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Emmanuel Hebey

Sobolev Spaces on Riemannian Manifolds

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Introduction

This monograph is devoted to the study of Sobolev spaces in the general setting of Riemannian manifolds. In addition to being very interesting mathematical structures in their own right, Sobolev spaces play a central role in many branches of mathematics. While analysis proves more and more to be a very powerful means for solving geometrical problems, it is striking that no global study of these spaces exists in the general context of Riemannian manifolds. The objective of this monograph is to fill this gap, at least partially. In so doing, it is intended to serve as a textbook and reference for graduate students and researchers. This monograph also hopes to convince the reader that the naive idea that what is valid for Euclidean spaces must be valid for Riemannian manifolds is completely false. Indeed, as one will see, several surprising phenomena appear when studying Sobolev spaces in the Riemannian context. Elementary questions now give rise to sophisticated developments, where the geometry of the manifolds plays a central role. This monograph is full of such examples.

In a certain sense, Sobolev spaces are studied here for their own interest. Needless to say, they are fundamental in the study of PDE's. A striking example where they have played a major role in the Riemannian context is given by the famous Yamabe problem. The concept of best constants appeared there as crucial for solving limiting cases of some partial differential equations. (Geometric problems often lead to limiting cases of known problems in analysis). While the theory of Sobolev spaces for (non compact) manifolds has its origin in the 70's with the works of Aubin and Cantor, many of the results presented in this monograph have been obtained in the '80's and '90's. As the reader will easily be convinced, the study of Sobolev spaces in the Riemannian context is a field currently undergoing great development !

This monograph presupposes a preliminary course in Riemannian geometry. Not much is assumed to be known so that chapter 1 of Aubin [Au6] should provide specialists in analysis who do not know Riemannian geometry with sufficient knowledge for what follows. Needless to say, many excellent books on Riemannian geometry exist. Although the following ones are not the only possible quality choices, we refer the reader to Chavel [Ch], Gallot-Hulin-Lafontaine [GaHL], Jost [Jo], Kobayashi-Nomizu [KoN], and Spivak [Sp] for more details on what is assumed to be known here.

The material is organized into five chapters and several new results are presented. More precisely, the plan of this monograph is as follows.

Chapter 1 is devoted to the presentation of recent developments of Anderson and Anderson-Cheeger concerning harmonic coordinates, as well as the presentation of a packing result that will be often used in the following chapters.

Chapter 2 is devoted to the presentation of Sobolev spaces on Riemannian manifolds, and to the study of density problems.

Chapter 3 is devoted to Sobolev embeddings. This includes the presentation of general results on the topic, and the study of Sobolev embeddings for Euclidean spaces, compact manifolds, and complete manifolds.

Chapter 4 is devoted to what is currently called the best constants problems. Several results are discussed here, including those concerning the resolution of Aubin's conjecture by Hebey-Vaugon.

Finally, chapter 5 is devoted to the study of the influence of symmetries on Sobolev embeddings.

It is my pleasure and privilege to express my deep thanks to my friend Michel Vaugon for his valuable comments and suggestions about the manuscript. It is also my pleasure and privilege to express my deep thanks to Ms Thanh-Ha Le Thi, and more generally to the staff of Springer-Verlag, for its patience and dedication.

Emmanuel Hebey
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Table of contents

Introduction ix

Chapter 1: Geometric preliminaries 1

1.1 A brief introduction to Riemannian geometry 1

1.2 Estimates on the components of the metric tensor 3

1.3 From local analysis to global analysis 6

Chapter 2: Sobolev spaces 10

2.1 First definitions 10

2.2 Density problems 12

Chapter 3: Sobolev embeddings 17

3.1 Preliminary results 17

3.2 The Sobolev embedding theorem for $\mathbf{R}^n$ 21

3.3 Sobolev embeddings for compact manifolds 23

3.4 Sobolev embeddings and convergence of manifolds 27

3.5 Sobolev embeddings for complete manifolds 29

3.6 Disturbed inequalities for complete manifolds 41

3.7 $\mathbf{R}^n$-type inequalities for complete manifolds 48

Chapter 4: The best constants problems 58

4.1 Program $\mathcal{B}$ 59

4.2 Program $\mathcal{A}$ 60

4.3 Proof of theorem 4.12 75

4.4 Program $\mathcal{A}$ - Various results and open questions 79

4.5 Improvement of the best constants 86

Chapter 5: Sobolev spaces in the presence of symmetries 90

5.1 Embeddings in higher L^p spaces 90

5.2 Improvement of the best constants 95

5.3 Compactness and symmetries 100

References 106

Notation index 114

Subject index 116

Chapter 1

Geometric preliminaries

This chapter is devoted to the presentation of geometric results we will use in the sequel. This includes a brief introduction to Riemannian geometry, the presentation of recent results of Anderson and Anderson-Cheeger concerning the harmonic radius of a Riemannian manifold, and the presentation of a packing lemma that will be often used in the following chapters.

1.1 A BRIEF INTRODUCTION TO RIEMANNIAN GEOMETRY

A Riemannian manifold (M,g) is a manifold M together with a $(2,0)$ tensor field g such that for any x in M, $g(x)$ is a scalar product on $T_x(M)$. Let $|.|_g$ be the norm on $T_x(M)$ with respect to $g(x)$. One can define a distance d_g on M and a positive Radon measure $f \to \int_M f dv(g)$. Basically, $d_g(x,y)$ is the infimum of the lengths $L(\gamma)$ of all piecewise C^1 curves $\gamma : [a,b] \to M$ from x to y, where

$$L(\gamma) = \int_a^b |(\frac{d\gamma}{dt})_t|_g dt ,$$

while the Riemannian volume element is given in any chart by

$$dv(g) = \sqrt{det(g_{ij})}dx ,$$

where the g_{ij}'s are the components of g in the chart, and dx is the Lebesgue's volume element of $\mathbf{R}^n$, $n = dimM$. One can also define the Levi-Civita connection of g as the unique linear connection on M which is torsion free and which is such that the covariant derivative of g is zero. The Christoffel symbols of the Levi-Civita connection are then given in any chart by

$$\Gamma_{ij}^k = \frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})g^{mk}$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}), and the Einstein's summation convention is adopted. The components in any chart of the Riemann curvature $Rm_{(M,g)}$ of (M,g), viewed as a $(4,0)$ tensor field, are given by the relation

$$R_{ijkl} = g_{i\alpha}(\partial_k \Gamma_{jl}^\alpha - \partial_l \Gamma_{jk}^\alpha + \Gamma_{k\beta}^\alpha \Gamma_{jl}^\beta - \Gamma_{l\beta}^\alpha \Gamma_{jk}^\beta)$$

Similarly, the components in any chart of the Ricci curvature $Rc_{(M,g)}$ of (M,g), $Rc_{(M,g)}$ is a $(2,0)$ tensor field, are given by the relation $R_{ij} = g^{\alpha\beta} R_{i\alpha j\beta}$.

It is well known that sign assumptions on the curvatures give topological and diffeomorphic informations on the manifold. Striking examples of this fact are given by Myers theorem (a complete Riemannian manifold whose Ricci curvature satisfies $Rc_{(M,g)} \geq kg$ as bilinear forms, $k > 0$ real, is compact), by the Cartan-Hadamard theorem (a complete simply connected Riemannian manifold whose sectional curvature is nonpositive is diffeomorphic to the euclidean space of same dimension), by the sphere theorem of Berger, Klingenberg, and Rauch (a compact simply connected Riemannian manifold whose sectional curvature is 1/4 pinched is homeomorphic to the sphere of same dimension), or by Hamilton's theorem (a compact simply connected 3-dimensional Riemannian manifold whose Ricci curvature is positive is diffeomorphic to the sphere of dimension 3). On the other hand, by a recent result of Lokhamp, any compact manifold carries a metric with negative Ricci curvature. Concerning these interactions between the curvature and the topology of the manifold, one can also think to the Euler caracteristic, which according to the work of Allendoerfer, Chern, Fenchel and Weil, can be expressed as the integral of some universal polynomial in the curvature.

Another important object one will meet in the sequel is the injectivity radius. If D denotes the Levi-Civita connection of g, a smooth curve γ is said to be a geodesic if $D_{(\frac{d\gamma}{dt})}(\frac{d\gamma}{dt}) = 0$. In local coordinates, this means that for any k,

$$(\gamma^k)''(t) + \Gamma_{ij}^k(\gamma(t))(\gamma^i)'(t)(\gamma^j)'(t) = 0$$

By the Hopf-Rinow's theorem, any geodesic on a complete Riemannian manifold (M, g) (that is with respect to the distance d_g) is defined on the whole of $\mathbf{R}$. Given (M, g) a complete Riemannian manifold and x a point of M, the injectivity radius $inj_{(M,g)}(x)$ at x is defined as the largest $r > 0$ for which any geodesic γ of length less than r and having x as an endpoint is minimizing. One has that $inj_{(M,g)}(x)$ is positive for any x. The injectivity radius of (M, g) is then defined as the infimum of $inj_{(M,g)}(x)$, $x \in M$. It may be zero.

Closely related to geodesics is the exponential map. Given (M, g) a complete Riemannian n-manifold, x a point of M, and $X \in T_x(M)$, one easily checks that there exists a unique geodesic γ such that $\gamma(0) = x$ and $(\frac{d\gamma}{dt})_0 = X$. Let $t \to \gamma_X(t)$ be this geodesic. The exponential map exp_x at x is then the map from $T_x(M)$ to M defined by $exp_x(X) = \gamma_X(1)$. Up to the identification of $T_x(M)$ with $\mathbf{R}^n$, it is smooth and it defines geodesic normal coordinates at x on $B_x(inj_{(M,g)}(x)) = \{y \in M \text{ s.t. } d_g(x, y) < inj_{(M,g)}(x)\}$. More generally, one can define the cut locus $Cut(x)$ of x, it is a subset of M, and prove that $Cut(x)$ has measure zero, that $inj_{(M,g)}(x) = d_g(x, Cut(x))$, and that exp_x is a diffeomorphism from some star-shaped domain of $T_x(M)$ at 0 onto $M \backslash Cut(x)$. In particular, this implies that the distance function r to a given point is differentiable almost everywhere, with the additional property that $|\nabla r| = 1$ a.e., where $|\nabla r|$ denotes the norm with respect to g of the first covariant derivative of r.

For more details we refer on one hand to [Au6, chapter 1], where a short interesting introduction to Riemannian geometry can be found, on the other hand to the excellent books of Chavel [Ch], Gallot-Hulin-Lafontaine [GaHL], Jost [Jo], Kobayashi-Nomizu [KoN], and Spivak [Sp]. Needless to say, these are not the only possible quality choices.

Once and for all,

ALL THE MANIFOLDS IN THIS MONOGRAPH ARE ASSUMED TO BE CONNECTED, SMOOTH, WITHOUT BOUNDARY, AND OF DIMENSION $n \geq 3$.

In the following, the Einstein's summation convention is adopted so that $\alpha_i x^i = \sum_i \alpha_i x^i$.

1.2 ESTIMATES ON THE COMPONENTS OF THE METRIC TENSOR

The purpose of this paragraph is to recall how one gets bounds on the components of the metric tensor from bounds on the curvature and the injectivity radius. In other words, how one can choose suitable coordinates so that the components g_{ij} of the metric in these coordinates are bounded in terms of bounds on the curvature and the injectivity radius. The first results in this direction were obtained by using geodesic normal coordinates. In such coordinates, one easily obtains C^0-bounds on the g_{ij}'s from lower and upper bounds on the sectional curvature, and from a lower bound on the injectivity radius. (See [Au6, chapter 1]. See also lemma 1.4 below for further developments). Independently, we know by now that harmonic coordinates are more adapted to this goal. These coordinates were first used by Einstein, then by Lanczos who observed that they simplify the formula for the Ricci tensor. Namely, in such coordinates,

$$R_{ij} = -\frac{1}{2} g^{mk} \partial_{km} g_{ij} \ + \ \text{terms involving at most one derivative of the metric}$$

where (g^{ij}) is the inverse matrix of (g_{ij}) and the R_{ij}'s denote the components of the Ricci curvature of the Riemannian manifold (M, g). Although we are not going to discuss that here, we mention that such a formula has many interesting consequences. We refer to DeTurck-Kazdan [DK] for some of them.

From now on, let Δ_g be the Laplace operator associated to g acting on functions. In local coordinates,

$$\Delta_g u = -g^{ij}\left(\partial_{ij} u - \Gamma_{ij}^k \partial_k u\right) = -\frac{1}{\sqrt{det(g_{ij})}} \partial_m \left(\sqrt{det(g_{ij})} g^{mk} \partial_k u\right)$$

where the Γ_{ij}^k's are the Christoffel symbols of the Levi-Civita connection, and where $det(g_{ij})$ stands for the determinant of the matrix (g_{ij}). One then has the following definition of harmonic coordinates.

Definition 1.1: *A coordinate chart $(x^1,\dots,x^n)$ on a Riemannian n-manifold (M,g) is called harmonic if $\Delta_g x^k = 0$ for all $k = 1,\dots,n$. Since $\Delta_g x^k = g^{ij}\Gamma_{ij}^k$, we get that a coordinate chart $(x^1,\dots,x^n)$ is harmonic if and only if for any $k = 1,\dots,n$, $g^{ij}\Gamma_{ij}^k = 0$.*

It is easy to prove that for any $x \in M$, there is a neighborhood of x in which harmonic coordinates exist. The proof of such a claim is based on the classical fact that there always exists a smooth solution of $\Delta_g u = 0$ with $u(x)$ and $\partial_i u(x)$ prescribed. The solutions y^j, $j = 1,\dots,n$, of

$$\begin{cases} \Delta_g y^j = 0 \\ y^j(x) = 0 \\ \partial_i y^j(x) = \delta_i^j \end{cases}$$

are then the desired harmonic coordinates. Furthermore, since composing with linear transformations do not affect the fact that coordinates are harmonic, one easily sees that we can choose the harmonic coordinate system such that $g_{ij}(x) = \delta_{ij}$ for any $i,j = 1,\dots,n$. We refer to DeTurck-Kazdan [DK] for more basic material on harmonic coordinates.

Let us now define the concept of harmonic radius.

Definition 1.2: *Let (M,g) be a Riemannian n-manifold and let $x \in M$. Given $Q > 1$, $k \in \mathbb{N}$, and $\alpha \in (0,1)$, we define the $C^{k,\alpha}$ harmonic radius at x as the largest number $r_H = r_H(Q,k,\alpha)(x)$ such that on the geodesic ball $B_x(r_H)$ of center x and radius r_H, there is a harmonic coordinate chart such that the metric tensor is $C^{k,\alpha}$ controlled in these coordinates. Namely, if g_{ij}, $i,j = 1,\dots,n$, are the components of g in these coordinates, then*

1) $Q^{-1}\delta_{ij} \le g_{ij} \le Q\delta_{ij}$ as bilinear forms

2) $$\sum_{1\le|\beta|\le k} r_H^{|\beta|} sup_y |\partial_\beta g_{ij}(y)| + \sum_{|\beta|=k} r_H^{k+\alpha} sup_{y\ne z} \frac{|\partial_\beta g_{ij}(z) - \partial_\beta g_{ij}(y)|}{d_g(y,z)^\alpha} \le Q-1$$

where d_g is the distance associated to g. The harmonic radius $r_H(Q,k,\alpha)(M)$ of (M,g) is now defined by $r_H(Q,k,\alpha)(M) = inf_{x\in M} r_H(Q,k,\alpha)(x)$.

One easily checks that the function $x \to r_H(Q,k,\alpha)(x)$ is 1-lipschitzian on M, since by definition, for any $x, y \in M$,

$$r_H(Q,k,\alpha)(y) \geq r_H(Q,k,\alpha)(x) - d_g(x,y)$$

According to what we have said above, one then gets that the harmonic radius is positive for any fixed smooth compact Riemannian manifold. Now, theorem 1.3 below shows that one obtains lower bounds on the harmonic radius in terms of bounds on the Ricci curvature and the injectivity radius. Roughly speaking, when changing from geodesic normal coordinates to harmonic coordinates, one controls the components of the metric in terms of the Ricci curvature instead of the whole Riemann curvature. As it is stated below, theorem 1.3 can be found in Hebey-Herzlich [HH], and we refer to [HH] for its proof. For original references, we refer to Anderson [An2], and Anderson-Cheeger [AC]. (See also Jost-Karcher [JK]). Let (M,g) be a Riemannian manifold. In the following, $Rc_{(M,g)}$ denotes its Ricci curvature, $\nabla^j Rc_{(M,g)}$ denotes the jth-covariant derivative of $Rc_{(M,g)}$, and if x is some point of M, $inj_{(M,g)}(x)$ denotes the injectivity radius of (M,g) at x.

Theorem 1.3: *Let $\alpha \in (0,1)$, $Q > 1$, $\delta > 0$. Let (M,g) be a Riemannian n-manifold, and Ω an open subset of M. Set*

$$\Omega(\delta) = \{x \in M \ s.t. \ d_g(x,\Omega) < \delta\}$$

where d_g is the distance associated to g. Suppose that for some $\lambda \in \mathbf{R}$ and $i > 0$, we have that for all $x \in \Omega(\delta)$,

$$Rc_{(M,g)}(x) \geq \lambda g_x \quad \text{and} \quad inj_{(M,g)}(x) \geq i$$

Then there exists a positive constant $C = C(n,Q,\alpha,\delta,i,\lambda)$, depending only on n, Q, α, δ, i, and λ, such that for any $x \in \Omega$, $r_H(Q,0,\alpha)(x) \geq C$. In addition, if instead of the bound $Rc_{(M,g)}(x) \geq \lambda g_x$ we assume that for some $k \in \mathbf{N}$ and some positive constants $C(j)$,

$$|\nabla^j Rc_{(M,g)}(x)| \leq C(j) \quad \text{for all} \quad j = 0, \ldots, k \text{ and all } \ x \in \Omega(\delta)$$

then, there exists a positive constant $C = C(n,Q,k,\alpha,\delta,i,C(j)_{0\leq j\leq k})$, depending only on n, Q, k, α, δ, i, and the $C(j)$'s, $0 \leq j \leq k$, such that for any $x \in \Omega$, $r_H(Q,k+1,\alpha)(x) \geq C$.

The proof of theorem 1.3 is by contradiction. It is too long to be developed here. Let us just say that the general idea is to construct a sequence of Riemannian n-manifolds with harmonic radius less than or equal to 1, to prove that it converges to the euclidean space $\mathbf{R}^n$, and to get the contradiction by noting that this would imply that the harmonic radius of $\mathbf{R}^n$ is less than or equal to 1. (Obviously, $\mathbf{R}^n$ has an infinite harmonic radius).

As already mentioned, analogous estimates to those of theorem 1.3 are available if one works with geodesic normal coordinates instead of harmonic coordinates. These estimates are rougher since, for instance, they involve the Riemann curvature instead of the Ricci curvature. Anyway, such results are sometimes useful, and, actually, lemma 1.4 below is used in the proof of theorem 4.12 of chapter 4. This is why we mention it. For details on its proof, we refer to Hebey-Vaugon [HV3, section III]. Let (M,g) be a Riemannian manifold. In the following, $Rm_{(M,g)}$ denotes its Riemann curvature (viewed as a (4,0) tensor field), $\nabla Rm_{(M,g)}$ denotes the first covariant derivative of $Rm_{(M,g)}$, and, as above, $inj_{(M,g)}(x)$ denotes the injectivity radius of (M,g) at x.

Lemma 1.4: *Let (M,g) be a Riemannian n-manifold. Suppose that for some $x \in M$ there exist positive constants Λ_1 and Λ_2 such that $|Rm_{(M,g)}| \leq \Lambda_1$ and $|\nabla Rm_{(M,g)}| \leq \Lambda_2$ on the geodesic ball $B_x\big(inj_{(M,g)}(x)\big)$ of center x and radius $inj_{(M,g)}(x)$. Then there exist positive constants $K = K(n,\Lambda_1,\Lambda_2)$ and $\delta = \delta(n,\Lambda_1,\Lambda_2)$, depending only on n, Λ_1 and Λ_2, such that the components g_{ij} of g in geodesic normal coordinates at x satisfy: for any $i,j,k = 1,\dots,n$ and any $y \in B_0^e\Big(min\big(\delta, inj_{(M,g)}(x)\big)\Big)$,*

(i) $\frac{1}{4}\delta_{ij} \leq g_{ij}\big(exp_x(y)\big) \leq 4\delta_{ij}$ *(as bilinear forms)*

(ii) $|g_{ij}\big(exp_x(y)\big) - \delta_{ij}| \leq K|y|^2$ *and* $|\partial_k g_{ij}\big(exp_x(y)\big)| \leq K|y|$

where for $t > 0$, $B_0^e(t)$ denotes the euclidean ball of $\mathbf{R}^n$ with center 0 and radius t, and $|y|$ is the euclidean distance from 0 to y. In addition, one has that

$$\lim_{(\Lambda_1,\Lambda_2)\to(0,0)} \delta(n,\Lambda_1,\Lambda_2) = +\infty \quad \textit{and} \quad \lim_{(\Lambda_1,\Lambda_2)\to(0,0)} K(n,\Lambda_1,\Lambda_2) = 0$$

The proof of lemma 1.4 starts with standard estimates of the theory of Jacobi fields. (See for instance [Au6, chapter 1]). Then it relies on a careful study of the formula for the curvature in polar coordinates.

1.3 From local analysis to global analysis

The purpose of this paragraph is to prove a packing lemma that will be used many times in the following chapters. This lemma is by now classical. It was first proved by Calabi (unpublished) under the assumptions that the sectional curvature of the manifold is bounded and that the injectivity radius of the manifold is positive. (See Aubin [Au2], and Cantor [Can]). By Croke's result [Cr, proposition 14] it was then possible to replace the assumption on the sectional curvature by a lower bound on the Ricci curvature. Finally, by an ingenious use of Gromov's theorem, theorem 1.5 below, one obtains the result under the more general form of lemma 1.6. When we will discuss Sobolev inequalities on complete manifolds, this lemma will be an important tool in the process of passing from local to global inequalities.

The following result, due to Gromov [GrLP], is by now classical. We refer the reader to [GaHL, theorem 4.19] for details on its proof.

Theorem 1.5: *Let (M,g) be a complete Riemannian n-manifold whose Ricci curvature satisfies $Rc_{(M,g)} \geq (n-1)kg$ for some $k \in \mathbf{R}$. Then, for any $0 < r < R$ and any $x \in M$,*

$$Vol_g(B_x(R)) \leq \frac{V_k(R)}{V_k(r)} Vol_g(B_x(r))$$

where $Vol_g(B_x(t))$ denotes the volume of the geodesic ball of center x and radius t, and where $V_k(t)$ denotes the volume of a ball of radius t in the complete simply connected Riemannian n-manifold of constant curvature k. In particular, for any $r > 0$ and any $x \in M$, $Vol_g(B_x(r)) \leq V_k(r)$.

Remark: Let b_n be the volume of the Euclidean ball of radius one. It is well known (see for instance [GaHL]) that for any $t > 0$,

$$V_{-1}(t) = nb_n \int_0^t (sinhs)^{n-1} ds$$

where, according to the notations of theorem 1.5, $V_{-1}(t)$ denotes the volume of a ball of radius t in the simply connected hyperbolic space of dimension n. It is then easy to prove that for any $k \geq 0$ and any $t > 0$,

$$b_n t^n \leq V_{-k}(t) \leq b_n t^n e^{(n-1)\sqrt{k}t}$$

One just has to note that for $s \geq 0$, $s \leq sinhs \leq se^s$, and that if $g' = \alpha^2 g$ are Riemannian metrics on a n-manifold M, where α is some positive real number, then for any $x \in M$ and any $t > 0$,

$$Vol_{g'}(B'_x(t)) = \alpha^n Vol_g(B_x(t/\alpha))$$

As a consequence, by theorem 1.5 and what we just said we get that if (M,g) is a complete Riemannian n-manifold whose Ricci curvature satisfies $Rc_{(M,g)} \geq kg$ for some $k \in \mathbf{R}$, then for any $x \in M$ and any $0 < r < R$,

$$Vol_g(B_x(R)) \leq e^{\sqrt{(n-1)|k|}R} \left(\frac{R}{r}\right)^n Vol_g(B_x(r))$$

This explicit inequality will be used sometimes in the sequel.

Let (M,g) be a Riemannian manifold. For any $x \in M$ and any $r > 0$, we denote by $B_x(r)$ the geodesic ball of center x and radius r. Independently, we say that a family (Ω_k) of open subsets of M is a uniformly locally finite covering of M if the following holds: $\cup_k \Omega_k = M$ and there exists an integer N such that each point $x \in M$ has a neighborhood which intersects at most N of the Ω_k's.

Lemma 1.6: *Let (M,g) be a complete Riemannian n-manifold with Ricci curvature bounded from below by some $k \in \mathbf{R}$, and let $\rho > 0$ be given. There exists a sequence (x_i) of points of M such that for any $r \geq \rho$:*

(i) the family $(B_{x_i}(r))$ is a uniformly locally finite covering of M, and there is an upper bound for N in terms of n, ρ, r, and k

(ii) for any $i \neq j$, $B_{x_i}(\rho/2) \cap B_{x_j}(\rho/2) = \emptyset$

Proof of lemma 1.6: By theorem 1.5 and the remark following this theorem we get that for any $x \in M$ and any $0 < r < R$,

$$Vol_g(B_x(r)) \geq e^{-\sqrt{(n-1)|k|}R}\left(\frac{r}{R}\right)^n Vol_g(B_x(R)) \tag{1}$$

Independently, we claim that there exists a sequence (x_i) of points of M such that

$$M = \bigcup_i B_{x_i}(\rho) \text{ and } \forall i \neq j,\ B_{x_i}(\rho/2) \cap B_{x_j}(\rho/2) = \emptyset \tag{2}$$

Actually, let

$$X_\rho = \left\{(x_i)_I,\ x_i \in M,\ \text{s.t. } I \text{ is countable and } \forall i \neq j,\ d_g(x_i,x_j) \geq \rho\right\}$$

where d_g is the Riemannian distance associated to g. X_ρ is partially ordered by inclusion and, obviously, every chain in X_ρ has an upper bound. Hence, by Zorn's lemma, X_ρ contains a maximal element (x_i), and (x_i) satisfies (2).

Let (x_i) be such that (2) is satisfied. For $r > 0$ and $x \in M$ we define

$$I_r(x) = \left\{i \text{ s.t. } x \in B_{x_i}(r)\right\}$$

By (1) we get that for $r \geq \rho$

$$\begin{aligned} Vol_g(B_x(r)) &\geq \frac{1}{2^n} e^{-2\sqrt{(n-1)|k|}r} Vol_g(B_x(2r)) \\ &\geq \frac{1}{2^n} e^{-2\sqrt{(n-1)|k|}r} \sum_{i \in I_r(x)} Vol_g(B_{x_i}(\rho/2)) \end{aligned}$$

since

$$\bigcup_{i\in I_r(x)} B_{x_i}(\rho/2) \subset B_x(2r) \quad \text{and} \quad B_{x_i}(\rho/2) \cap B_{x_j}(\rho/2) = \emptyset \text{ if } i \neq j$$

But, again by (1),

$$Vol_g\big(B_{x_i}(\rho/2)\big) \geq e^{-2\sqrt{(n-1)|k|}r}\big(\frac{\rho}{4r}\big)^n Vol_g\big(B_{x_i}(2r)\big)$$

and since for any $i \in I_r(x)$, $B_x(r) \subset B_{x_i}(2r)$, we get that

$$Vol_g\big(B_x(r)\big) \geq \big(\frac{\rho}{8r}\big)^n e^{-4\sqrt{(n-1)|k|}r} Card I_r(x) Vol_g\big(B_x(r)\big)$$

where $Card$ stands for the cardinality. As a consequence, for any $r \geq \rho$ there exists $C = C(n, \rho, r, k)$ such that for any $x \in M$, $Card I_r(x) \leq C$.

Now, let $B_{x_i}(r)$ be given, $r \geq \rho$, and suppose that N balls $B_{x_j}(r)$ have a nonempty intersection with $B_{x_i}(r)$, $j \neq i$. Then, obviously, $Card I_{2r}(x_i) \geq N+1$. Hence,

$$N \leq C(n, \rho, 2r, k) - 1$$

and this proves the lemma.