CONTRIBUTIONS TO THE THEORY OF RIEMANN SURFACES

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Centennial Celebration of Riemann's Dissertation

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FOREWORD

On December 26, 1851, Bernard Riemann presented his inaugural dissertation. On December 14, 15, 1951, a Conference on Riemann Surfaces was held in Princeton, New Jersey in commemoration of this event. The Institute for Advanced Study and Princeton University acted as joint sponsors. The present volume of the Annals of Mathematics Studies contains the papers presented at this conference, and the papers are published in the order in which they were presented.

The diversity of mathematical interest and approach apparent in the papers presented here is a small indication of the breadth of influence of Riemann's ideas upon modern mathematics.

Committee on Arrangements
Marston Morse, Chairman
Lars V. Ahlfors
Eugenio Calabi
Leo Sario
D. C. Spencer

CONTENTS

CONTRIBUTIONS TO THE THEORY OF RIEMANN SURFACES

DEVELOPMENT OF THE THEORY OF CONFORMAL MAPPING AND RIEMANN SURFACES THROUGH A CENTURY

LARS V. AHLFORS

This conference has been called to celebrate the hundredth anniversary of the presentation of Bernhard Riemann's inaugural dissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse".

Very few mathematical papers have exercised an influence on the later development of mathematics which is comparable to the stimulus received from Riemann's dissertation. It contains the germ to a major part of the modern theory of analytic functions, it initiated the systematic study of topology, it revolutionized algebraic geometry, and it paved the way for Riemann's own approach to differential geometry.

To deal with all these aspects in the short time at our disposal would be impossible. It is therefore necessary to concentrate on the central idea in Riemann's thesis which is that of combining geometric thought with complex analysis. In this introductory lecture my task will be to trace the main lines along which geometric function theory has expanded from Riemann's time to ours.

Geometric function theory

Riemann's paper marks the birth of geometric function theory. At the time of its appearance Cauchy had already laid the foundation of analytic function theory in the modern sense, but its use was not widespread. It is clear that complex integration had introduced a certain amount of geometric content in analysis, but it would be wrong to say that Riemann's ideas were in any way anticipated. Riemann was the first to recognize the fundamental connection between conformal mapping and complex function theory: to Gauss, conformal mapping had definitely been a problem in differential geometry.

The most astonishing feature in Riemann's paper is the breathtaking generality with which he attacks the problem of conformal mapping. He has no thought of illustrating his methods by simple examples which to lesser mathematicians would have seemed such an excellent preparation and undoubtedly would have helped his paper to much earlier recognition. On the contrary, Riemann's writings are full of almost cryptic messages to the future. For instance, Riemann's mapping theorem is ultimately formulated in terms which would defy any attempt of proof, even with modern methods.

Riemann surfaces

Among the creative ideas in Riemann's thesis none is so simple and at the same time so profound as his introduction of multiply covered regions, or Riemann surfaces. The reader is led to believe that this is a commonplace convention, but there is no record of anyone having used a similar device before. As used by Riemann it is a skillful fusion of two distinct and equally important ideas: 1) A purely topological notion of covering surface, necessary to clarify the concept of mapping in the case of multiple correspondence; 2) An abstract conception of the space of the variable, with a local structure defined by a uniformizing parameter. The latter aspect comes to the foreground in the treatment of branchpoints.

From a modern point of view the introduction of Riemann surfaces foreshadows the use of arbitrary topological spaces, spaces with a structure, and covering spaces.

Existence and uniqueness theorems

There is a characteristic feature of Riemann's thesis which should not be underestimated. The whole paper is built around existence and uniqueness theorems. To us, this seems the most natural thing in the world, for this is what we expect from a paper which introduces a new theory. But we must realize that Riemann is, to say the least, one of the earliest and strongest proponents of this point of view. Again and again, explicitly and between the lines, he emphasizes that a function can be defined by its singularities. This approach calls for existence and uniqueness theorems, in contrast to the classical conception of a function as a closed analytic expression. There is no doubt that Riemann's point of view has had a decisive influence on modern mathematics.

Potential theory

Next to the geometric interpretation, the leading mathematical idea in Riemann's paper is the importance attached to Laplace's equation. He virtually puts equality signs between two-dimensional potential theory and complex function theory. Riemann's aim was to make complex function theory a powerful tool in real analysis, especially in the theory of partial differential equations and thereby in mathematical physics. It must be remembered that Riemann was in no sense confined to a mathematical hothouse atmosphere; his broad mind was prone to accept all the inspiration he could gather from his unorthodox, but suggestive, conception of contemporary physics.

Dirichlet's principle

Riemann's proof of the fundamental existence theorems was based on an uncritical use of the Dirichlet principle. It is perhaps wrong to call Riemann uncritical, for he made definite attempts to exclude a degenerating extremal function. In any case, if he missed the correct proof, he made up for it by giving a very general formulation. In modern language, his approach, which is clearer in the paper on Abelian integrals, is the following:

Given a closed differential α with given periods, singularities and boundary values, he assumes the existence of a closed differential β such that $\alpha + \beta^*$ (β^* denotes the conjugate differential) has a finite Dirichlet norm. Then he determines an exact differential ω_1, with zero boundary values, whose norm distance from $\alpha + \beta^*$ is a minimum. But this is equivalent to an orthogonal decomposition

$$\alpha + \beta^* = \omega_1 + \omega_2^* \qquad (\omega_1 \text{ exact, } \omega_2 \text{ closed})$$

from which it follows that

$$\alpha - \omega_1 = \omega_2^* - \beta^*$$

is simultaneously closed and co-closed, that is to say harmonic. Hence the existence theorem: there exists a harmonic differential with given boundary values, periods, and singularities.

The easiest way to make the reasoning exact is to complete the differentials to a Hilbert space. Closed and exact differentials can be defined by orthogonality, and in the final step one needs a lemma of Herman Weyl for which there is now a very short proof. Riemann, who did not have these tools, was nevertheless able to choose this beautiful approach which unifies the problems of boundary values, periods, and singularities.

Schwarz-Neumann

Riemann's failure to provide a rigorous proof for the Dirichlet principle was beneficial in causing a flurry of attempts to prove the main existence theorems by other methods. The first to be successful was H. A. Schwarz who devised the alternating method. Minor improvements were contributed by C. Neumann who is most notable as a popularizer of Riemann's ideas. The alternating method proved to be sufficient to dispose of the existence problems on closed surfaces.

The method of Schwarz is a linear method, and in principle it amounts to solving a linear integral equation by iteration. The difficulties in adapting the method to the existence theorems are of a practical nature, but they are considerable. The advantages are that the method is successful, and constructive, but in simplicity and elegance it does not compare, even remotely, with Riemann's method.

Poincaré-Klein

In the next generation the leaders were Poincaré and F. Klein. Their important innovation is the introduction of the problem of uniformization of algebraic and general analytic curves. The study of Riemann surfaces in this light leads to automorphic functions and the use of non-euclidean geometry. The method is extremely beautiful; as H. Weyl puts it, the nature of Riemann surfaces is reflected in the non-euclidean crystal. It has also the advantage of leading to very explicit representations by way of Poincaré's theta-series and their generalizations.

The disadvantage is that the existence proofs are quite difficult. For compact surfaces Poincaré produces a correct proof based on a continuity method which is simple in principle, but technically very involved. It is interesting to note that Poincaré concentrates his efforts on proving the general uniformization theorem. In spite of its generality this theorem would not even include the Riemann mapping theorem. Poincaré has been extremely influential in developing methods which ultimately led to proofs of the mapping and uniformization theorems, but he did not himself produce a complete proof until 1908, having been preceded by Osgood who proved the Riemann mapping theorem in 1900, and Koebe who proved the uniformization theorem in 1907.

Osgoods' proof is very remarkable, because it is so clear and concise and does not leave any room for doubt. It is based on an idea of Poincaré, but Osgood deserves full credit for making the idea work. The proof uses the modular function, and is thus not elementary.

Koebe

The crowning glory was achieved by Koebe when he proved the general uniformization theorem. This is the theorem which asserts that every simply connected Riemann surface is conformally equivalent with the sphere, the disk, or the plane. It immediately takes care of the uniformization of the most general surface, for it is sufficient to map the universal covering surface. As a tool he uses his famous "Verzerrungssatz".

The stage was now set for deeper investigations of the problem of conformal mapping. The standard theorems which concern the canonical mappings of multiply connected regions are from this time. Koebe was an undisputed leader, and Leipzig a center for conformal mapping.

Looking back one cannot help being impressed by Koebe's life work. His methods were completely different from those of his predecessors, and when the initial difficulties were conquered he did not hesitate to attack new problems of ever increasing complexity.

Idee der Riemannschen Fläche

In the classical literature no clear definition of a Riemann surface is ever given. Primarily, the classical authors thought in terms of multiply covered regions with branch-points, but applications to surfaces in space are not uncommon. It is of course true that F. Klein had a general conception of a Riemann surface which is quite close to modern ideas, but his conception is still partially based on geometric evidence.

H. Weyl's book "Die Idee der Riemannschen Fläche", first published in 1913, was the real eye-opener. Pursuing the ideas of Klein it brings, for the first time, a rigorous and general definition of a Riemann surface, and it marks the death of the glue-and-scissors period. The pioneer qualities of this book should not be forgotten. It is a forerunner which has served as a model for the axiomatization of many mathematical topics. For his definition of the abstract Riemann surface Weyl uses the power series approach. The equivalent definition of Radó is perhaps a little smoother, and Radó added the important recognition that the separability is a consequence of the conformal structure.

Weyl was able to base the existence theorems on the Dirichlet principle which had been salvaged by Hilbert. The book is a reminder of how Riemann's original idea still provides the easiest access to the existence theorems. It has exerted a strong implicit influence by its change of emphasis which has led to a strengthening of the ties between the theory of Riemann surfaces and differential geometry.

Topological aspects

The abstract approach to Riemann surfaces, with all its advantages, tends to neglect the covering surface aspect. With the advances made in topology the notions of fundamental group and universal covering space had become thoroughly familiar, and accordingly the case of smooth covering surfaces was well covered in Weyl's book. Stoilow filled the gap with a study of covering surfaces with branch-points. Whyburn completed the work of Stoilow and based it more firmly on pure topology.

Higher dimensions

Finally, the important question of generalizing Riemann's work to several dimensions has made enormous strides in the last decades. The greenest laurels belong to Hodge for his pioneer research on harmonic integrals on Riemannian manifolds. Through his initiative, and the parallel work of de Rham in topology, it was discovered that the problems of Riemann have a significant counterpart on more-dimensional closed manifolds with a Riemannian metric. The existence theorems presented initial difficulties, but it was finally found that Hilbert's integral equation method as well as Riemann's own method of orthogonal projection can both be made successful (Weyl, de Rham, Kodaira). For the purpose of pursuing the function-theoretic analogy Kählerian manifolds have the most desirable properties, and for such manifolds the problem of singularities has been successfully attacked (A. Weil, Kodaira). Recent advances referring to boundary values will be discussed during this conference.

Meromorphic functions

In discussing the generalizations to several variables I have rushed ahead of the chronological order. I will devote the rest of the lecture to progress in the one variable theory after Koebe's active period. In giving so much space to the modern theory of conformal mapping I may be guilty of overemphasizing the questions which I personally happen to know most about, but there are also objective reasons for being partial to the topics which lie nearest to the central theme in Riemann's thesis.

It is time to recall the important advances made in the theory of entire and meromorphic functions about 1925 and shortly thereafter. Borel, Hadamard, and many others had investigated the properties of entire functions by powerful methods which made use of the canonical representations and power series developments. Their results had an air of being quite definitive, but in an almost sensational paper R. Nevanlinna was able to show that surprisingly elementary potential theoretic methods make it possible to push the study of meromorphic functions very much further. Nevanlinna's theory is quite a show-piece of modern mathematics on a classical basis, and it marked a victory of Riemann's method over the methods which go back to Weierstrass. Nevanlinna's theory of meromorphic functions, and perhaps even more the joint work of the two brothers Nevanlinna, represent a revival of geometric function theory in a direction quite different from the one pursued by Koebe. Among the important tools introduced at this time it is sufficient to mention the introduction of harmonic measure, used independently by Carleman, Ostrowski and the Nevanlinnas.

The type problem

Nevanlinna's main results are generalizations of Picard's theorem. Inasmuch as a meromorphic function maps the plane onto a covering surface they have the character of distortion theorems. This point of view leads almost immediately to the problem of type. The problem is to determine whether a simply connected open Riemann surface, ordinarily given as a covering surface of the sphere, can be mapped conformally onto the whole plane or onto a disk (parabolic or hyperbolic case).

Picard's theorem is the only classical theorem of this sort: if the surface fails to cover three points it must be hyperbolic. In its first phase the problem of type centered about generalizations of this theorem. Generally speaking, if the surface is strongly ramified, it tends to be hyperbolic. It is difficult, however, to measure the ramification. In the special case where all branch-points project into a finite number of points the surface can be described in combinatorial terms with the help of a graph introduced by Speiser. There is a rich literature on this special subject which has produced remarkable results, although the gap between sufficient and necessary conditions is still wide, and likely to remain so.

General methods

There is little point in giving a list of individual results related to the type problem. It is more interesting to discuss the methods by which these results were obtained, especially since they are by no means restricted to this particular problem.

On a Riemann surface it is important to consider all metrics whose element of length is of the form $\rho|dz|$, where z is the local uniformizer. The totality of such metrics is a conformal invariant. It is a classical result that only hyperbolic surfaces can carry a metric with constant negative curvature, and it is not difficult to see that the existence of a metric whose curvature is negative and bounded away from zero is sufficient to imply that the surface is hyperbolic. More generally, the same can be expected if the curvature is negative in the mean, but this property is difficult to formulate in the absence of a natural method to prescribe the weights. What one can do is to consider properties in the large which are roughly equivalent to negative curvature. Such properties are expressed through relations between length and area, for instance in the form of isoperimetric inequalities. In such terms it is possible to formulate necessary and sufficient conditions for the type which have proved very useful.

Method of Grötzsch

Comparisons between length and area in conformal mapping, and the obvious connection derived from the Schwarz inequality, had been used before, notably by Hurwitz and Courant. The first to make systematic use of this relation was H. Grötzsch, a pupil of Koebe. The speaker hit upon the same method independently of Grötzsch and may, unwittingly, have detracted some of the credit that is his due. Actually, Grötzsch had a more sophisticated point of view, but one which did not immediately pay off in the form of simple results.

Extremal length

The strip method of Grötzsch has finally given way to the much more flexible and concise considerations of Beurling. Although the method remains essentially the same, the original idea is given a new twist by putting the emphasis on numerical conformal invariants which obey very simple rules of composition and majoration. This makes the theory so easy to apply that many applications become practically trivial. Above all, it leads to a systematic search for extremal metrics which in many cases are associated with important canonical mappings. The details of this theory of extremal length were developed jointly by Beurling and Ahlfors.

Grunsky

In Germany the tradition of geometric function theory was also carried on by H. Grunsky and O. Teichmüller. The former was a student of

E. Schmidt, and his thesis is a truly remarkable piece of work. He shows, rather surprisingly, that some important extremal problems in the theory of conformal mapping can be successfully attacked by the time-honored device of contour integration. I cannot resist quoting the most beautiful of these results. A region of finite connectivity can be mapped, with a proper normalization, on a slit-region bounded by horizontal or vertical slits. If these mapping functions are $p(z)$ and $q(z)$, Grunsky shows that $p - q$ and $p + q$ have important extremal properties. With a given normalization $p - q$ maps the region onto a Riemann surface of smallest area, while $p + q$ yields a schlicht mapping onto a region whose complement has maximal area. What is more, the contours are convex curves, the same for both mappings (up to a translation and reflection). Part of this result must be accredited to Schiffer, for Grunsky failed to notice that $p + q$ is schlicht. In any case, the idea of applying contour integration to functions of the form $p \pm q$ is due to Grunsky. The same idea has been used later by Schiffer, Spencer and the speaker in the case of general Riemann surfaces with analytic contours.

Teichmüller

In the premature death of Teichmüller geometric function theory, like other branches of mathematics, suffered a grievous loss. He spotted the importance of Grötzsch's technique, and made numerous applications of it which it would take me too long to list. Even more important, he made systematic use of extremal quasi-conformal mappings, a concept that Grötzsch had introduced in a very simple special case. Quasi-conformal mappings are not only a valuable tool in questions connected with the type problem, but through the fundamental although difficult work of Teichmüller it has become clear that they are instrumental in the study of modules of closed surfaces. Incidentally, this study led to a better understanding of the role of quadratic differentials which in somewhat mysterious fashion seem to enter in all extremal problems connected with conformal mapping.

The Finnish school

We now turn our attention to the progress made by the Finnish school under the leadership of R. Nevanlinna and P. Myrberg. The original problem of type had been formulated for simply connected surfaces. While this was natural as long as the problem was coupled with the theory of meromorphic functions, it soon turned out to be an artificial restriction in the general theory of conformal mapping. A very obvious way would have been to consider, for arbitrary surfaces, the type of their universal covering surface, but this does not lead anywhere, for except in the very simplest topological cases the universal covering surface is always hyperbolic.

In the study of general Riemann surfaces it is a good policy to pay careful attention to what happens in the case of plane regions, which are of course considered as special Riemann surfaces. With some care it is usually not too difficult to generalize a theory from plane regions to Riemann surfaces, provided that the theory can be expressed in conformally invariant terms. The notion of potential, and the derived notion of capacity, in the precise definition of Wiener, are not strict conformal invariants, but they have related properties. Myrberg observed that a plane region has a Green's function if and only if its complement is of capacity zero, a property which had not been explicitly stated before. From there it was only a short step to introduce the fundamental dichotomy by which an arbitrary Riemann surface is said to be hyperbolic if it has a Green's function, parabolic if it does not.

Simultaneously, Nevanlinna had looked at the same question from another angle. He found that a closed set of zero capacity has always vanishing harmonic measure, and he was led to consider surfaces whose "ideal boundary" has the harmonic measure zero. It was easy to see that this was identical with Myrberg's classification, surfaces with a null-boundary corresponding to the class of parabolic surfaces.

Both authors proceeded to study the theory of Abelian differentials on parabolic surfaces. Nevanlinna produced a very complete theory of Abelian differentials of the first kind, and Myrberg studied in even greater detail the case of hyperelliptic surfaces of infinite genus. These studies had been preceded by work of Hornich who had investigated some examples.

Classification theory

Parabolic surfaces are degenerate in that they share some of the properties of closed surfaces. One notes, however, that there are certain properties of closed surfaces which carry over to some, but not all, parabolic surfaces. As a continuation of the work begun by Myrberg and Nevanlinna there has lately been much discussion of a more general classification theory of Riemann surfaces. This theory studies different kinds of degeneracy. For instance, one may ask whether a surface can carry bounded analytic functions other than constants, whether it can carry bounded harmonic functions, and so on. It is of interest to relate these properties to each other. It has been demonstrated, for instance, that every surface with a non-trivial bounded harmonic function is hyperbolic, but it is not known whether all hyperbolic surfaces have bounded harmonic functions. Many problems of this nature are still unsolved and seem to present quite a challenge.

Variational methods

Very important progress has also been made in the use of variational methods. I have frequently mentioned extremal problems in conformal mapping, and I believe their importance cannot be overestimated. It is evident that extremal mappings must be the cornerstone in any theory which tries to classify conformal mappings according to invariant properties.

Dirichlet's principle, which is the classical variational problem in function theory, has already been discussed and will not be mentioned further. An interesting attempt to utilize the method of calculus of variations was made by Hadamard who determined the variation of the Green's function for regions with a very regular boundary. The next initiative was taken by Löwner whose most spectacular result was the proof of the inequality $|a_3| \leqq 3$ in the theory of schlicht functions. This was the first decisive advance beyond Koebe's distortion theorem, and it is a result of approximately the same depth as the most recent achievements.

As a systematic tool the method of variation was first introduced by Schiffer. It has also been used by Spencer and Schaeffer in their work on schlicht functions. Important results have been obtained, but what counts more is the creation of a new tool which can be applied with comparative ease to problems which definitely do not lie near the surface.

To give an idea of the method I will describe it in quite general terms, without regard to the fact that this is not precisely the form in which it has been used. I wish only to make the principle clear, without entering in details.

The main application is to extremal problems for schlicht conformal mappings. If one wants to solve such a problem the main difficulty one has to cope with is the effective construction of schlicht variations. Suppose that we are dealing with schlicht mappings of a surface Z into a surface W. If f is such a mapping, and δf a variation, how can we make sure that $f + \delta f$ is again schlicht?

The methods used by Schiffer and Spencer-Schaeffer can be interpreted as follows: On W we introduce a new metric of the form

$$ds^2 = |dw + \epsilon h d\bar{w}|^2,$$

where h behaves properly under changes of the local variable. This metric is not conformal with the metric on W, and thus determines a new Riemann surface W^* with the same points. Through f a corresponding metric will be defined on Z, which is such that f defines a conformal mapping of the new surface Z^* into W^*. Now we define $f + \delta f$ through the series of mappings

$$Z \xrightarrow{\alpha} Z^* \xrightarrow{f} W^* \xrightarrow{\beta} W,$$

where α and β are conformal mappings. This is possible only if Z, Z^* and W, W^* are conformally equivalent. Naturally, the conditions for conformal equivalence lead to the vanishing of certain linear functionals $L_i(h)$[1]. Any side-conditions can be satisfied in the same manner, and if

1) More precisely, linear functionals of h and $\bar{h}$.

$F(f)$ is the functional of f that we try to extremalize, the variation δF is likewise a linear functional of h, and the condition for an extremum can be expressed in the form

$$\delta F(h) = \Sigma \lambda_i L_i(h).$$

This is merely a rough sketch, and it is by no means sure that the program could be carried out in anything like this generality. In any case it should be possible to determine the functionals $L_i(h)$, which express the conformal equivalence, explicitly, and something in this direction has already been done by Garabedian and by Schiffer and Spencer. A serious difficulty is connected with the finding of the mappings which satisfy the variational condition, and finally one is faced with the problem of eliminating the solutions which give only local extremes.

Future progress

To mark the end of this lecture, it is perhaps not useless to point to the directions in which future progress can be expected. Geometric function theory of one variable is already a highly developed branch of mathematics, and it is not one in which an easily formulated classical problem awaits its solution. On the contrary it is a field in which the formulation of essential problems is almost as important as their solution; it is a subject in which methods and principles are all-important, while an isolated result, however pretty and however difficult to prove, carries little weight.

Nothing could be more false than to say that classical function theory has solved its problems and has therefore outlived itself. Even without the introduction of completely new ideas the classical problem of modules, vague as it is, and - to mention a more recent example - the investigation of the true role played by Teichmüller's extremal quasi-conformal mappings, are questions which can keep generations busy. The interaction of topology and function theory is likewise a field which has only been scratched on the surface. Above all, it has happened before that the whole outlook on function theory has changed abruptly, and it will happen again. The spirit of Reimann will move future generations as it has moved us.

Remark

This survey has necessarily followed a narrow central line. Since many names and important contributions have been omitted, it must be underscored that such omissions are in no way indicative of relative merit in the eyes of the writer.