Lecture Notes in Mathematics

Roger Mansuy
Marc Yor

Random Times and Enlargements of Filtrations in a Brownian Setting

1873

$$\mathcal{F}_t^{\sigma(X)} = \mathcal{F}_t \vee \sigma(X)$$
$$\mathcal{F}_t^{\Lambda} = \mathcal{F}_t \vee \sigma(\Lambda \wedge t)$$

Springer

Roger Mansuy · Marc Yor

Random Times and Enlargements of Filtrations in a Brownian Setting

Springer

Authors

Roger Mansuy
Marc Yor
Université Paris VI
Pierre et Marie Curie
Laboratoire de Probabilités et Modéles Aléatoires
4 Place Jussieu - Casier 188
F-75252 Paris Cedex 05
e-mail: mansuy@ccr.jussieu.fr

Library of Congress Control Number: 2005934037

Mathematics Subject Classification (2000): 60-02, 60G40, 60G44, 60J65

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN-10 3-540-29407-4 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-29407-8 Springer Berlin Heidelberg New York

DOI 10.1007/11415558

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the authors and Techbooks using a Springer LaTeX package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11415558 41/TechBooks 5 4 3 2 1 0

Lecture Notes in Mathematics

Edited by J.-M. Morel, F. Takens and B. Teissier

Editorial Policy for Multi-Author Publications: Summer Schools / Intensive Courses

1. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome. Manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They should provide sufficient motivation, examples and applications. There should also be an introduction making the text comprehensible to a wider audience. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this „lecture notes" character.

2. In general SUMMER SCHOOLS and other similar INTENSIVE COURSES are held to present mathematical topics that are close to the frontiers of recent research to an audience at the beginning or intermediate graduate level, who may want to continue with this area of work, for a thesis or later. This makes demands on the didactic aspects of the presentation. Because the subjects of such schools are advanced, there often exists no textbook, and so ideally, the publication resulting from such a school could be a first approximation to such a textbook.

   Usually several authors are involved in the writing, so it is not always simple to obtain a unified approach to the presentation.

   For prospective publication in LNM, the resulting manuscript should not be just a collection of course notes, each of which has been developed by an individual author with little or no co-ordination with the others, and with little or no common concept. The subject matter should dictate the structure of the book, and the authorship of each part or chapter should take secondary importance. Of course the choice of authors is crucial to the quality of the material at the school and in the book, and the intention here is not to belittle their impact, but simply to say that the book should be planned to be written by these authors jointly, and not just assembled as a result of what these authors happen to submit.

   This represents considerable preparatory work (as it is imperative to ensure that the authors know these criteria before they invest work on a manuscript), and also considerable editing work afterwards, to get the book into final shape. Still it is the form that holds the most promise of a successful book that will be used by its intended audience, rather than yet another volume of proceedings for the library shelf.

3. Manuscripts should be submitted (preferably in duplicate) either to Springer's mathematics editorial in Heidelberg, or to one of the series editors (with a copy to Springer). Volume editors are expected to arrange for the refereeing, to the usual scientific standards, of the individual contributions. If the resulting reports can be forwarded to us (series editors or Springer) this is very helpful. If no reports are forwarded or if other questions remain unclear in respect of homogeneity etc, the series editors may wish to consult external referees for an overall evaluation of the volume. A final decision to publish can be made only on the basis of the complete manuscript; however a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter.

   Volume editors and authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer evaluation times. They should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

Continued on inside back-cover

Lecture Notes in Mathematics 1873

Dedicated to the memory of J.L. Doob (1910 – 2004)
and P.-A. Meyer (1934 – 2003)

In the neighborhood of the second author's family home...

Preface

These notes represent accurately the contents of the six lectures we gave in the Statistics Department of Columbia University, between the 10th and the 25th of November 2004.

The audience was a mix of faculty, most of whom were fine "connoisseurs" of stochastic calculus, excursion theory, and so on, and graduate students who were basically acquainted with Brownian motion.

Our aim in teaching this course was two-fold:

- on one hand, to give the audience some familiarity with the theory and main examples of enlargements of filtrations, either of the initial or the progressive kinds;
- on the other hand, to update the relevant Chapters[1] of Part II [Yor97b] of the Zürich volumes, precisely, those which were devoted to martingale and filtration problems, i.e. **Chapters** 12 to 17 in Part II.

Each lecture was followed by an exercises session.
Here is the detailed organization of these lecture notes:

- as a set of **Preliminaries**, the basic operations of stochastic calculus and of the (Strasbourg) general theory of processes are recalled; no doubt that this is too sketchy, only a first aid tool kit is being presented, and the reader will want to read much more, e.g. [Del72] and the last volume, by Dellacherie, Maisonneuve and Meyer, of Probabilités et Potentiel [DMM92];
- in **Chapter** 1, the transformation of martingales in a "small" filtration into semimartingales in a bigger filtration is being studied; an important number of, by now, classical examples, drawn more or less from Jeulin's monograph [Jeu80] or Jeulin-Yor [JY85], are presented, and then collected in an appendix at the end of the chapter: this appendix consists in two

[1] An updated, revisited version of **Chapters** 1 to 11, corresponding to Part I [Yor92a], is being published in the Springer Universitext collection under the title *Aspects of Brownian motion* [MY05a].

tables, the first one for progressive enlargements, the second one for initial enlargements; we tried to gather there some most important examples, which often come up in the discussion of various Brownian path decompositions and their applications. This presentation is close to the effort made in the *Récapitulatif* in [JY85] pp. 305-313;

- in **Chapter** 2, we examine what remains of a number of classical results in martingale theory when, instead of dealing with a stopping time, one works up to a general random time;
- the main topic of **Chapter** 3 consists in the comparison of $\mathbb{E}[X|\mathcal{F}_\gamma]$ and $X_\gamma := \mathbb{E}[X|\mathcal{F}_t]_{|t=\gamma}$ where, for the simplicity of our exposition, γ is the last zero before 1 of an underlying Brownian motion, and X is a generic integrable random variable. Note how easily one may be confusing the two quantities, which indeed are identical when γ is replaced by a stopping time. Moreover, in our set-up with γ, one of these quantities is equal to 0 if and only if the other one is, and this remark leads naturally to the description of all martingales which vanish on the (random) set of the Brownian zeroes;
- **Chapter** 4 discusses the predictable and chaotic representation properties (abbreviated respectively as PRP and CRP) for a given martingale with respect to a filtration. Although the CRP is rarer than the PRP, a much better understanding of the CRP, and many examples, have been obtained since the unexpected discovery by Émery [Éme89] that Azéma's martingale enjoys the CRP. In particular, we introduce in this chapter the Dunkl martingales, which also enjoy the CRP.
- the two next **Chapters** 5 and 6 are devoted to questions of filtrations. They are tightly knit with the preceding chapters, e.g. in **Chapter** 5, Azéma's martingale plays a central role, and in **Chapter** 6, ends of predictable sets are being discussed in the framework of the Brownian filtration. In more details, the deep roots of **Chapter** 5 are to be found in excursion theory where, traditionally, a level, e.g. level 0, is being singled out from the start, and excursions away from this level are studied. It was then natural to consider how quantities and concepts related to a given level based excursion theory vary with that level. Two different suggestions for this kind of study were made, the first one by D. Williams, with following studies by J. Walsh and C. Rogers, the second one by J. Azéma, which provoked answers from Y. Hu. Both set-ups are being examined in **Chapter** 5.
  Chapter 6 develops our present understanding of the Brownian filtration, or rather, of some fundamental properties which are necessary for a given filtration to be generated by a Brownian motion. The results are due mainly to B. Tsirel'son, and collaborators, between 1996 and 2000 (roughly). In particular, it was established during this period that:
  - the filtration of a N-legged Brownian spider ($N \geq 3$) is not strongly Brownian.

   - there exist probability measures Q equivalent to Wiener measure, such that under Q, the natural filtration of the coordinate process is not strongly Brownian.

  Tsirel'son original "hands on" method of attack of these questions later developed into the search of "invariants of filtration", e.g. the notions of standard filtration, cosy filtration,..., which were studied by M. Emery and co-workers, and Tsirel'son himself, and which we briefly present at the end of **Chapter** 6.
- Each chapter ends with some exercises, which complement the content of that chapter. A standard feature of these exercises, as well as the style of their solutions, is an illustration of general "principles", which we present in the framework of explicit examples. The solutions-presented in **Chapter** 7 – are succinctly written, but should contain sufficient details for the reader. As much as possible, the arguments in the proposed solutions are closely connected with the material found in the corresponding chapters. We also took the opportunity to include some open questions, sometimes in the form of exercises, which are then indicated with the symbol ✠.

We are both very grateful for the warm hospitality we received during our stay in Columbia University as well as the strong motivation of the audience during the sessions. Thanks to everyone involved, and special thanks to Peter Bank and Ioannis Karatzas.

Paris,
November 3, 2005

Roger Mansuy
Marc Yor

Contents

-1

Notation and Convention

Here is a short list of current notation and convention used in the different chapters.

- We shall always assume that the underlying probability space $(\Omega, \mathcal{F}, P)$ is separable.
- Let $\mathcal{A}(\subset \mathcal{F})$ be a σ-field, X an $\mathcal{A}$-measurable random variable and Y a random variable independent of $\mathcal{A}$. Then, for any Borel function f, the conditional expectation $\mathbb{E}\left[f(X,Y)|\mathcal{A}\right]$ shall be denoted as $\hat{\mathbb{E}}\big[f(X,\hat{Y})\big]$. In other words, the expectation concerns the hat-variables with all others remaining *frozen*.
- We sometimes make the abuse of notation:
  $X \in \mathcal{A}$, meaning that the random variable X is $\mathcal{A}$-measurable.
- The symbol γ_t (resp. δ_t) will only be used to denote the last (resp. the first) zero of a certain process, usually Brownian motion, before (resp. after) the time t. We often abbreviate γ_1 and δ_1 by γ and δ.
- In general, to a process N, we associate $\overline{N}$, its one sided supremum process; namely, for $t \geq 0$, $\overline{N}_t := \sup_{s\leq t} N_s$. However, for Brownian motion $(B_t;\ t \geq 0)$, we keep the usual notation $(S_t;\ t \geq 0)$ for its one-sided supremum.
- In this book, studies of the law of a process $(X_t;\ t \geq 0)$ often begin with: "For any bounded functional F, $\mathbb{E}\left[F(X_s;\ s \leq t)\right]\ldots$". By this sentence, we mean that F is a measurable functional on $\mathcal{C}([0,t],\mathbb{R})$ if X is assumed to be continuous, on $\mathcal{D}([0,t],\mathbb{R})$ otherwise.
- $\mathcal{F}^{\sigma(X)}$ will denote the initial enlargement of the filtration $(\mathcal{F}_t;\ t \geq 0)$ with the random variable X, that is the filtration defined by

$$\boxed{\mathcal{F}_t^{\sigma(X)} := \bigcap_{\varepsilon>0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma(X)\right), \qquad t \geq 0}$$

  We shall sometimes use the terminology: X-initial enlargement of $(\mathcal{F}_t; t \geq 0)$.
- For $\Lambda : \Omega \to [0,\infty]$, a random time, we denote by $\mathcal{F}^{\Lambda}$ the smallest filtration which contains $(\mathcal{F}_t;\ t \geq 0)$, and makes Λ a stopping time, i.e.

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 1–2 (2006)
www.springerlink.com

$$\boxed{\mathcal{F}_t^{\Lambda} := \bigcap_{\varepsilon>0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma(\Lambda \wedge (t+\varepsilon))\right), \qquad t \geq 0}$$

We shall sometimes use the terminology: Λ-progressive enlargement of $(\mathcal{F}_t;\ t \geq 0)$.

- All martingales considered in this volume are assumed to be càdlàg (i.e. right-continuous and left-limited); in a number of cases, they are even assumed to be continuous, but this will always be specified.
- **e** (resp. $\mathcal{N}$) will often denote a standard exponentially distributed variable (resp. a standard normal variable).
- The symbol $\hookrightarrow$ (resp. $\not\hookrightarrow$) denotes immersion (resp. non-immersion) between two filtrations $(\mathcal{F}_t;\ t \geq 0)$ and $(\mathcal{G}_t;\ t \geq 0)$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for every t; $(\mathcal{F}_t;\ t \geq 0)$ is said to be immersed in $(\mathcal{G}_t;\ t \geq 0)$ if all $(\mathcal{F}_t;\ t \geq 0)$-martingales are $(\mathcal{G}_t;\ t \geq 0)$-martingales. This notion will be studied in Chapter 5, but we already note that the more general situation when some (perhaps all...) $(\mathcal{F}_t;\ t \geq 0)$-martingales are $(\mathcal{G}_t;\ t \geq 0)$-semimartingales will be a recurrent subject of study in these lecture notes.

0

Preliminaries

Throughout these preliminaries, we are working with an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$. We insist that most of the notions introduced below are relative to this filtered probability space.

0.1 Doob's Maximal Identity

The following lemmas are variants of Doob's optional stopping theorem.

Lemma 0.1 *Let N be a $\mathbb{R}_+$-valued continuous local martingale with $N_0 = 1$, and $N_t \underset{t\to\infty}{\longrightarrow} 0$.*
Denote $\overline{N}_t = \sup_{s \leq t} N_s$ and $\overline{N}^t = \sup_{s \geq t} N_s$.
Then $\overline{N}_\infty \overset{(law)}{=} 1/U$ where U denotes a uniformly distributed variable.
More generally, for every finite stopping time T such that $N_T > 0$ a.s., $N_T/\overline{N}^T$ is a uniform variable independent of $\mathcal{F}_T$.

Proof

- Define, for $a > 1$, $T_a = \inf\{t \geq 0,\ N_t = a\}$.
  Then: $1 = \mathbb{E}[N_0] = \mathbb{E}[N_{T_a}] = aP(\overline{N}_\infty \geq a)(= aP(T_a < \infty))$
  Thus, $P(\overline{N}_\infty \geq a) = \frac{1}{a}$ i.e. $P\left(1/\overline{N}_\infty \leq \frac{1}{a}\right) = \frac{1}{a}$
- For any finite stopping time T such that $N_T > 0$ a.s., consider the local martingale constructed from N by shifting time from T, namely $(N_{u+T}/N_T;\ u \geq 0)$. We can apply the first step of this proof to this local martingale whose supremum is $\overline{N}^T/N_T$. The result follows easily.

■

The next lemma completes, in some sense, Lemma 0.1. $(N_t;\ t \geq 0)$ is now replaced by a general continuous semi-martingale $(X_t;\ t \geq 0)$, which is not necessarily positive.

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 3–9 (2006)
www.springerlink.com

Lemma 0.2 *Let $h : \mathbb{R} \to \mathbb{R}$ be a locally integrable function and set $H(x) = \int_0^x dy h(y)$.*
Then $H(\overline{X}_t) - h(\overline{X}_t)(\overline{X}_t - X_t) = \int_0^t h(\overline{X}_s) dX_s$; hence, if $(X_t;\ t \geq 0)$ is a local martingale, so is $(H(\overline{X}_t) - h(\overline{X}_t)(\overline{X}_t - X_t),\ t \geq 0)$.

Proof
This result is easily obtained when h is regular thanks to Itô formula, and the essential fact that $d\overline{X}_t$ is carried by $\{t;\ \overline{X}_t = X_t\}$. The general result follows from a monotone class argument.

■

Comment 0.1 *For $h(x) = 1_{x \leq a}$ and $(X_t;\ t \geq 0)$ a continuous local martingale, Lemma 0.2 yields that*

$$(a 1_{\overline{X}_t \geq a} + X_t 1_{\overline{X}_t \leq a};\ t \geq 0)$$

is a local martingale, from which the result of Lemma 0.1 follows.

Example 0.1 *(Doob's inequality in L^p for positive submartingales)*
We consider $(\Sigma_t;\ t \geq 0)$ a positive continuous submartingale.
Taking $F(x) = x^p$ with $p > 1$, Lemma 0.2 implies that $\overline{\Sigma}_t^p - p\overline{\Sigma}_t^{p-1}(\overline{\Sigma}_t - \Sigma_t)$ is a local submartingale.
Up to a localization argument, we obtain

$$\begin{aligned} \mathbb{E}\left[\overline{\Sigma}_t^p\right] &\leq \frac{p}{p-1} \mathbb{E}\left[\overline{\Sigma}_t^{p-1} \Sigma_t\right] \\ &\leq \frac{p}{p-1} \mathbb{E}\left[\overline{\Sigma}_t^p\right]^{(p-1)/p} \mathbb{E}\left[\Sigma_t^p\right]^{1/p} \qquad \textit{(Hölder)} \end{aligned}$$

Thus

$$\|\overline{\Sigma}_t\|_p \leq \frac{p}{p-1} \|\Sigma_t\|_p$$

0.2 Balayage Formula

The result of Lemma 0.2 may be understood in a more general framework. Let $(k_u;\ u \geq 0)$ be a locally bounded, predictable process, $(Y_u;\ u \geq 0)$ a continuous semi-martingale starting at 0.
Denote by γ_t and δ_t respectively the last zero of Y before t and the first zero of Y after t, namely:

$$\begin{aligned} \gamma_t &= \sup\{u \leq t;\ Y_u = 0\} \\ \delta_t &= \inf\{u \geq t;\ Y_u = 0\} \end{aligned}$$

Then