Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

Subseries: Fondazione C.I.M.E., Firenze
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Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held at Varenna, Italy, June 15–23, 1989

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To the memory of Enrico Bompiani

Founder of the CIME, on the Centennial of his birth

PREFACE

To a large extent, the development of non-linear analysis has been based on convexity: many existence theorems, in areas ranging from Fixed Point Theory to the Calculus of Variations, depend on this crucial assumption. Moreover, Convex Analysis, with its emphasis on epigraphs and subgradients, has been able to extend many classical results substituting for assumptions of smoothness (differentiability) assumptions of convexity. In this sense convexity is also akin to non-smoothness.

There are, in different domains, successful even if scarce attempts to avoid the assumption of convexity. The tools employed may become the basis for what could be called Non-convex Analysis: not a well defined subject but, rather, an active domain of research. Purpose of this CIME Session was to bring together mathematicians who, with different aims in mind, try to avoid the limitations of convexity. We hope that the presentation of the basic tools and the discussion of the underlying fundamental ideas will stimulate further advances in this promising area.

Arrigo Cellina

Trieste, November 20, 1989

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The ε-variational principle revisited.

I.Ekeland

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The ε-variational principle appeared in 1972, and has been widely used in nonlinear analysis. We refer to the survey [E2] for the state of the art in 1980. On the invitation of A.Cellina, it has appeared worthwhile to update this survey by describing more recent results. Some of them have been very useful in the study of periodic solutions of Hamiltonian systems; we refer to [E2] for proofs and applications.

1. THE ε-VARIATIONAL PRINCIPLE.

THEOREM 1.1. (ε-principle). *Let* (X, d) *be a complete metric space and let* $F : X \to \mathbf{R} \cup \{+\infty\}$ *be a closed semi-continuous function bounded from below. Then, for every* $x_0 \in X$ *and* $\varepsilon > 0$ *such that*

$$F(x_0) \leq \inf_X F + \varepsilon$$

there exists an $x_\varepsilon \in X$ *such that*

$$F(x_\varepsilon) + \varepsilon d(x_\varepsilon, x) \leq F(x_0) \tag{1.1}$$

$$F(x) > F(x_\varepsilon) - \varepsilon d(x, x_\varepsilon), \qquad \forall x \neq x_\varepsilon . \tag{1.2}$$

REMARK 1.1. Formula (1.1) usually splits in

$$F(x_\varepsilon) \leq F(x_0) \tag{1.1)'$$

$$d(x_\varepsilon, x_0) \leq 1 . \tag{1.1)''$$

So x_ε improves x_0 from the point of view of minimization and it is located in a neighborhood of x_0 .

REMARK 1.2. Formula (1.2) says that the downward slope of F in x_ε is smaller than ε , that is

$$\sup_{x \neq x_\varepsilon} \max \left(\frac{F(x_\varepsilon)-F(x)}{d(x,x_\varepsilon)} , 0 \right) \leq \varepsilon .$$

Observe that if x_ε is a minimum of F , then its downward slope vanishes.

If X is a Banach space and F is differentiable, then (1.2) implies that

$$\|F'(x_\varepsilon)\|_{X^*} \leq \varepsilon ;$$

so if ε is small, the derivative in x_ε is almost flat.

REMARK 1.3. We can replace the distance d with any $d' = \lambda d$ $(\lambda > 0)$, since (X, d') is still a complete metric space. Taking $\lambda = \frac{1}{\sqrt{\varepsilon}}$ we obtain

$$d(x_\varepsilon, x_0) \leq \sqrt{\varepsilon}$$

$$\|F'(x_\varepsilon)\|_{X^*} \leq \sqrt{\varepsilon} .$$

Hence, if $(x_n)_n$ is a minimizing sequence (that is $\lim_{n\to+\infty} F(x_n) = \inf_X F$) , then there exists another sequence (y_n) such that

$$F(y_n) \leq F(x_n)$$

$$\lim_{n\to+\infty} d(x_n, y_n) = 0$$

$$\lim_{n\to+\infty} \|F'(y_n)\|_{X^*} = 0 .$$

The following examples show how the ε-principle can be used to prove some classical results relying on completeness (rather than compactness).

EXAMPLE 1.1. *Fixed point theorems (Banach).* Let (X, d) be a complete metric space, and let $f : X \to X$ be a contraction (i.e. $d(f(x), f(y)) \leq L\, d(x, y)$ with $L < 1$, $\forall x, y \in X$). Then f has a fixed point.
Define

$$F(x) = d(x, f(x)) ,$$

and apply Theorem 1.1 with $\varepsilon = 1 - L$ and any x_0 such that $F(x_0) \leq \inf_X F + \varepsilon$.
Let x_ε be such that (2) holds: then x_ε is the fixed point of f . If not, for $x = f(x_\varepsilon)$, (2) leads to

$$L\, d(x_\varepsilon, f(x_\varepsilon)) \geq d(f(x_\varepsilon) , f^2(x_\varepsilon)) > d(f(x_\varepsilon), x_\varepsilon) - \varepsilon d(f(x_\varepsilon), x_\varepsilon)$$

that is $1 - \varepsilon > 1 - \varepsilon$. ♦

EXAMPLE 1.2. *Inverse function Theorem.* Let X, Y be two Banach spaces, and $f : X \to Y$ a C^1 function such that $f'(x_0)$ is an isomorphism. Then f is a local homeomorphism between a neighborhood of x_0 and a neighborhood of $f(x_0)$. Usually the proof is divided in two steps, and the injectivity is the easiest to be obtained. Let us prove the open mapping theorem.

Let $\eta, \varepsilon > 0$ be such that

$$\forall x ,\ \|x - x_0\| < \eta \ , \text{ then}$$
$$\|f'(x)\xi\| \geq \varepsilon\|\xi\| \qquad \forall \xi \in Y .$$

Writing

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x) ,$$

we find that $r : B(x_0, \mu) \to Y$ is Lipschitz continuous with constant $\gamma < \frac{\varepsilon}{2}$ (if μ is small enough).

Let $y \in B(f(x_0) , \inf(\frac{\varepsilon^2}{4}, \mu))$, and define $F(x) = \|y - f(x)\|$. Then Theorem 1.1 says that there is an x_ε such that

(1.3)
$$\|x_\varepsilon - x_0\| \leq \mu$$
$$\|y - f(x)\| \geq \|y - f(x_\varepsilon)\| - \frac{\varepsilon}{2}\|x - x_\varepsilon\|, \qquad \forall x \neq x_\varepsilon .$$

When $y \neq f(x_\varepsilon)$, i.e. $x_\varepsilon \neq \bar{x} = [f^{-1}(x_0)]^{-1}\ (y - f(x_0) - f'(x_0)\ x_0 - r(x_\varepsilon)]$, then (1.3) leads to $\|y - f(\bar{x})\| = \|r(\bar{x}) - r(x_\varepsilon)\| > \|y - f(x_\varepsilon)\| - \varepsilon\ \|\bar{x} - x_\varepsilon\|$, i.e.

$$\|y - f(x_\varepsilon)\| \leq (\frac{\varepsilon}{2} + \gamma)\frac{1}{\varepsilon}\ \|y - f(x_\varepsilon)\| < \|y - f(x_\varepsilon)\|$$

a contradiction. Then $y = f(x_\varepsilon)$. ♦

EXAMPLE 1.3. *Cauchy Problems.* Consider the problem (in a Banach space E)

(CP)
$$\begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(0) = \xi \end{cases}$$

with $\|f(t, x) - f(t, y)\| \leq k\ \|x - y\|$. Then (CP) has a solution in [0, T], $T < \frac{1}{k}$.

Let $X = C^0([0, T], E)$ (a Banach space), and define $T : C^0 \to C^0$ by

$$(Tx)(t) = \xi + \int_0^t f(s, x(s))\, ds \quad .$$

Then T is Lipschitz continuous with constant $L = kT < 1$. Define $F(x) = \|Tx - x\|_{C^0}$, and apply Theorem 1.1 with $\varepsilon = 1 - kT$,to find an x_ε such that (2) holds. Reasoning as in the Example 1.1, one finds that x_ε is the solution of (CP).

EXAMPLE 1.4. *Dynamical systems.* Let (M, d) be a complete Riemannian manifold (possibly infinite dimensional). Given $f \in C^1(M, M)$, a (discrete) *dynamical system* is the semi-group $\{f^n\}_{n \in \mathbf{N}}$.

For example, the dynamical system naturally associated to the ODE

$$\begin{cases} \frac{dx}{dt} = g(x) \\ x(0) = \xi \end{cases}$$

is the group $\{x(n)\}_{n\in \mathbf{N}}$.
A point $x \in M$ is called *periodic* if there is $N \in \mathbf{N}$ such that $f^N(x) = x$. Denote by Per (f) the set $\{x \in M$, x periodic$\}$.
A point $x \in M$ is called *non wandering* if

$$\forall V \text{ neighborhood of } x \text{ , } \forall N \in \mathbf{N}, \exists n \geq N \text{ s.t } V \cap f^N(V) \neq \emptyset .$$

Let $\Omega(f)$ denote the set of all the non wandering points of M and note that

$$\Omega(f) \supseteq \overline{Per(f)} .$$

If the dynamical system arises from a Hamiltonian differential system

$$\begin{cases} \dot{p}_i = \frac{\partial H}{\partial q_i} \\ -\dot{q}_i = \frac{\partial H}{\partial p_i} \end{cases} ,$$

then Poincaré's Theorem says that, if for every h the set $\{(p, q) / H(p, q) \leq h\}$ is bounded, then every (p, q) is non wandering. Therefore it is very interesting to investigate the structure of the set $\Omega(f)$. More precisely, a natural question is whether $\Omega(f) = \overline{Per(f)}$ is satisfied. The following result can be proved using the ε-principle.

THEOREM 1.2. *Assume* $\bar{x} \in \Omega(f)$. *Then,* $\forall \varepsilon > 0$, $\forall N \in \mathbf{N}$, *there exist* $x_\varepsilon \in M$ *and* $n \geq N$ *such that*

$$d(\bar{x}, x_\varepsilon) \leq \varepsilon(1 + \frac{\varepsilon}{2})$$

$$d(f^n(x_\varepsilon), x_\varepsilon) \leq \varepsilon^2$$

(1.4) $$\|(I - Tf^n(x_\varepsilon))^* y_\varepsilon\| \leq \varepsilon \|y_\varepsilon\| \quad \textit{where } y_\varepsilon = f^n(x_\varepsilon) - x_\varepsilon .$$ ♦

Theorem 1.2 states the existence of one "almost" eigenvector of $(Tf^n(x_\varepsilon))^*$ associated to the eigenvalue $\lambda = 1$. This result is useful in the study of a special class of dynamical systems for which (1.4) never holds unless x_ε is periodic.

DEFINITION 1.1 A dynamical system associated to $f \in C^1(M, M)$ is *hyperbolic* if for every $x \in M$, T_xM splits into the sum of a stable subspace E_x^s and an unstable subspace E_x^n $T_xM = E_x^s \oplus E_x^n$ (depending continuously on x) such that

$$\forall \xi \in E_x^s \quad \|Tf^n(x)\xi\| \leq \lambda^{-n} \|\xi\|$$

$$\forall \xi \in E_x^n \quad \|Tf^n(x)\xi\| \geq \lambda^{n} \|\xi\|$$

for some $\lambda > 0$.

THEOREM 1.3. *If* f *is hyperbolic then* $\Omega(f) = \overline{Per(f)}$.

Let us observe that, if f is not hyperbolic, Theorem 1.3 is in general false. Indeed, consider the Hamiltonian system defined by $H(p_1, p_2, q_1, q_2) = \frac{\alpha_1}{2}(p_1^2 + q_1^2) + \frac{\alpha_2}{2}(p_2^2 + q_2^2)$ (it is the harmonic oscillator with periods $T_i = \frac{2\pi}{\alpha_i}$). Then, every solution is either periodic or quasi-periodic (hence non wandering), so that $\Omega(f) = \mathbf{R}^4$; on the other hand, the only periodic solutions are of the form $(p_1, q_1) = 0$ or $(p_2, q_2) = 0$. This example provides a case when $\Omega(f) \underset{\neq}{\supset} \overline{Per(f)}$.

2. SMOOTHNESS AND ε-PRINCIPLE.

A smooth ε-principle.

Let X be a Banach space. The main property of x_ε in Theorem 1.1 is that

$$F(x) \geq F(x_\varepsilon) - \varepsilon \|x - x_\varepsilon\|, \qquad \forall x \in X .$$

The graph of the mapping appearing at the right hand side of the above inequality is a cone, which is a non smooth object. Our goal is to replace it with the graph of a smooth function. (This problem has been solved by Borwein and Preiss [B-P]). Let us recall some basic definitions:

DEFINITION 2.1 A function $F : X \to \mathbf{R}$ is *Gateaux differentiable* (G-differentiable) in $x_0 \in X$ if there exists a continuous functional $F'(x_0) \in X^*$ such that

$$\lim_{h \to 0} \frac{F(x_0 + hy) - F(x_0)}{h} = \langle F'(x_0), y \rangle, \qquad \forall y \in X .$$

DEFINITION 2.2 $F : X \to \mathbf{R}$ is *Fréchet differentiable* (F-differentiable) in $x_0 \in X$ if there exist $F'(x_0) \in X$ and $\varepsilon : X \to \mathbf{R}$ with $\lim_{y \to 0} \varepsilon(y) = 0$ such that

$$F(x_0 + y) = F(x) + \langle F'(x_0), y \rangle + \varepsilon(y) \|y\|$$

Note that the G-differentiability is weaker than the F-differentiability.

We define the family

$$\mathcal{F} = \left\{ \phi : X \to \mathbf{R} \,,\, \phi = \frac{1}{2}\sum_{n=1}^{\infty} \varphi_n \, \|x - x_n\| \,,\, \varphi_n \leq 0 \,,\, \sum_{n=1}^{\infty} \varphi_n = 1 \quad , \quad \lim_{n \to +\infty} x_n = \bar{x} \right\} .$$

THEOREM 2.1. *Assume that for* $x \neq 0$ *the norm* $\|\cdot\|$ *is G-differentiable (resp. F-differentiable) with derivative* $J(x)$. *Then each* $\phi \in \mathcal{F}$ *is G-differentiable (resp. F-differentiable) and the derivative is*

$$\phi'(x) = \sum_{n=1}^{\infty} \varphi_n \, \|x - x_n\| \, J(x - x_n) \,. \qquad \blacklozenge$$

THEOREM 2.2. (Borwein-Preiss). *Let* $F : X \to \mathbf{R} \cup \{+\infty\}$ *be a lower semicontinuous function such that* $\inf_X F > -\infty$. *Let* $\varepsilon > 0$ *and* $x_0 \in X$ *be such that*

$$F(x_0) < \inf_X F + \varepsilon \,.$$

Then there exist $x_\varepsilon \in X$ *and* $\phi \in \mathcal{F}$ *such that*

(2.1) $$F(x_\varepsilon) < \inf_X F + \varepsilon$$

(2.2) $$\|x_\varepsilon - x_0\| < 1$$

(2.3) $$F(x) \geq F(x_\varepsilon) + 2\varepsilon\, [\phi(x_\varepsilon) - \phi(x)], \qquad \forall x \in X \,.$$

$\blacklozenge$

REMARK 2.1. If F is differentiable then

$$F'(x_\varepsilon) = -\, 2\varepsilon\, \phi'(x_\varepsilon) \,.$$

Moreover, the sequence $(x_n)_n$ can be chosen so that $\|x_n - x_\varepsilon\| \leq 1$ $(\forall n)$; hence, since $\|J(x)\|_{X^*} = 1$, we have

$$\|F'(x_\varepsilon)\|_{X^*} \leq 2\varepsilon \,. \qquad \blacklozenge$$

REMARK 2.3. A reflexive space can always be renormed so that the new norm is differentiable. In this case, Theorem 2.1 holds with some $\phi \in \mathcal{F}_0$, where $\mathcal{F}_0$ contains all the elements of $\mathcal{F}$ having a single non zero term:

$$\mathcal{F}_0 = \{\phi \in \mathcal{F} \,/\, \phi(x) = \frac{1}{2}\|x - \bar{x}\|^2\} .$$

A Hilbert space is reflexive and the standard norm is differentiable (twice for $x \neq 0$). Therefore (2.3) becomes

$$F(x) \geq F(x_\varepsilon) + \varepsilon\,[\|x_\varepsilon - \bar{x}\|^2 - \|x - \bar{x}\|^2] ,$$

and moreover we obtain

$$F''(x_\varepsilon) \geq -2\varepsilon I .$$

As a consequence of the above remarks we have the following

COROLLARY 2.1. *Let* X *be a Hilbert space, let* $F : X \to \mathbf{R}$ *be twice differentiable and bounded from below. Then there exists a sequence* $(x_n)_n$ *such that*

$$\lim_{n\to+\infty} F(x_n) = \inf_X F$$

$$\lim_{n\to+\infty} \|F'(x_n)\|_{X^*} = 0$$

$$\lim_{n\to+\infty} F''(x_n) \geq 0 . \qquad \blacklozenge$$

REMARK 2.4. The main idea in proving the existence of ϕ in Theorem 2.2 consists in the following iterative construction:

step 0
$$\begin{cases} F_0 = F \\ x_0 \text{ prescribed} \end{cases}$$

step n + 1
$$\begin{cases} F_{n+1} = F_n(x) + \delta\mu^n \|x - x_n\|^2 \\ x_{n+1} \text{ is chosen such that} \\ F_{n+1}(x_{n+1}) \leq \nu\, F_n(x_n) + (1-\nu) \inf F_{n+1} \end{cases}$$

where μ, ν, δ are chosen as follows:

$$F(x_0) - \inf F < \varepsilon_2 < \varepsilon_1 < \varepsilon$$

$$0 < \mu < 1 - \frac{\varepsilon_1}{\varepsilon}$$

$$0 < \frac{\nu}{\mu} < [1 - (\frac{\varepsilon_2}{\varepsilon_1})^{1/2}]^2$$

$$\delta = (1 - \mu)\, \varepsilon \, .$$

Observe that $F_{n+1}(x) = F(x) + \sum_{K=0}^{h} \delta\mu^k \, \|x - x_k\|^2$. The sequence $(x_n)_n$ can be chosen so that $x_n \to \overline{x}$, and it is shown that $\phi = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \mu^n (1 - \mu) \, \|x - x_n\|^2$ satisfies (2.3). ♦

ε-principle and differentiability.

Let $F : X \to \mathbf{R}$ be convex and semicontinuous; a basic result of convex analysis is the following reciprocity formula:

$$F(x) = \sup_{x \in X^*} \{<x, x^*> - F^*(x^*)\} \tag{2.4}$$

where $F^* : X^* \to \mathbf{R}$ is defined as

$$F^*(x^*) = \sup_{x \in X} \{<x; x^*> - F(x)\} \, .$$

There is a connection between the differentiability of F and the existence and uniqueness of the maximum for (2.4). (For example, it is easy to see that if F is differentiable in $\overline{x}$ with derivative $F'(\overline{x})$ and the maximum of the right hand side is achieved in $\overline{x}^*$, then $\overline{x}^* = F'(\overline{x})$).

The problem of differentiability on a Banach space X can be handled with the ε-principle. Below, we give some results on this subject.

THEOREM 2.3.(Ekeland-Lebourg). *If X admits an F-differentiable norm, then every continuous convex $F : X \to \mathbf{R}$ is F-differentiable on a dense G_δ set.* ♦

THEOREM 2.4. (Borwein-Preiss). *If X admits a G-differentiable norm, then every continuous convex $F : X \to \mathbf{R}$ is G-differentiable on a dense set.* ♦

THEOREM 2.5. (Preiss). *If X admits an F-differentiable norm, then every locally Lipschitz continuous function $F : X \to \mathbf{R}$ is F-differentiable on a dense set.* ♦

(Note that the norm is never differentiable at the origin, so the differentiability is always assumed for $x \neq 0$).

Linear perturbations.

Let X be a Banach space, and $C \subseteq X$ be bounded (possibly not closed). Let $F : X \to \mathbf{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below. Given $x^* \in X^*$, we consider the minimization problem

$$(P_{x^*}) \qquad \inf_{x \in C} \{F(x) + \langle x^*, x\rangle\} .$$

Note that C is not assumed to be either closed or convex, so that the minimum cannot be achieved for each $x^* \in X^*$.

THEOREM 2.6. (Ekeland-Lebourg). *Let C be closed and let X, X* be uniformly convex. Then,* $\forall \varepsilon > 0$, $\exists x^* \in X^*$ $\|x^*\| < \varepsilon$ *such that* (P_{x^*}) *has a solution. Moreover every minimizing sequence converges.* ♦

THEOREM 2.7. (Stegall). *Let C be closed and let X have the Radon-Nikodym property. Then the same conclusion of Theorem 2.6 holds.* ♦

These problems have been studied systematically by Ghoussoub and Maurey, who provide an extension of the ε-principle to the case when two topologies are present on the underlying space X .

3. MINIMIZATION OF FUNCTIONALS AND CRITICAL POINTS THEORY.

Let X be a Banach space, and $F : X \to \mathbf{R} \cup \{+\infty\}$ be bounded from below. In the classical approach to the problem of minimizing F over X, one assumes X to be reflexive, and F to be coercive and convex (so that F is weakly lower semicontinuous). The coercivity implies that every minimizing sequence is bounded in X , the reflexivity of X implies that any bounded sequence has a weakly convergent subsequence and, from the convexity of F, one concludes that the weak limit is actually a minimum of F . So, assuming F to be convex, one assumes that every minimizing sequence possesses a weakly converging subsequence.

On the other hand, the ε-principle states the existence of a special minimizing sequence having the further property that

$$(3.1) \qquad F(x) \geq F(x_n) - \varepsilon_n \|x - x_n\|, \qquad \forall x \in X , \varepsilon_n \to 0 .$$

A much weaker assumption than the convexity, but still sufficient to have a minimum, is that every minimizing sequence satisfying (3.1) has a convergent subsequence. It might happen that not every minimizing sequence converges, but there exists a special minimizing sequence that does.

This idea has been applied by Marcellini and Sbordone, [M-S], in order to minimize functionals of the type

$$F(u) = \int_\Omega f(x, u, Du)$$

$$u : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}^n$$

with f non convex in Du .
When F is C^1 , then (3.1) leads to $\|F'(x_n)\| \le \varepsilon_n$. Therefore a good compactness condition is the Palais-Smale condition

$$(P\text{-}S)_c \quad \begin{cases} \text{Every sequence } (x_n)_n \text{ in } X \text{ such that} \\ \lim\limits_{n\to+\infty} F(x_n) = c \\ \lim\limits_{n\to+\infty} F'(x_n) = 0 \\ \text{possesses a converging subsequence.} \end{cases}$$

The Palais-Smale condition has been introduced for the search of critical points of unbounded functionals. The Mountain Pass Theorem and its generalizations state the existence of a sequence satisfying $(P\text{-}S.)_c$ (a Palais-Smale sequence) under some geometrical assumptions, which replace the boundedness of the functional. The first version (Theorem 3.1) has been proved by Ambrosetti and Rabinowitz in [A-R]. We also give a generalized Mountain Pass Theorem due to Ghoussoub and Priess [G-B], which can be useful when dealing with some limiting cases.

THEOREM 3.1. (Ambrosetti-Rabinowitz) *Let* X *be a Banach space, and let* $F \in C^1(X ; \mathbf{R})$. *Assume that there are:* $\rho > 0$ *and* $\alpha > F(0)$ *such that*

$$\inf_{\|x\|=\rho} F \ge \alpha \tag{3.2}$$

$$\exists \bar{x} \in X ,\ \|\bar{x}\| > \rho \text{ such that } F(\bar{x}) < \alpha . \tag{3.3}$$

Then there exists a sequence $(x_n)_n$ *such that*

$$\lim_{n\to+\infty} F(x_n) = c \ge \alpha$$

and

$$\lim_{n\to+\infty} F'(x_n) = 0 .$$

♦

COROLLARY 3.1. *In addition to the assumptions of Theorem 3.1, assume that the Palais-Smale condition* $(P.S.)_c$ *is satisfied. Then there exists* $\bar{x} \in X$ *such that*

$$F(\bar{x}) = c$$

and

$$F'(\bar{x}) = 0 .$$

♦

PROOF OF THEOREM 3.1.(Sketch) Consider the metric space

(3.4) $$C = \{\gamma \in C^0([0,1]; X) \ / \ \gamma(0) = 0 \ , \ \gamma(1) = \bar{x}\}$$

endowed with the uniform topology, and define $\psi : C \to \mathbf{R}$ as

$$\psi(\gamma) = \max_{t \in [0,1]} F(\gamma(t)) .$$

Since the function $\tilde{\psi} : C \times [0, 1] \to \mathbf{R}$, $\tilde{\psi}(\gamma, t) = F(\gamma(t))$ is continuous, then ψ is lower semicontinuous. The assumption (3.2) gives

$$c = \inf_C \psi \geq \alpha ,$$

so ψ is bounded from below.

By applying Theorem 1.1, for every $\varepsilon > 0$ we obtain the existence of $\gamma_\varepsilon \in C$ such that

(3.5) $$\begin{aligned} &\psi(\gamma_\varepsilon) \leq c + \varepsilon^2 \\ &\psi(\gamma) \geq \psi(\gamma_\varepsilon) - \varepsilon \|\gamma - \gamma_\varepsilon\|_{C^0}, \qquad \forall \gamma \in C . \end{aligned}$$

Define

$$I_\varepsilon = \{t \in [0, 1] \ / \ F(\gamma_\varepsilon(t)) = \max_{s \in [0,1]} F(\gamma_\varepsilon(s)) .$$

From (3.5) one can deduce that there exists at least one $t_\varepsilon \in I_\varepsilon$ such that $\|F'(\gamma_\varepsilon(t_\varepsilon))\| \leq \varepsilon$. Finally, it is sufficient to take $x_\varepsilon = \gamma_\varepsilon(t_\varepsilon)$ and (3.2), (3.3) are satisfied. ♦

REMARK 3.1. The critical value c is expressed by

(3.6) $$c = \inf_{\gamma \in C} \max_{t \in [0,1]} F(\gamma(t)) .$$ ♦

REMARK 3.2. It has been proved that there exists $\gamma_\varepsilon \in C$ $(\forall \varepsilon > 0)$ such that every $t_\varepsilon \in I_\varepsilon$ has

$$\|F'(\gamma_\varepsilon(t_\varepsilon))\| \leq \varepsilon .$$ ♦

REMARK 3.3. The topological meaning of (3.2), (3.3) is that the set $\{x \ / \ F(x) < F(0)\}$ is not path connected and the sphere $S_\rho = \{\|x\| = \rho\}$ "separates" 0 and $\bar{x}$. Indeed every path $\gamma \in C$ has to intersect S_ρ and hence the set $\{x \ / \ F(x) \geq F(0)\}$. ♦