Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

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Author

Steve Wright
Department of Mathematics, Oakland University
Rochester, MI, 48309-4401, USA

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Introduction

These notes are based on the content of lectures delivered to the Seminar on Operator Algebras at Oakland University during the Winter semesters of 1985 and 1986. They are a detailed exposition of [10] and Section 2 of [16], which together constitute a proof of the uniqueness of the (separably acting) injective III_1 factor. The exposition contains nothing that is not already in [10] and [16], but merely fills in details in some of the arguments appearing there. Our hope is that the notes will contribute in some small way to an understanding and appreciation of these profound and beautiful results of Connes and Haagerup.

The following rather informal discussion is intended to define terms and fix some notation relevant for the sequel and to historically orient the results with which it deals. We concentrate exclusively on only selected developments that focus directly on the classification of factors, and apologize here for the many serious omissions of developments in the general theory which consequently result.

The classification of von Neumann algebras to within isomorphism has been the fundamental problem in their study and has motivated much of the work in the subject. (We will only consider von Neumann algebras acting in a *separable* Hilbert space.) Indeed, this was the underlying theme of the initial work in the 1930's of the founding fathers Murray and von Neumann [20], [30]. They isolated the factors, the von Neumann algebras with trivial center, as the key objects for the classification program and showed that the factors could be divided into the three basic types I, II, and III. They showed that all type I factors arise as the full algebra $B(H)$ of all bounded linear operators on a Hilbert space H and are completely classified by the range of their dimension functions. A consequence of these results is that all type I factors are *hyperfinite*, meaning that they contain an ascending sequence of finite-dimensional *-subalgebras whose union is dense in the weak operator topology. It is hence very natural to consider the hyperfinite type II and type III factors as the next goal in the classification program, and in one of the great early triumphs of the theory (Part IV of [20], Theorem XIV), Murray and von Neumann succeeded in proving that all hyperfinite II_1 factors are isomorphic (this factor is usually denoted by R_1).

Attention now turned to the structure of the hyperfinite factors of type II_∞ and type III, and suddenly progress came to a screeching halt. Von Neumann [30] succeeded with a great deal of effort in exhibiting two non isomorphic type III factors, and later J. T. Schwartz and L. Pukansky [24], [23] added two more examples, but the situation remained in this very primitive state throughout the 50's and most of the 60's. However, in 1967 R. T. Powers [22] made an important breakthrough when he exhibited an uncountable family $\{R_\lambda : 0 < \lambda < 1\}$ of nonisomorphic, hyperfinite, type III factors. Powers' examples are constructed as certain infinite tensor products of countably many copies of the algebra of 2×2 matrices, and in order to better understand these examples, Araki and Woods [2] conducted a detailed study in 1968 of infinite tensor products of finite-dimensional matrix algebras, the $ITPFI$ factors. They found a useful isomorphism invariant for such factors, the asymptotic ratio set, and proved that the Powers factor R_λ is characterized as the type III $ITPFI$ factor with asymptotic ratio set $\{0\} \cup \{\lambda^n : n = 0, \pm 1, \pm 2, \ldots\}, 0 < \lambda < 1$. They also discovered a new type III $ITPFI$ factor, with asymptotic ratio set $[0, +\infty)$, commonly denoted by R_∞.

Meanwhile, two developments that would prove crucial for progress on the classification program were occuring in Japan. A problem of interest at the time concerned the determination of the commutant of the (spatial) tensor product of two von Neumann algebras: if M_1 and M_2 are von Neumann algebras, is $(M_1 \otimes M_2)' = M_1' \otimes M_2'$? In 1967, M. Tomita [27], [28] answered this question affirmatively by a new and original analysis of the spatial relationship between a von Neumann algebra and its commutant. The exposition of [27] and [28] was somewhat obscure, however, and in 1970, M. Takesaki published his seminal monograph [25] which explained and extended Tomita's earlier work. Let M be a von Neumann algebra with a vector that is both cyclic and separating for M. Takesaki associated a closed, densely-defined, positive operator Δ with M, the so-called ***modular operator***, which has two very important properties. The first is that $\{\Delta^{it} : t \in (-\infty, \infty)\}$ forms a one-parameter unitary group for which $\Delta^{-it} M \Delta^{it} = M, t \in (-\infty, \infty)$, and so Δ gives rise in this way to a one-parameter group of *-automorphisms of M, the *modular automorphism group*. The second is that Δ induces a conjugate-linear, involutive isometry J of the underlying Hilbert space for which $JMJ = M'$. This shows in particular that M and M' are anti-isomorphic, and is the key to Tomita's proof of the commutation theorem for tensor products. In actuality, the existence of a cyclic and separating vector for M is not necessary for the definition of the modular automorphism group, and in fact if φ is any faithful, normal, positive linear functional on M, then the modular operator Δ_φ and the modular automorphism group $\sigma_t^\varphi = \pi_\varphi^{-1} \circ Ad\Delta_\varphi^{it} \circ \pi_\varphi$ corresponding to φ can be constructed in H_φ and M, where (π_φ, H_φ) is the GNS representation of M induced by φ. A very nice construction of the modular operator and the verification of its main properties can be found in the book [3] of Bratteli and Robinson, and a treatment of the full-strength version is given of course in Takesaki's original memoir [25].

The second major development we referred to above occured also in 1967 when Hakeda and Tomiyama [17] introduced and began the study of the class of injective von Neumann algebras. A von Neumann algebra M is *injective* if whenever A and B are C^*-algebras with $A \subseteq B$, each completely positive map of A into M extends to a completely positive map of B into M, i.e., M is an injective object in the category of C^*-algebras with completely positive maps as the morphisms. Hakeda and Tomiyama's original definition of injectivity (which they called the extension property) was different: they said a von Neumann algebra M, M acting on a Hilbert space H, has the extension property if M is complemented in $B(H)$ by a projection of norm 1. They showed that this definition was independent of the Hilbert space H and that all hyperfinite von Neumann algebras have this property. It is a consequence of work by W. B Arveson in 1969 that injectivity and the extension property are one and the same ([14], Theorem 5.3). Later work of Effros and Lance [14] and Choi and Effros [4], showed that the injective algebras behaved very well with respect to many of the natural operations performed on von Neumann algebras (inductive limits, commutants, direct integrals, tensor products). It was noticed in particular that if injectivity implied hyperfiniteness, then the stability of injectivity with respect to tensor products would give a simple proof of the uniqueness of the hyperfinite II_∞ factor. Attention was thus focused on the relationship between injectivity and hyperfiniteness, with Effros and Lance explicitly conjecturing in 1973 that the former implied the latter.

A new epoch in the classification program opened with the appearance of Alain Connes' thesis [5] in 1973. This memoir contained, among many other results of fundamental importance, a revolutionary refinement of the structure of type III factors. Motivated by the work of Araki and Woods, Connes used the Tomita-Takesaki theory to discover a generalization of the asymptotic ratio set which applied to any type III factor M. This invariant, the *modular spectrum* $S(M)$

of M, is defined as the intersection of the spectra of all modular operators of M which arise from a faithful, normal state of M. It is a remarkable fact that $S(M)\setminus\{0\}$ is a closed subgroup of the multiplicative group of positive real numbers, and $S(M)$ is thus of the form (a) $\{0,\ 1\}$, (b)$_\lambda$ $\{0\}\cup\{\lambda^n;\ n=\pm1,\pm2,\ldots\}$, $0<\lambda<1$, or (c) $[0,+\infty)$. Connes hence refined the classification of type III factors by defining such a factor M to be of type III_0, type III_λ, or type III_1 if $S(M)$ is respectively of the form (a), (b)$_\lambda$, or (c). He observed that certain type III factors which arise from the classical group - measure space construction of von Neumann are type III_0, the Powers factor R_λ is type III_λ, $0<\lambda<1$, and the Araki-Woods factor R_∞ is type III_1.

Armed with his new structure theory for factors, Connes embarked on a remarkable attack on the classification problem which after three years resulted in his monumental paper [9] of 1976 on the classification of injective factors. Combining a very deep and penetrating analysis of the group of *-automorphisms of II_1 factors with his structure theory for type III factors, Connes deduced several fundamental results: (a) all injective factors are hyperfinite (b) all injective factors of type II_1 are isomorphic (c) all injective factors of type II_∞ are isomorphic (d) for each $\lambda\in(0,1)$, all injective factors of type III_λ are isomorphic (to the Powers factor R_λ), and (e) all injective factors of type III_0 come from the group-measure space construction via a discrete cyclic group and are hence by results of W. Krieger [19] classified by ergodic, nontransitive flows on a standard measure space. Thus in a single virtuoso performance, Connes brought almost all of the injective factors to heel!

Almost all, but not all. The injective type III_1 case refused to yield to Connes' onslaught. The betting was that the only injective factor of type III_1 was the Araki-Woods factor R_∞, and for the next two years, Connes worked intensively trying to prove this. He discovered several sufficient conditions for the uniqueness of the injective III_1 factor but was not able to verify any of them. One, however, was singled out for special emphasis. Because this condition is central to our purposes in these notes, we will now describe it in detail.

Let M be a factor, φ a faithful, normal state of M. Connes defined the *bicentralizer* B_φ of φ to be the set of all elements x of M such that $\lim_n(x_n\ x - x x_n) = 0$, *-strongly, for each norm-bounded sequence (x_n) of M for which $\lim_n \|x_n\varphi - \varphi x_n\| = 0$. Notice that B_φ is contained in the relative commutant of the centralizer $\{x\in M : x\varphi = \varphi x\}$ of φ in M, whence the term bicentralizer. The relevance of this for our purposes comes from the following fact, due to Connes, and proved by him in [10]: if M is an injective III_1 factor that has a faithful, normal state with a trivial bicentralizer ($B_\varphi = \mathbf{C}\cdot 1$), then M is isomorphic to R_∞. Thus to prove uniqueness of the injective III_1 factor one needs to verify the hypothesis of this statement. To do this became known among the operator algebras faithful as the bicentralizer problem, and stood as a great challenge to all the experts. Finally, in 1987, U. Haagerup in a brilliant analytical *tour de force* ([16]) proved that every injective III_1 factor has a normal state with a trivial bicentralizer, and hence that all such factors are isomorphic. Thus the classification program for the hyperfinite factors was at last completed, and a detailed account of the final episode of this great story is the main goal of these notes.

We end this introduction by reminding the reader of a few basic facts about modular automorphisms and crossed products of von Neumann algebras that will be used extensively in Chapter 1 of Part I.

Let G be a locally compact group, M a von Neumann algebra acting on a Hilbert space H. A *continuous action of G on M* is a homomorphism α of G into the group $\mathrm{Aut}(M)$ of *-automorphisms of M such that for each $x \in M$, the mapping $g \to \alpha_g(x), g \in G$, is *-strongly continuous. Let λ denote Haar measure on G, and let $L^2(G,H)$ denote the Hilbert space of all H-valued, λ-square integrable functions on G. We define the representations π_α of M and λ of G on $L^2(G,H)$ as follows:

$$\begin{aligned}(\pi_\alpha(x)\xi)(h) &= \alpha_h^{-1}(x)\xi(h)\ ,\ x \in M, h \in G,\ \xi \in L^2(G,H),\\ (\lambda(g)\xi)(h) &= \xi(g^{-1}h)\ ,\ g,\ h \in G,\ \xi \in L^2(G,H).\end{aligned}$$

The *crossed product* $M \times_\alpha G$ *of* M *by* α is the von Neumann subalgebra of $B\big(L^2(G,H)\big)$ generated by $\{\pi_\alpha(x) : x \in M\}$ and $\{\lambda(g) : g \in G\}$. If $\theta \in \mathrm{Aut}(M)$ is a single automorphism, then $M \times_\theta \mathbf{Z}$ will denote the crossed product of M by the $\mathbf{Z}$-action on M defined by $n \to \theta^n, n \in \mathbf{Z}$.

A fundamental principle in the modern theory of von Neumann algebras is Takesaki's duality theorem for crossed products by abelian groups. We now suppose that α is a continuous action of an abelian group G on a von Neumann algebra M with $\hat{G}$ denoting the dual group of G. We define a unitary representation u of $\hat{G}$ on $L^2(G,H)$ by

$$u(p)\xi(g) = \overline{\langle g,p\rangle}\xi(g), g \in G, p \in \hat{G}, \xi \in L^2(G,H).$$

We clearly have

$$u(p)\pi_\alpha(x)u(-p) = \pi_\alpha(x) , \; x \in M , \; p \in \hat{G},$$

$$u(p)\, \lambda(g)u(-p) = \overline{\langle g, p\rangle}\lambda(g) , \; g \in G , \; p \in \hat{G},$$

and so we can define a continuous action $\hat{\alpha}$ of $\hat{G}$ on $M \times_\alpha G$ by

$$\hat{\alpha}_p(x) = u(p)xu(-p) , \; x \in M \times_\alpha G , \; p \in \hat{G}.$$

$\hat{\alpha}$ is called the *dual action of* $\hat{G}$ *on* $M \times_\alpha G$. Takesaki's duality theorem ([26], Theorems 4.5 and 4.6) now asserts that the double crossed product $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ is isomorphic to $M \otimes B(L^2(G))$ via an isomorphism which sends the double-dual action $\hat{\hat{\alpha}}$ onto the action of G on $M \otimes B(L^2(G))$ defined by $g \to \alpha_g \otimes \mathrm{Ad}\rho(g)$, $g \in G$, where $\rho(g)$ denotes the right regular representation of G on $L^2(G)$.

We also need to recall some information about weights and their modular automorphism groups. A (faithful, normal, semifinite) *weight* on a von Neumann algebra M is an additive, positively homogeneous mapping ω from the set M_+ of positive elements of M into $[0, +\infty]$ such that $\omega(x^*x) = 0$ only if $x = 0$, $\omega(\sup_i x_i) = \sup_i \omega(x_i)$ for every norm-bounded, increasing net (x_i) in M_+, and such that the linear span of the elements of M_+ on which ω is finite is σ-weakly dense in M. Each such weight ω gives rise to a faithful, normal GNS representation π_ω and a modular automorphism group σ^ω constructed in the usual way (for the details of these constructions, we refer the reader to Section I. 1.1 of [5]). The *centralizer* M^ω *of* ω is defined to be the fixed-point algebra of σ^ω, and we note that if ω is the restriction of a normal positive linear functional φ to M_+, then M^ω agrees with the centralizer $\{x \in M : x\varphi = \varphi x\}$ of φ. Given a weight ω on M and an action α of an abelian group G on M, we can associate with ω a canonical weight $\hat{\omega}$ on the crossed product $M \times_\alpha G$, called the *dual weight.* The construction of $\hat{\omega}$ is given in Section 5 of [26], and we note that the set of dual weights on $M \times_\alpha G$ consists of precisely those weights which are invariant for the dual action $\hat{\alpha}$. The action of the modular automorphism group of the dual weight $\hat{\omega}$ on $M \times_\alpha G$ is characterized by the relations

$$\sigma_r^{\hat{\omega}}(\pi_\alpha(x)) = \pi_\alpha(\sigma_r^\omega(x)) , \; x \in M , \; r \in \mathbf{R},$$

$$\sigma_r^{\hat{\omega}}(\lambda(g)) = \lambda(g) , \; g \in G , \; r \in \mathbf{R}.$$

If G is discrete, then there is a canonical normal, faithful, norm-1 projection E of $M \times_\alpha G$ onto $\pi_\alpha(M) \approx M$, and the dual weight $\hat{\omega}$ is in this case given by $\hat{\omega}(x) = (\omega \circ \pi_\alpha^{-1} \circ E)(x)$, $x \in M \times_\alpha G$.

An important relationship between actions of a locally compact group G on a von Neumann algebra M is given by the notion of exterior equivalence. Given such an action α, an *$\alpha-$ twisted, unitary $1-$ cocycle on M* is a strongly continuous mapping $g \to u_g$ of G into the unitary group of M such that $u_{gh} = u_g \alpha_g(u_h)$ for all $g, h \in G$. Two actions α and β of G are *exterior equivalent* if there is a β-twisted, unitary 1-cocycle $g \to u_g$ on M such that $\alpha_g(x) = u_g \beta_g(x) u_g^*$, $x \in M$, $g \in G$. Given two exterior equivalent actions α and β of G on M, the 1-cocycle implementing the equivalence induces in a natural way a (spatial) isomorphism between the crossed products $M \times_\alpha G$ and $M \times_\beta G$ ([26], Proposition 3.5). A remarkable theorem of A. Connes ([5], Theorémè 1.2.1) asserts that if φ and ψ are any two weights on M, then the modular automorphism groups σ^ψ and σ^φ are exterior equivalent, and hence the isomorphism type of $M \times_{\sigma^\omega} \mathbf{R}$ is independent of the weight ω. The unitary 1-cocycle u_t , $t \in \mathbf{R}$, which implements the equivalence between σ^ψ and σ^φ is called the *cocycle Radon-Nikodym derivative of ψ with respect to φ,* and is denoted by $[D\psi : D\varphi]_t$, $t \in \mathbf{R}$. For several useful facts about the cocycle Radon-Nikodym derivative, the reader should consult Section 1.2 of [5].

As in most situations involving von Neumann algebras, the predual of all normal linear functionals will play an important role in our considerations. If N is a von Neumann algebra, we will let N_* (respectively, N_*^+) denote the set of all normal (respectively, normal, positive) linear functionals on N. The dual space N^* of N becomes a Banach N-bimodule with the bimodule multiplication defined in the usual way by

$$(a\varphi)(x) \;=\; \varphi(xa),\; (\varphi a)(x) \;=\; \varphi(ax),\; x \in N,$$

for each $a \in N$, $\varphi \in N^*$. It is a standard fact that N_* is a Banach N-submodule of N^* with this bimodule structure. Given any N-bimodule M, we will denote the commutator $mx - xm$ (resp., $xm - mx$) for $x \in N$, $m \in M$ by $[m, x]$ (resp., $[x, m]$).

A very important representation theorem for the positive elements of N_*, due to H. Araki, will be used repeatedly in our work. Recall that each factor M can be represented on a Hilbert space H in such a way that there exists a self-dual cone $P^\natural$ in H and a conjugate-linear isometry J of order two on H for which $JMJ = M'$, $J\xi = \xi$, $\xi \in P^\natural$, and $xJx(P^\natural) \subseteq P^\natural$, $x \in M$ ([15], Theorem 1.6). Given M and H, J and $P^\natural$ are uniquely determined by these conditions ([15], Theorem 2.3). Araki showed that for each positive element φ of M_*, there is a unique vector $\xi_\varphi \in P^\natural$ such that $\varphi(x) = \langle x\xi_\varphi,\, \xi_\varphi \rangle$, $x \in M$ ([3], Theorem 2.5.31) (here and in all that follows, $\langle , \rangle$

will denote the inner product in an appropriate Hilbert space). We call ξ_φ the ***representing vector for*** φ (in $P^\natural$). Whenever φ is a positive element of M_*, ξ_φ will always denote this representing vector unless specifically indicated otherwise.

Part I

Connes' Reduction of the Uniqueness Proof to the Bicentralizer Problem

Chapter 1: Connes' Argument: outline and preliminary lemmas

Basic Lemma. Let M be a hyperfinite III_1 factor. Suppose $\lambda \in (0,\ 1), T_0 = -2\pi/log\lambda$. Let σ_{T_0} be a modular automorphism of M at T_0.

(a) [Lemma 1(a), p.190 of [10]] The crossed product $M \times_{\sigma_{T_0}} \mathbf{Z}$ is a factor of type III_λ, necessarily hyperfinite, and so isomorphic to the Powers factor R_λ.

(b) [Corollary 4, p.191 of [10]] Let (R_λ, θ) be the dynamical system dual to $(M,\ \sigma_{T_0})$ (Part (a) is used here to identify the dual of (M, σ_{T_0}) as $(R_\lambda,\ \theta)$). Suppose that

(α) $\theta \otimes id_{R_\lambda}$ is outer conjugate to θ;

(β) $\theta \otimes \beta$ is outer conjugate to θ where β is an infinite product action of the unit circle on the hyperfinite II_1 factor R. (Note that $R_\lambda \otimes R$ and $R_\lambda \otimes R_\lambda$ are both isomorphic to R_λ).

Then M is isomorphic to the Araki-Woods factor R_∞.

Proof. The proof of this is based on the following two lemmas. To state them, we recall that if Int (M) denotes the normal subgroup of inner automorphisms in the group Aut (M) of *-automorphisms of the von Neumann algebra M, then $\overline{\text{Int}}\ M$ denotes the closure of Int (M) in the topology on Aut (M) of point-norm convergence in the predual M_* of M with respect to the action $\alpha \to \varphi \circ \alpha^{-1}$, $\alpha \in$ Aut (M), $\varphi \in M_*$. In what follows, $\approx$ will mean "is isomorphic to" and $\sim$ will mean "is outer conjugate to".

Lemma 1 [Lemma 5, p.192 of [10]]. Let M be a hyperfinite III_1 factor such that

(a) $M \approx M \otimes R_\lambda$;

(b) $\sigma_{T_0} \in \overline{\text{Int}}\ M$ $(T_0 = -2\pi/log\lambda)$.

Then the dual system satisfies (α) and (β) of the Basic Lemma.

Lemma 2 [Proposition 3, p.191 of [10]]. Let M_i be hyperfinite III_1 factors, (R_λ, θ_i) dual to (M_i, σ_{T_0}), $i = 1,2$. Then the action of $S^1 = \hat{\mathbf{Z}}$ on $R_\lambda \otimes R_\lambda$ given by $\theta_1(t) \otimes \theta_2(-t)$, $t \in S^1$, is outer conjugate to $id_{R_\lambda} \otimes \beta$ where β is an infinite product action of S^1 on R.

We will take up the proofs of Lemmas 1, 2, and part (a) of the Basic Lemma momentarily. For now, let us see how these yield a proof of part (b) of the Basic Lemma.

Let $(R_\lambda,\ \theta_0)$ be dual to $(R_\infty,\ \sigma_{T_0})$. If $\theta_0 \sim \theta$, then $R_\lambda \times_{\theta_0} S^1 \approx R_\lambda \times_\theta S^1$. But then by Takesaki duality ([26], Theorem 4.5), we have

$$M \approx R_\lambda \times_\theta S^1 \approx R_\lambda \times_{\theta_0} S^1 \approx R_\infty,$$

and so we must prove that $\theta \sim \theta_0$.

By well-known facts, $R_\infty \otimes R_\lambda \approx R_\infty$ and $\overline{\text{Int}}\ R_\infty = \text{Aut}\ (R_\infty)$, so that by Lemma 1, (α) and (β) of the Basic Lemma hold for the dual system $(R_\lambda,\ \theta_0)$. Hence

$$\theta_0 \otimes (\text{id}_{R_\lambda} \otimes \beta) \sim (\theta_0 \otimes \text{id}_{R_\lambda}) \otimes \beta \sim \theta_0 \otimes \beta \sim \theta_0, \tag{1.1}$$

$$\theta \otimes (\text{id}_{R_\lambda} \otimes \beta) \sim (\theta \otimes \text{id}_{R_\lambda}) \otimes \beta \sim \theta \otimes \beta \sim \theta. \tag{1.2}$$

(As we shall see, the infinite-tensor product actions of R that occur in the statements of Lemmas 1, 2, and the Basic Lemma can all be taken to be a fixed canonical product action of S^1 on R). Now set $\tilde{\theta}(t) = \theta(-t)$, $\tilde{\theta}_0(t) = \theta_0(-t)$, $t \in S^1$. By Lemma 2,

$$\tilde{\theta} \otimes \theta_0 \sim \text{id}_{R_\lambda} \otimes \beta, \tag{1.3}$$

$$\theta \otimes \tilde{\theta} \sim \text{id}_{R_\lambda} \otimes \beta. \tag{1.4}$$

Thus from (1.1) - (1.4), we deduce that

$$\begin{aligned} \theta \sim \theta \otimes (\text{id}_{R_\lambda} \otimes \beta) \sim\ & \theta \otimes (\tilde{\theta} \otimes \theta_0) \\ \sim\ & (\theta \otimes \tilde{\theta}) \otimes \theta_0 \\ \sim\ & (\text{id}_{R_\lambda} \otimes \beta) \otimes \theta_0 \\ \sim\ & \theta_0. \end{aligned}$$

Q.E.D

Thus by Lemma 1 and the Basic Lemma, to prove that a hyperfinite III_1 factor M is isomorphic to R_∞, we must prove that for some $\lambda \in (0,\ 1)$, (a) and (b) of Lemma 1 always hold. Connes' proof of the implication (M a hyperfinite III_1 factor with trivial bicentralizer) $\Rightarrow M \approx R_\infty$ hence proceeds like so: in Chapter 2, he uses a new characterization of Araki's property L'_λ obtained in that chapter to prove that whenever M is a hyperfinite III_1 factor which satisfies (b), then (a) in fact holds. This reduces the proof of the above implication to proving (M a hyperfinite III_1 factor with trivial bicentralizer) $\Rightarrow$ (b) of Lemma 1 holds. Connes proves this is Chapter 4 by using a characterization of $\overline{\text{Int}M}$ for type III factors that he developes in Chapter 3.

Proof of part (a) of the Basic Lemma. The proof of this is based on the flow of weights of a properly infinite von Neumann algebra, which we will now describe. The basic reference for this is of course [11].

Let M be a properly infinite von Neumann algebra. Let ω be a dominant weight on M, i.e., a faithful, normal, semifinite weight on M with properly infinite centralizer, and with the property that ω is unitarily equivalent to $\lambda\omega$ for all $\lambda > 0$ (the existence of such a weight on M is guaranteed by Theorem II. 1.1 of [11]). Let $(N,\ \theta,\ \tau)$ be a continuous decomposition of M relative to ω, i. e., N is the centralizer M^ω of ω, θ is a continuous action of the real line $\mathbf{R}$ on N such that $\tau \circ \theta_r = e^{-r}\tau$, $r \in \mathbf{R}$ for the faithful, normal, semifinite trace τ on N, and M is isomorphic to the crossed product $N \times_\theta \mathbf{R}$. Connes and Takesaki [11] showed that the action of the multiplicative group $\mathbf{R}_+^*$ of positive real numbers on the center $Z(N)$ of N defined by $s \to \theta_{-\log s}|_{Z(N)}$, $s \in \mathbf{R}_+^*$, does not depend on the weight ω, and this action, denoted by F^M, is called the (smooth) *flow of weights on M*. Abstractly, the restriction to the projection lattice in $Z(N)$ of the flow of weights represents the action of multiplication by elements of $\mathbf{R}_+^*$ on the unitary equivalence classes of all integrable, faithful, normal, semifinite weights on M with properly infinite centralizer, whence the name.

We will need two facts about the flow of weights, one general, one particular. The general fact is given by Theorem II. 3.1 of [11]: the kernel of F^M is precisely the modular spectrum $S(M)$ of M. The particular fact computes the flow of weights for a factor of type III_λ, $0 < \lambda \leq 1$: if $\lambda = 1$, the flow of weights is trivial, and if $0 < \lambda < 1$, the flow of weights is canonically isomorphic to the action of $\mathbf{R}_+^*$ on $\mathbf{R}_+^*/(S(M) \cap \mathbf{R}_+^*)$ given by multiplication ([11], Section II.3).

We begin the proof of (a) of the Basic Lemma by recalling that for any factor M, the outer conjugacy class σ_{t_0} of the modular automorphism $\sigma_{t_0}^\varphi$ in Out (M) = Aut (M) / Int (M) for any $t_0 \in \mathbf{R}$ is independent of the faithful, normal, semifinite weight φ, and so therefore is the isomorphism class of $M \times_{\sigma^{\varphi}_{t_0}} \mathbf{Z}$.

Now, let M be a III_1 factor, $T_0 = -2\pi/log\lambda, 0 < \lambda < 1$. Since $S(M) = [0, +\infty)$, it follows from Théorème 3.4.1 of [5] that $\sigma^n_{T_0}$ is outer for each nonzero integer n, and so by [5], Proposition 4.1.1, $N = M \times_{\sigma_{T_0}} \mathbf{Z}$ is a factor.

Let ψ be a fixed dominant weight on M, and let φ (resp., ω) be the weight on N (resp., $P = M \times_{\sigma^{\psi}} \mathbf{R}$) dual to ψ ([26], Section 5). One easily checks that φ is dominant on N, and so it follows from Takesaki duality that the flow of weights on N is given by the action of $\hat{\mathbf{R}} = \mathbf{R}^*_+$ (duality here given by $(r, s) = s^{ir}, r \in \mathbf{R}, s \in \mathbf{R}^*_+$) via the dual action $\hat{\sigma}^{\varphi}$ on the center of $N \times_{\sigma^{\varphi}} \mathbf{R}$. It is straightforward to see that

$$N \times_{\sigma^{\varphi}} \mathbf{R} = (M \times_{\sigma^{\psi}_{T_0}} \mathbf{Z}) \times_{\sigma^{\varphi}} \mathbf{R} \approx (M \times_{\sigma^{\psi}} \mathbf{R}) \times_{\sigma^{\omega}_{T_0}} \mathbf{Z}. \tag{1.5}$$

By Theorem 7.1 of [26], with $\alpha = \sigma^{\omega}$, $(M \times_{\sigma^{\psi}} \mathbf{R}) \times_{\alpha_{T_0}} \mathbf{Z} = P \times_{\alpha_{T_0}} \mathbf{Z}$ can be viewed as the fixed-point algebra in $P \times_{\alpha} \mathbf{R}$ of the group G of automorphisms $\{\hat{\alpha}_s : s \in H^{\perp}\}$, where $H^{\perp}$ is the annihilator in $\mathbf{R}^*_+$ of the subgroup H of $\mathbf{R}$ generated by T_0. Since the dual action $\hat{\sigma}^{\varphi}$ on $N \times_{\sigma^{\varphi}} \mathbf{R}$ passes under the isomorphism (1.5) to the restriction of the dual action $\hat{\alpha}$ to $P \times_{\alpha_{T_0}} \mathbf{Z}$, the flow of weights of N is isomorphic to the action of $\mathbf{R}^*_+$ via $\hat{\alpha}$ on the center of $P \times_{\alpha_{T_0}} \mathbf{Z}$. Since M is type III_1, P is a factor ([26], Corollary 9.7), and so $\hat{\alpha}_s$ is trivial on $P \times_{\alpha_{T_0}} \mathbf{Z}$ if and only if it is trivial on its center. Since $P \times_{\alpha_{T_0}} \mathbf{Z}$ is the fixed-point algebra of G in $P \times_{\alpha} \mathbf{R}$, we conclude that the kernel of the flow of weights on N is $H^{\perp} = \{\lambda^n : n \in \mathbf{Z}\}$. By Theorem II. 3.1 of [11],

$$S(N) \cap \mathbf{R}^*_+ = \{\lambda^n : n \in \mathbf{Z}\},$$

and so N is type III_λ. Q.E.D.

Proof of Lemma 1. We will now define the infinite-tensor-product action of the circle on the hyperfinite II_1 factor R that appears in Lemmas 1, 2, and the Basic Lemma. Let M_2 denote the algebra of 2×2 matrices, and for each $t \in S^1$, set

$$\beta^0_t\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & e^{-it}b \\ e^{it}c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2.$$

Then β^0 defines an action of S^1 on M_2. If we realize R as the infinite tensor product of countably many copies of M_2 relative to the normalized trace on M_2, then $\beta = \beta^0 \otimes \beta^0 \otimes \beta^0 \otimes \cdots$ defines an infinite-tensor-product action of S^1 on R, the one that will arise in the arguments to come.

In order to verify that the conditions in Lemma 1 guarantee that (α) of the Basic Lemma is satisfied, we first recall the definition of the *modular homomorphism* $\delta : \mathbf{R} \to \mathrm{Out}(M)$ of a factor M (this will also be used in the proof of Lemma 2). We take a faithful, normal, semifinite weight ω on M and set $\delta(r) = \epsilon(\sigma_r^\omega)$, $r \in \mathbf{R}$, where $\epsilon : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ is the canonical quotient map. By Connes' cocycle Radon-Nikodym theorem([5], Théorème 1.2.1), this definition is independent of the weight ω chosen. The kernel of δ is by definition the Connes T-invariant $T(M)$ of M, and if M is of type $III_\lambda, 0 < \lambda < 1$, then $T(M)$ is the subgroup of $\mathbf{R}$ generated by $T_0 = -\frac{2\pi}{\log \lambda}$ ([5], Théorème 3.4.1).

Now, let M satisfy the hypotheses of Lemma 1. Since $T_0 \in T(R_\lambda) =$ kernel of δ, all modular automorphisms σ_{T_0} of R_λ at T_0 are inner, so that if $M \approx M \otimes R_\lambda$, then $\sigma_{T_0} \sim \sigma_{T_0} \otimes \mathrm{id}_{R_\lambda}$, and passing to the dual actions we obtain (α) of the Basic Lemma.

To verify (β) of the Basic Lemma, we note first that $R_\lambda \approx R_\lambda \otimes R$ since R_λ is ITPFI ([2]), whence

$$M \approx M \otimes R_\lambda \approx (M \otimes R_\lambda) \otimes R \approx M \otimes R.$$

M hence has property $L'_{\frac{1}{2}}$ of [1], and so by Theorem 3.1 of [1], there is a sequence (u_n) of partial isometries in M such that

$$u_n^2 = 0, \; u_n u_n^* + u_n^* u_n = 1, \forall n; \tag{1.6}$$

$$||\,[u_n, \psi]\,|| \to 0, \; \forall \psi \in M_*, \text{ i.e., } (u_n) \text{ is a } \textit{centralizing sequence} \text{ in } M. \tag{1.7}$$

Let φ be a fixed faithful, normal state on M, and consider (u_n) as a sequence in $N = M \times_{\sigma^\varphi_{T_0}} \mathbf{Z}$. We claim that

$$||\,[u_n, \, \psi]\,|| \to 0, \; \forall \, \psi \in N_*, \tag{1.8}$$

$$\theta_t\,(u_n) = u_n, \; n = 1, 2, 3, \ldots, \; t \in S^1. \tag{1.9}$$