their contribution to the success of the course. I would also like to express my special thanks to all the authors for undertaking the heavy task of writing the text of their lectures.

Mariano Giaquinta

Firenze, June 1988

C.I.M.E. Session on "Topics in Calculus of Variations"

List of Participants

L. AMBROSIO, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 Pisa

G. ANZELLOTTI, Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento)

A. AROSIO, Via Cocchi 10, 50131 Firenze

M. ATHANASSENAS, Math. Institut, Beringstrasse 4, D-5300 Bonn

P. AVILES, Department of Mathematics, University of California at San Diego, La Jolla, CA 92003, USA

M. BADIALE, Via G. Conti 13, 30014 Cavarzere (VE)

G. BARTUZEL, Institute of Mathematics, Technical University of Warsaw, Plac Jednosci Robotniczej 1, 00661 Warsaw

M.L. BERTOTTI, Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento)

I. BIRINDELLI, Via dei Coronari 82, 00185 Roma

G. BUSONI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

L. CAFFARELLI, Institute for Advanced Study, Princeton, N.J. 08540, USA

P. CANNARSA, Dipartimento di Matematica, II Università di Roma, Via O. Raimondo, 00173 Roma

F. CATANESE, Dipartimento di Matematica, Università, Via Buonarroti 2, 56100 Pisa

G. CERAMI, Via G. Sciutti 156, 90144 Palermo

L. FU CHEUNG, Mathematisches Institut der Universitat Bonn, Beringstr.4, 5300 Bonn 1

G. CHITI, Dipartimento di Matematica del Politecnico, Corso Duca degli Abruzzi 24, 10129 Torino

E. COMPARINI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

F. DAL FABBRO CROSTA, Dipartimento di Matematica del Politecnico, Piazza L. da Vinci, 32, 20133 Milano

R. DAL PASSO, IAC-CNR, Viale del Policlinico 137, 00161 Roma

G. DEB RAY, St. Xavier's College, 30 Park Street, 700016 Calcutta, India

F. DEL BUONO, Via Eden 15, 52010 Badia Prataglia (Arezzo)

E. DI BENEDETTO, Facoltà di Ingegneria, Università degli Studi di Tor Vergata, Via O. Raimondo, 00173 Roma

B. D'ONOFRIO, Via Veneto 21, 04020 SS. Cosma e Damiano (Latina)

F. DUZAAR, Mathematisches Institut, Universitat Dusseldorf, Universitatsstr. 1, D-4000 Dusseldorf

J. EELLS, I.C.T.P., Box 586, 34100 Miramare, Trieste

H. EGNELL, Department of Mathematics, Thunbergsvagen 3, 752 38 Uppsala, Sweden

J. ESCOBAR, The University of Chicago, Department of Mathematics, 5734 University Av. Chicago, Illinois 60637

A. FASANO, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

M. FERRARIS, Dipartimento di Matematica, Via Ospedale 72, 090100 Cagliari

G. FIORITO, Via XX Settembre 25, Pal. I, 95017 San Gregorio (Catania)

M. FUCHS, Mathematisches Institut, Universitat Dusseldorf, Universitatsstr. 1, D-4000 Dusseldorf

M. GIAQUINTA, Istituto di Matematica Applicata, Via S. Marta 3, 50139 Firenze

M. GIRARDI, Via Mercalli 21, 00197 Roma

E. GIUSTI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

G. GREGORI, Via Venosa 8, 00178 Roma

M. GRUTER, Math. Institut, Universitat Bonn, Beringstr. 6, D-5300 Bonn

F. HELEIN, Ecole Polytechnique, Dept. de Mathematiques, 91128 Palaiseau, France

S. HILDEBRANDT, Drachenfelsstrasse 23, D-5205 St. Augustin 2, W. Germany

L.A. IBORT, Dpto. de Fisica Teorica, Universidad de Zaragoza, 50009 Zaragoza, Spain

P.-A. IVERT, Matematiska Institutionen, Universitetet i Linkoping, S-581 83 Linkoping, Sweden

F. JOSELLIS, Institut f. Reine u. Angewandte Mathematik RWTH Aachen, D-5100 Aachen, W. Germany

M. KOISO, Max-Planck-Institut fur Mathematik, Gottfried Str. 26, D-5300 3, W. Germany

N. KOISO, Max-Planck-Institut fur Mathematik, Gottfried Str. 26, D-5300 Bonn 3, W. Germany

J. KONDERAK, I.C.T.P., P.O. Box 586, Strada Costiera 11, 34100 Miramare, Trieste

P. KUMLIN, Department of Mathematics CTH, S-412 96 Goteborg, Sweden

P. LAURENCE, Courant Institute, 251 Mercer Str., New York, N.Y. 10012, USA

F. LEONETTI, Dipartimento di Matematica pura e applicata, Università de L'Aquila, Via Roma, 67100 L'Aquila

H. LINDBLAD, Matematiska Institutionen, Lunds University, Box 118, 22100 Lund, Sweden

M. LONGINETTI, IAGA-CNR, Via S. Marta 13/A, 50139 Firenze

R. MAGNANINI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

G. MANCINI, Dipartimento di Matematica, Piazza di Porta S.Donato 5, 40127 Bologna

P. MARCATI, Dipartimento di Matematica pura e applicata, Università de L'Aquila, Via Roma, 67100 L'Aquila

M. MATZEU, Dipartimento di Matematica, Università "La Sapienza", Città Universitaria, 00185 Roma

A. MAUGERI, Via Etnea 688, 95128 Catania

M. MEIER, Mathematisches Institut, Beringstr. 4, D-5300 Bonn 1

G. MODICA, Istituto di Matematica Applicata, Via S. Marta 3, 50139 Firenze

J. MOSER, ETH-Zentrum, CH-8092 Zurich, Switzerland

M.K.V. MURTHY, Dipartimento di Matematica, Via Buonarroti 2, 56100 Pisa

R. MUSINA, S.I.S.S.A., Strada Costiera, 34014 Grignano, Trieste

L. NIRENBERG, Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N.Y. 10012, USA

F. NJOKU, S.I.S.S.A., Strada Costiera 11, 34014 Grignano, Trieste

L. NOTARANTONIO, Viale Tirreno 187, 00141 Roma

A. OLVERA, Depto. de Matematicas Aplicadas y Analisis, Facultad de Matematicas, Universidad de Barcelona, Gran Via 585, Barcelona 08071, Spain

F. PACELLA, Via Valsolda 111, 00141 Roma

D. PALLARA, Via Vittorio Veneto 3, 73100 Lecce

G. PALMIERI, Dipartimento di Matematica, Via G. Fortunato, 70125 Bari

B. PELLONI, Via Ronciglione 5, 00191 Roma

M.A. POZIO, Dipartimento di Matematica, II Università di Roma, Via O. Raimondo, 00173 Roma

C. PUCCI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

R. PUTTER, Mathematisches Institut, Universitat Bonn, Beringstr. 6, D-5300 Bonn

A. RATTO, Via De Amicis 8/17, 17100 Savona

O. REY, Ecole Polytechnique, Departement de Mathématiques, 91128 Palaiseau, France

R. RICCI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

E. ROSSET, Via Isonzo 38, Codroipo (Udine)

R. SCHOEN, Department of Mathematics, Stanford University, Stanford, Ca. 94305, USA

P. SMITH, Max-Planck-Institut fur Mathematik, Gottfried-Claren-Str. 26, D-5300 Bonn 3, W. Germany

K. STEFFEN, Mathematisches Institut, Universitat Dusseldorf, Universitatsstr. 1, D-4000 Dusseldorf

I. STRATIS, Department of Mathematics, University of Athens-Panepistimiopolis, GR 15781 Athens, Greece

D. STROTTMAN, Los Alamos National Laboratory, Los Alamos, New Mexico 87544, USA

M. STRUWE, Mathematik, ETH Zentrum, CH-8092 Zurich, Switzerland

J. SZTAJNIC, Institute of Mathematics, Lodz University, ul. S.Banacha 22, 80-238 Lodz, Poland

G. TALENTI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

A. TARSIA, Dipartimento di Matematica, Via Buonarroti 2, 56100 Pisa

N.A. TCHOU, Dipartimento di Matematica, Università "La Sapienza", Città Universitaria, 00815 Roma

B. TERRENI, Piazza Toniolo 10, 56100 Pisa

E. TOMAINI, Via Cimarosa 2, 45100 Rovigo

V.M. TORTORELLI, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 Pisa

F. TRICERRI, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

A. TROMBA, Max-Planck-Institut fur Mathematik, Gottfried-Claren-Str. 26, D-5300 Bonn 3, W. Germany

L. TUBARO, Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento)

R. TURNER, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

M. UGHI, Dipartimento di Scienze Matematiche, Piazzale Europa 1, 34127 Trieste

G. VALLI, University of Warwick, Coventry, CV4 7AL, (G.B.)

V. VESPRI, Dipartimento di Matematica, II Università di Roma, Via O. Raimondo, 00173 Roma

G. WEILL, 23 Avenue Marceau, 75016 Paris, France

R. WENTE, Department of Mathematics, University of Toledo, Toledo, Ohio 43606, USA

C.M. WOOD, Department of Pure Mathematics, University of Liverpool, P.O.Box 147, Liverpool L69 3BX, England

R. YE, Mathematisches Institut der Universitat, Beringstr. 4, 5300 Bonn 1, W. Germany

F. ZIRILLI, Dipartimento di Matematica, Università "La Sapienza", Città Universitaria, 00185 Roma

TABLE OF CONTENTS

S^k – VALUED MAPS WITH SINGULARITIES

Haïm Brézis
Département de Mathématiques, Université Paris 6
4, pl. Jussieu, 75252 Paris Cedex 05
and
Rutgers University, New Brunswick, NJ 08903

The purpose of these notes is to present a survey of some recent results and open problems dealing with the "energy" of S^k – valued maps. The original motivation comes from the theory of liquid crystals; such materials are composed of rod–like molecules with a well defined orientation, except at isolated points (the "defects") which are observed by the physicists. The optic axis φ is a vector of unit length (in $\mathbb{R}^3$) defined in the domain $\Omega \subset \mathbb{R}^3$ (the container of the liquid crystal); so that φ is a map from Ω into S^2. Associated with a configuration φ is a deformation energy which we shall usually take to be

$$E(\varphi) = \int_\Omega |\nabla\varphi|^2 \, dx. \tag{0.1}$$

Physicists consider more general energies, such as,

$$\tilde{E}(\varphi) = \int_\Omega k_1(\operatorname{div}\varphi)^2 + k_2(\varphi\cdot\operatorname{curl}\varphi)^2 + k_3|\varphi\wedge\operatorname{curl}\varphi|^2 + \alpha[\operatorname{tr}(\nabla\varphi)^2-(\operatorname{div}\varphi)^2]dx \tag{0.2}$$

where k_1, k_2, k_3 and α are positive constants. In the special case where $k_1 = k_2 = k_3 = \alpha = 1$, then it is easy to see that $\tilde{E} = E$. While much progress has been achieved for the energy E, little is known so far for $\tilde{E}$. Stable equilibrium configurations correspond to minima of E (or $\tilde{E}$) and therefore it is essential to study the properties of minimizers. For a detailed discussion of the physical background we refer e.g. to [9], [10], [13], [16], [17], [18] and [33]. However we feel that the mathematical questions involved in this field are of great interest for their own sake, an interest which goes much beyond the original motivation. In fact, it is remarkable that progress has been achieved through the joint efforts of experts in Nonlinear Partial Differential Equations, Functional Analysis, Differential Geometry, Geometric Measure Theory, Topology, Numerical Analysis, Graph Theory, etc.

The plan is the following:

I. The problem of prescribed singularities.

I.1. Point singularities in $\mathbb{R}^3$.

I.2. Various generalizations:

1) A domain $\Omega \subset \mathbb{R}^3$ with constant boundary condition.

2) Holes in $\mathbb{R}^3$.

3) An example related to minimal surfaces.

I.3. Some open problems.

II. The problem of free singularities.

II.1. $x/|x|$ is a minimizer.

II.2. The analysis of point singularities.

II.3. Energy estimates for maps which are odd on the boundary.

II.4. The gap phenomenon. Density and nondensity of smooth maps between manifolds. Traces.

II.5. Some open problems.

I. The problem of prescribed singularities.

I.1 Point singularities in $\mathbb{R}^3$.

We start with a simple question (originally raised by J. Ericksen). Let $a_1, a_2, \dots a_N$ be N points given in $\mathbb{R}^3$ (the desired location of the singularities). Define the class of admissible maps $\mathcal{E}$ to be

$$\mathcal{E} = \{\varphi \epsilon C^1(\mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \{a_i\}; S^2); \int_{\mathbb{R}^3} |\nabla\varphi|^2 < \infty \text{ and } \deg(\varphi, a_i) = d_i \quad \forall i\} \tag{1.1}$$

where the d_i's are given integers, $d_i \epsilon Z$ with $d_i \neq 0$. Here $\deg(\varphi, a_i)$ denotes the Brouwer degree of φ restricted to any small sphere around a_i. [Stable singularities observed by the physicists have always degree ± 1 and the reason why this is so will be given in Section II.2. However it makes sense to formulate the mathematical question with general degrees].

The problem is to study the quantity

$$E = \operatorname*{Inf}_{\varphi \in \mathcal{E}} \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx \tag{1.2}$$

i.e. the least deformation energy needed to produce singularities at a given location with a given degree. Such a question may seem unrealistic because the container Ω is all of $\mathbb{R}^3$ and also because one cannot prescribe physically the location of the singularities. Nevertheless this model problem has interesting features; it has led to the development of new tools which are useful in more realistic questions.

Surprisingly, there is a simple formula for E:

<u>Theorem 1</u> ([8]). We have

$$E = 8\pi L \tag{1.3}$$

where L is the length of a minimal connection (in a sense to be defined below).

So far, we have made no restriction about the d_i's. However, we must assume that

$$\sum_{i=1}^{N} d_i = 0 \tag{1.4}$$

because of the following:

<u>Lemma 1.</u> $\mathcal{E}$ is nonempty if and only if (1.4) holds.

<u>Sketch of the proof.</u> Suppose first that $\mathcal{E}$ is nonempty and let $\varphi \in \mathcal{E}$. We claim that φ restricted to a large sphere S_R of radius R has degree zero. Intuitively, this is clear because $\int_{\mathbb{R}^3} |\nabla \varphi|^2 < \infty$ implies that, roughly speaking, φ goes to a constant at infinity. More precisely, we recall (see e.g. [36]) that if S is a closed two dimensional surface in $\mathbb{R}^3$ and ψ is a C^1 map from S into S^2 then

$$\deg \psi = \frac{1}{4\pi} \int_S J_\psi \, d\xi \tag{1.5}$$

where J_ψ denotes the Jacobian determinant of ψ. A useful way to write J_ψ is

$$J_\psi = \psi \cdot \psi_x \wedge \psi_y \tag{1.6}$$

where (x,y) are normal coordinates on S. This follows from the fact that $\psi \cdot \psi_x = \psi \cdot \psi_y = 0$, and thus $\psi_x \wedge \psi_y = (J_\psi)\, \psi$. We deduce from (1.5) and (1.6) that

$$|\deg \psi| \leq \frac{1}{8\pi} \int_S |\nabla_T \psi|^2 d\xi$$

where $|\nabla_T \psi|^2 = |\psi_x|^2 + |\psi_y|^2$.

We now return to φ and choose R_1 large enough so that B_{R_1} contains all the singularities a_i. By continuity, $\deg(\varphi_{|S_r})$ is constant for $r > R_1$. We have for any $R_2 > R_1$,

$$\int_{R_1<|x|<R_2} |\nabla\varphi|^2 = \int_{R_1}^{R_2} dr \int_{S_r} |\nabla\varphi|^2 d\xi \geq 8\pi |\deg\varphi_{|S_r}| (R_2 - R_1).$$

Letting $R_2 \to \infty$ we see that $\deg \varphi_{|S_r} = 0$ for $r > R_1$.

From the additivity of the degree we conclude that (1.4) holds. The converse is more delicate and follows from an explicit construction sketched in the proof of Theorem 1.

Definition of L, the length of a minimal connection.

It is convenient to start with simple cases:

Case 1: There are only two singularities, a_1 with degree $+1$ and a_2 with degree -1. We shall call this a dipole. Here

$$L = |a_1 - a_2|$$

is the (Euclidean) distance between the two points. Note that it was easy to guess, from dimensional analysis, that E is proportional to a length.

Case 2: All the degrees d_i are equal ± 1. Because of (1.5) there are as many $+$ signs as $-$ signs. We rename the points (a_i) as positive points $p_1, p_2, \ldots, p_k$ and negative points $n_1, n_2, \ldots, n_k$. Then

$$L = \underset{\sigma}{\mathrm{Min}} \sum_{i=1}^{k} |p_i - n_{\sigma(i)}| \tag{1.7}$$

where the minimum is taken over all permutations σ of the integers 1 to k.

Case 3: In the general case, proceed as above except that in the list (p_i, n_i) points are repeated according to their multiplicity $|d_i|$.

Sketch of the proof of Theorem 1. The proof consists of two independent steps:

A) $E \leq 8\pi L$,

B) $E \geq 8\pi L$.

Step A. The main ingredient is the following basic dipole construction:

Lemma 2. Let (a_1, a_2) be a dipole. Given any $\epsilon > 0$ there is a map φ_ϵ which is smooth on $\mathbb{R}^3$, except at (a_1, a_2), such that

$$\deg(\varphi_\epsilon, a_1) = +1, \quad \deg(\varphi_\epsilon, a_2) = -1, \tag{1.8}$$

$$\int |\nabla\varphi_\epsilon|^2 \leq 8\pi|a_1-a_2| + \epsilon, \tag{1.9}$$

and moreover

φ_ϵ is constant outside an ϵ–neighborhood of the line segment $[a_1, a_2]$.

(1.10)

In fact, given any positive integer d there is a map φ_ϵ which is smooth on $\mathbb{R}^3$, except at (a_1, a_2), such that

$$\deg(\varphi_\epsilon, a_1) = d, \quad \deg(\varphi_\epsilon, a_2) = -1, \tag{1.8'}$$

$$\int |\nabla\varphi_\epsilon|^2 \leq 8\pi|a_1-a_2|\, d + \epsilon, \tag{1.9'}$$

and (1.10) holds. Such a map is constructed explicitly in [8] (see also [7]). Putting together these basic dipoles over a minimal connection it is easy to prove that $E \leq 8\pi L$. Clearly, this construction also shows that $\mathcal{E}$ is nonempty when (1.4) holds.

Step B. There are two different methods for proving the lower bound $E \geq 8\pi L$. Each one has its own flavor and I will describe both of them.

Proof of (B) via the D–field approach. This is the original method introduced in [8]. To every map φ we associate the vector field D defined as follows

$$D = (\varphi\cdot\varphi_y\wedge\varphi_z,\ \varphi\cdot\varphi_z\wedge\varphi_x,\ \varphi\cdot\varphi_x\wedge\varphi_y) \tag{1.11}$$

where $\varphi_x, \varphi_y, \varphi_z$ denote partial derivatives of φ with respect to x, y, z. A more intrinsic way to define D is to say that D is the pull–back under φ of the canonical 2–form on S^2. The main properties of D are the following:

$$|D| \leq \tfrac{1}{2}\, |\nabla\varphi|^2 \quad \text{on } \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \{a_i\} \tag{1.12}$$

and

$$\operatorname{div} D = 4\pi \sum_{i=1}^{N} d_i\, \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{1.13}$$

Proof of (1.12). Changing coordinates at a given point we may always assume that

$$\varphi = (0, 0, 1).$$

Since $|\varphi|^2 = 1$ we have $\varphi\cdot\varphi_x = \varphi\cdot\varphi_y = \varphi\cdot\varphi_z = 0$ and thus we may write

$$\varphi_x = (a_1, b_1, 0),\ \varphi_y = (a_2, b_2, 0) \text{ and } \varphi_3 = (a_3, b_3, 0).$$

We see that

$$D = a \wedge b$$

with $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

It follows that

$$|D| \le |a|\,|b| \le \tfrac{1}{2}(|a|^2 + |b|^2) = \tfrac{1}{2}|\nabla\varphi|^2.$$

<u>Proof of (1.13).</u> On $\mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \{a_i\}$ we have

$$\operatorname{div} D = 3\ \varphi_x \cdot \varphi_y \wedge \varphi_z = 0$$

since φ_x, φ_y and φ_z are in the same plane (perpendicular to φ). In view of a celebrated Theorem of L. Schwartz we find

$$\operatorname{div} D = \sum_{\alpha, i} c_\alpha \partial^\alpha \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

On the other hand, since $D \in L^1(\mathbb{R}^3)$, we must have

$$\operatorname{div} D = \sum_{i=1}^{N} c_i \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^3). \tag{1.14}$$

Integrating (1.14) over a small ball B around a_i we see that

$$\int_S D \cdot n = c_i$$

where $S = \partial B$ and n is the outward normal to S. On the other hand, it follows from the definition of D that $D \cdot n = J_\varphi$ where φ is considered as a map *restricted* to S and J_φ denotes its 2×2 Jacobian determinant. Applying (1.5) we find that $c_i = 4\pi \deg(\varphi, a_i)$.

The proof of (B) then proceeds as follows. Let $\zeta\colon \mathbb{R}^3 \to \mathbb{R}$ be any function such that

$$\|\zeta\|_{Lip} = \sup_{x \ne y} \frac{|\zeta(x) - \zeta(y)|}{|x-y|} \le 1,$$

so that $\|\nabla\zeta\|_{L^\infty} \le 1$. We have

$$\int |\nabla\varphi|^2 \ge 2 \int |D| \ge -2 \int D \cdot \nabla\zeta = 2 \sum_{i=1}^{N} 4\pi d_i \zeta(a_i). \tag{1.15}$$

Relabelling the points (a_i) as positive and negative points (p_i, n_i) and taking into account their multiplicity we have

$$\sum_{i=1}^{N} d_i \zeta(a_i) = \sum_{i=1}^{k} \zeta(p_i) - \sum_{i=1}^{k} \zeta(n_i).$$

The conclusion of Step B is a direct consequence of the following:

Lemma 3. Let M be a metric space and let $p_1, p_2, \cdots p_k$ and $n_1, n_2, \cdots n_k$ be $2k$ points in M. Then

$$\operatorname*{Max}_{\substack{\zeta: M\to\mathbb{R} \\ \|\zeta\|_{Lip}\le 1}} \left\{ \sum_{i=1}^{k} \zeta(p_i) - \sum_{i=1}^{k} \zeta(n_i) \right\} = L$$

where $\|\zeta\|_{Lip} = \operatorname*{Sup}_{x\neq y} \frac{|\zeta(x)-\zeta(y)|}{d(x,y)}$ and $L = \operatorname*{Min}_{\sigma} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)})$.

A quick proof of Lemma 3 relies on the Kantorovich min–max principle (see [32] or [37]) and the Birkhoff Theorem which asserts that the extreme points of the doubly stochastic matrices are the permutation matrices (see [8] for details). Another self contained proof of Lemma 3 is given in [7].

Proof of (B) via the coarea formula. This new proof discovered by F. Almgren – W. Browder – E. Lieb (see [2]) relies heavily on Federer's coarea formula (see [20], [24] or [41]), which we recall for the convenience of the reader. Suppose φ is a C^1 map from a domain $\Omega \subset \mathbb{R}^n$ into a manifold N of dimension $p \leq n$. (Think, for example, of N as being a sphere). The differential of φ, $D\varphi$, is a $(p\times n)$ matrix. Set

$$J_p\varphi = \sqrt{\det(D\varphi\cdot(D\varphi)^t)}$$

where det denotes the determinant of the $p\times p$ matrix $D\varphi\cdot(D\varphi)^t$; $J_p\varphi$ is called the p–Jacobian of φ. We have

$$\int_\Omega J_p\varphi \, dx = \int_N \mathcal{H}^{n-p}(\varphi^{-1}(\xi))\, d\xi \tag{1.16}$$

where $\mathcal{H}^{n-p}$ is the $(n-p)$–dimensional Hausdorff measure in $\mathbb{R}^n$. In the special case where $N = \mathbb{R}^n$, then $J_n\varphi$ is the usual Jacobian determinant of φ and (1.16) becomes

$$\int_\Omega J_\varphi \, dx = \int_{\varphi(\Omega)} \operatorname{card}(\varphi^{-1}(\xi))\, d\xi. \tag{1.17}$$

In the case of interest to us we take $\varphi \in \mathcal{E}$, $\Omega = \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \{a_i\}$ and $N = S^2$. Therefore we find

$$\int_{\mathbb{R}^3} J_2\varphi \, dx = \int_{S^2} \mathcal{H}^1(\varphi^{-1}(\xi))\, d\xi. \tag{1.18}$$

First, we claim that

$$J_2\varphi \leq \tfrac{1}{2}|\nabla\varphi|^2. \tag{1.19}$$

With the same notations as in the proof of (1.12) we have

$$J_2\varphi = \sqrt{|a|^2|b|^2-(a\cdot b)^2} = |a\wedge b| = |D| \le \tfrac{1}{2}|\nabla\varphi|^2.$$

Next, we claim that, for a.e. $\xi\in S^2$,

$$\mathscr{H}^1(\varphi^{-1}(\xi)) \geq L. \tag{1.20}$$

This will complete the proof of (B) via the coarea formula.

Proof of (1.20). By Sard's Theorem we know that a.e. $\xi\in S^2$ is a regular value of φ. When ξ is regular value, the Implicit Function Theorm implies that $\varphi^{-1}(\xi)$ is a collection of curves which either connect the points (a_i), or go to infinity, or are closed loops. Here $\mathscr{H}^1(\varphi^{-1}(\xi))$ is the total length of these curves. In view of (1.18), (1.19) and since $\int|\nabla\varphi|^2 < \infty$, the total length is finite and hence there is no curve going to infinity. Furthermore, we shall disregard the closed loops (since they only increase the total length). We are left with a finite collection of curves connecting the points (a_i). Since $\deg(\varphi,a_i) = d_i \neq 0$, at least one curve emanates from each a_i, but there could be more than one. The simplest situation is the case where each positive point p_i is connected by one of the curves to a negative point $n_{\sigma(i)}$, for some permutation σ. Then, clearly $\mathscr{H}^1(\varphi^{-1}(\xi)) \geq L$. Unfortunately, the general situation could be more complicated. For example, a bad configuration would be if we have 4 points p_1, p_2, n_1, n_2 and $\varphi^{-1}(\xi)$ consists only of two curves: one joining p_1 to p_2 and the other n_1 to n_2. We could not conclude, because $|p_1-p_2| + |n_1-n_2|$ might be smaller than L! We shall see that such a configuration is excluded. For this purpose it is convenient to introduce an arrow (i.e. an orientation) on each curve C.

Let x be any point on C and let (e_1, e_2, e_3) be a direct basis with e_1 tangent to C at x. Consider φ restricted to the plane (e_2, e_3) and its (2×2) Jacobian determinant $J_\varphi(x)$. Note that $J_\varphi(x) \neq 0$ since ξ is a regular value. If $J_\varphi(x) > 0$ the orientation of C is given by e_1, and if $J_\varphi(x) < 0$ take the orientation opposite to e_1.

With this convention, and using the properties of the degree, one can see that at every point a_i, one has the basic relation:

$$d_i=\deg(\varphi,a_i)=(\#\text{outgoing arrows})-(\#\text{incoming arrows}). \tag{1.21}$$

For example, an admissible configuration is given by the following figure