Introduction to General Topology

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(Revised)

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To

My Mother

Preface to the Revised Edition

This is mostly a reprint of the first edition except for correcting a rather large number of misprints. I have also added a few more exercises and made slight changes in the text where the original statements were erroneous and/or misleading. I am indebted to all those who pointed them out to me.

K. D. JOSHI

Preface

The present book, as its title indicates, is meant as an introduction to general topology. It is especially written for the average M.A./M.Sc. or the advanced undergraduate student in educational institutions in India. It can also be used by persons from other areas of mathematics or from computer science, physics etc., who want to get a working knowledge of topology. However, it is not meant as a reference book on the subject.

It was relatively recently that general topology was introduced in the M.A./M.Sc. syllabi of most of our educational institutions offering postgraduate degrees in mathematics. There is a growing and as yet unfulfilled demand for a good textbook on this subject catering especially to the needs of the students in this country, despite the fact that there is a large number of books written on this subject.

In the present book an attempt is made to present the basic material in general topology as simply as possible, with a special emphasis on motivating the definitions and on discussing the significance of the concepts defined and the theorems proved. The topics covered are all standard, it is the presentation which, I believe, is new. For example, instead of starting with a ready-made definition of a topological space, one whole chapter is devoted to developing a motivation for the definition, that is, to convincing the reader why the definition is natural and important. Similarly, while defining compactness, it is painstakingly pointed out how it is the next best thing to finiteness. The same applies to other important concepts in the book. General topology is usually taught to second year students of M.A./M.Sc. and as such it is assumed that the students are familiar with elementary properties of metric spaces through their first year courses in analysis. However, this is not a stringent prerequisite for reading the book as most of it is quite independent of metric spaces and their important properties are proved here anyway. Prerequisites from set theory and analysis are reviewed in Chapter 2, while Chapter 1 gives a warm-up in logic so that the student can do justice to the deductive nature of the subject.

Many books on topology assume a certain degree of maturity on the part of the reader. My experience shows that for the average M.A./M.Sc. students, a presumption of mathematical maturity is grossly incorrect and leads to disastrous consequences. In this book, therefore, I have made an attempt to develop, along with the material, some of the qualities that constitute maturity. This is done both through the exposition of the material and through the exercises at the end of each section. While some exercises are routine applications of the theorems in the text, or as the reader to fill the gaps in the proofs of some theorems, the others are intended to provoke some thoughts. A few exercises have no clear cut answers as they ask the reader to discuss or comment upon something. Their intentional vagueness should not, however, be taken to imply that they are of peripheral interest.

A word or two would probably be in order regarding the language used and the style of presentation. I have tried to avoid stodgy formalism, without being sloppy or too colloquial. Where the subject matter so demands, I have freely indulged into long discussions stressing some points even at the risk of sounding platitudinous to a reader who is already mature enough. Keeping in mind the type of students for whom this book is meant, I would prefer to be guilty of inclusion of such discussions rather than of their omission.

As the contents of the book reveal, it covers most of the material which is standard in a one-year course in general topology. However, occasionally I have exercised my personal choice. An entire chapter is devoted to category theory in order to acquaint the student with an important language in which current mathematics is expressed. On the other hand, I have omitted such topics as paracompactness of metric spaces and the general metrisation theorem. Although their importance can hardly be denied, these topics turn out to be rather too involved for the average student and are often skipped even when they form a part of the syllabus. It may be argued and with good sense that they deserve a better place in a book on general topology than, say, category theory does. The only defence I can put up against such a charge is that the present book is written more for the sake of the student than for the sake of the subject.

The book contains fifteen chapters, more or less of equal length (except Chapter 9). Each chapter is divided into four (sometimes three) sections. Each section generally concentrates itself on one or two major ideas and theorems. It is expected that in an average class, the gist of the material in a section can be covered in one lecture of sixty minutes, allowing one more lecture to discuss some of the exercises. With this pace, there should be little difficulty in completing the book in one academic year in a class meeting three to four times a week.

The first two chapters are strictly preparatory and may be assigned as a self-study. Chapter 3 provides a motivation for topology through geometry. The core of topology is contained in Chapters 4 to 11. The remaining four chapters are relatively independent of each other; they can be read in any order without a significant loss of continuity. Also any of them may be omitted in the interest of time in a short or a rapid course.

All the definitions, propositions, lemmas and theorems occurring in a section are numbered together in the order in which they appear, the section number always coming first, thus: (2.1) Definition, (2.2) Proposition, etc. These numbers are also used in referring to them in the same chapter. However, if they are referred to in another chapter, they will be numbered as (a.b.c) where a is the number of the chapter in which they occur. Thus Theorem (1.4) of Section 1, Chapter 6 will be designated by Theorem (1.4) in Chapter 6 and by Theorem (6.1.4) in all other chapters. The symbol ▮ is used to mark the end of a proof. Where a proof is either omitted or comes prior to the statement, the symbol ▮ is put immediately after the statement. The abbreviations 'iff' for 'if and only if' and 'w.r.t.' for 'with respect to' are also commonly used. In definitions, the concept to be defined is put in **bold face**.

Exercises form an integral part of the book. Each section has ten to twelve exercises on the average, ranging from the very routine to the challenging. Many of them are referred to in the body of the text. Quite often the student is asked to complete a proof or to supply the details of an argument by way of an exercise. Sometimes the queries such as 'why?' or 'how?' are made in the text where we want to alert the reader that some justification is needed, even though it may be a routine one. Generous hints are provided and the reader is well advised to try at least the less demanding exercises. Following the usual practice, an asterisk (*) is used to mark a challenging problem, realising of course, that there will be a room for disagreement as to whether a particular problem is worth so marking or not. Occasionally, two stars are used to mark a problem. This is usually the case where the problem is unusually demanding or requires extraneous techniques for its solution. A few exercises are designed to acquaint the reader with certain important concepts which could not be introduced in the body of the text for considerations of space. Miscellaneous exercises are provided at the end.

At the end of nearly every section the reader will find notes and a guide to the literature relevant to the material in that section. They are generally intended to direct the interested reader to appropriate references, for further information on certain concepts which are only briefly touched at or sometimes to acknowledge the credit to the original sources I have borrowed from. No attempt is made to trace the origin of each and every result and counter-example. Occasionally a historical remark is also made. However, as the present book is not meant as a reference book on topology, it is not very fruitful—nor is it really practicable—to give an exhaustive bibliography.

A word is in order about the sequence in which certain topics appear especially where it differs from the conventional one. It is customary in books on topology to put off things like convergence of nets, products and compactness until relatively late and then to give concentrated doses of them. We have deviated from this practice. The average reader of the book is sure to be familiar with convergence of sequences at least for real numbers. We have counted on this familiarity and defined convergence of sequences in topological spaces as early as in Chapter 4. Many results about sequences are proved and all this is used as a motivation for the theory of nets in Chapter 10. Similarly, instead of packing all theorems about compactness in a single chapter, we have defined compactness very early in the book and have spread the results about it over several chapters. We hope this will enable the student to absorb this concept without haste. As for products, although there is no conceptual difference between finite products and arbitrary ones, experience shows that while the student can generally handle finite products, often even the definition of an arbitrary product is not clear to him. In the usual treatment, even before the student has sufficient time to develop a feeling for set-theoretic products, the concept of the weak topology generated by a family of functions is thrust upon him and theorems about product spaces are proved at an alarming rate. As a result, the average student often has a very hazy understanding of the product topology. To remedy this, we have defined products of finitely many spaces very early in the book and have proved a number of theorems about them. It is hoped that this approach will allow the student ample time to grasp the essence of the product topology. When he sees arbitrary products in Chapter 8, he would be in a position to accept them as a natural extension of what he already knows.

This procedure has undoubtedly resulted in some repetitions which could have been avoided otherwise. But we feel the clarity gained more than compensates for it.

As for diagrams, I have tried to strike a balance between the two extremes typified on one side by Kelley (without a single figure) and on the other by Munkres (with almost every proof illustrated with a diagram). It is true that most of the arguments in topology have a euclidean or at least a metrical intuition. As a result the importance of drawing appropriate diagrams as an aid to understanding the proofs in topology is indisputable. However, I feel this is a habit which a reader should develop on his own. If this book is used as a text, the instructor, instead of duplicating the proofs, may illustrate them with diagrams (and a lot of hand-waving). As a result I have confined the diagrams only to drawing some important spaces. They are numbered consecutively in each chapter.

It was Professor M. N. Vartak who suggested that I write a book on Topology. Financial assistance to prepare the manuscript was given by the Curriculum Development Centre of the Quality Improvement Programme at I.I.T. Bombay. My sincere thanks to both.

In a book of this type any claim to originality of the results per se must necessarily fail and save for a few exercises and counter-examples, almost all results are borrowed from various sources. The most prominent among these is the book 'General Topology' by J. L. Kelley. I had my first course in topology through this book and its influence on the present work is inevitable. Other books I have occasionally borrowed from are by Dugundji, Hocking and Young, Willard and Munkres. The case of Munkres' book especially deserves a comment. It appeared after the first version of this book was prepared. The similarity between the two was noticeable although perhaps not very surprising since probably the same convictions have guided Munkres as me. In preparing the second version, I have tried to minimize the overlap with Munkres. However a few of his exercises were too tempting not to borrow. As regards the style, I am influenced by I. N. Herstein. All this is of course not to suggest that any of the authors mentioned here is responsible for deficiencies on my part.

Several colleagues and the referee made important suggestions to improve the book. The typing was done mostly by M. Parameswaran.

K. D. Joshi

Contents

Chapter One

Logical Warm-up

If mathematics is regarded as a language, then logic is its grammar. In other words, logical precision has the same importance in mathematics as grammatical correctness in a language. Both can be acquired either through a formal study or through good practice. In this chapter we do not proceed to study formal logic. Rather, we emphasise those aspects of it which are vital to the study of mathematics. Indeed, the essence of what we are going to say is the same as what any layman would tend to think as commonsense, provided he is in the habit of carefully weighing his statements. Some people are naturally good at these sorts of things just as some privileged persons have a musical ear. The moment they come across a flaw in some reasoning, some 'inner sense' within them keeps them on the alert. Unlike musical expertise, however, anyone can acquire this skill through careful practice.

1. Statements and Their Truth Values

It is by no means an easy task to give a complete and rigorous definition of a statement. Indeed, such a definition would cut deep into formal logic, linguistics and philosophy. For our purpose, a statement is a declarative sentence, conveying a definite meaning which may be either true or false but not both simultaneously. Incomplete sentences, questions and exclamations are not statements.

Some examples of statements are:

(1) John is intelligent.
(2) If there is life on Mars then the postman delivers a letter.
(3) Either grandmother chews gum or missiles are not costly.
(4) Every man is mortal.
(5) All men are mortal.
(6) There is a man who is eight feet tall.
(7) Every even integer greater than 2 can be expressed as a sum of two prime numbers.
(8) Every man with six legs is intelligent.

Notice that (2) and (3) are perfectly valid statements although they sound ridiculous, because there is no rational correlation between their components. Statements (4) and (5) can be logically regarded as same or equivalent statements. Note that they are statements about members of a class (in this case the class of all men) as opposed to the first statement which is about a particular individual called John. Statement (6) is an example of what is known as an **existence statement.** It asserts the existence of something but does not do anything more. For example in the present case, statement (6) does not say anything about the name or the whereabouts of the eight-footer nor does it say that he is the only such man. There may or may not be other men who are eight feet tall. The statement is silent on this point.

Statements must be either 'true' or 'false' but not both. The truth value of a statement is said to be 'True' or T or 1 if it is true and 'False' or F or 0 if the statement is false. The beginner is warned not to be misled by the word 'value' here which has the connotation of numerical value. In particular, 1 and 0 are merely symbols here and not the integers which they usually represent. There is indeed some good reason why they are used here despite the likely confusion, but it is beyond our scope to discuss it here. For our purpose, giving the truth value of a statement is merely a fancy way of saying whether it is true or false. What is important is that there is no in-between stage. There is no such thing as saying that a statement is 'somewhat true' or 'almost true'. True means absolutely and completely true, without qualifications. This is especially important to understand in the case of statements such as (4) and (5) which deal with members of a class. Such statements are to be regarded as false even when there is just one instance in which they fail. Indeed this is where a logician or a mathematician differs from a layman for whom the exception proves the rule. In mathematics, even a single exception (a **counter-example** as it is called) renders false a statement about members of a class.

Of course, the truth value of a statement may depend upon the way it is interpreted. For example in statement (1) above, the truth value depends upon what one means by 'intelligent'. But once intelligence is clearly defined then the statement must be declared as either completely true or false. Actually unless such a precise definition is implied and understood, (1) is not a valid statement. Truth values of statements, such as (2) or (3) will be discussed later.

Statement (7) is the famous conjecture of Goldbach. **Conjecture** means a statement whose truth value is not known at present. So far nobody has proved the statement nor has anyone disproved it. (Note that since the statement is about a class of integers, in order to disprove it, that is, to prove that it is false, all one has to do is to exhibit just one even integer greater than 2 which cannot be expressed as a sum of two primes. But no one has found such an integer so far.) Thus, at present no one knows whether (7) is true or false. However, this does not prevent it from being a statement. For it does have a definite truth value, the only trouble is that we do not quite know what it is. This is indeed a fine point. As was remarked earlier, the existence of something is to be distinguished from other knowledge about the thing. This observation applies to the truth value of a statement.

Another interesting point is presented by statement (8), which sounds as a statement from mythology or from fairy tales. We know no man has six legs. And, therefore, the statement (8) (and indeed any statement about a six-legged man) is logically true. Such a statement is said to be **vacuously true.** This certainly appears very puzzling to a layman. A moment's reflexion will, however, clear the mystery. For, if statement (8) is not true, it must be false, as there is no other possibility. But the only way statement (8) can be shown to be false is by finding (that is, proving the existence of) a six-legged man who is not intelligent. Since no man has six legs, this is impossible, the question of intelligence does not arise at all. So the statement (8) cannot be proved to be false. Therefore, we are forced to conclude that it is true. (This argument is not convincing to all persons. Such persons have founded their own school of mathematics called the **constructivist** school. The main point where they differ from the classical mathematical logic is that they do not accept proofs by contradiction, or the so-called *reductio ad absurdum* arguments of which the present one is an example. We shall, however, agree to bend our logical conscience if necessary in order to accept that (8) is true.)

As a practical application of vacuous truth, a student who has not appeared for any subject in an examination can boast that he secured a first class in every subject that he appeared for. Logically, his claim is absolutely correct. It is easy to set him right, however, if one merely notes that in such a case it is equally true to say that he has failed miserably in every subject he appeared for. Both statements are equally true and there is no logical contradiction, because both statements deal with something non-existent. We shall return to this point later when we shall discuss logical implications.

To conclude this section we give some phrases or sentences which are not statements.

(1) Will it rain?
(2) O! Those heavy rains.
(3) I am telling you a lie.

The first two are not statements because they are not declarative sentences. The third one is not a statement because it cannot be assigned any truth value. If it is true then according to its contents it is false. On the other hand, if it is false then according to its contents it is true. The trouble with this sentence is that it refers to itself. Many paradoxical situations arise because of self-referencing and the reader must have come across some popular puzzles which are tacitly based upon self-reference or some variations thereof.

Exercises

1.1 Which of the following expressions are statements? Why?
(1) If it rains the streets get wet.
(2) If it rains the streets remain dry.
(3) It rains.
(4) If it rains.

(5) John is intelligent and John is not intelligent.
(6) If John is intelligent then John is intelligent.
(7) If John is intelligent then John is not intelligent.
(8) For every man there is a woman who loves him.
(9) There exists a woman for whom there exists no man who loves her.
(10) There exists a woman such that no man loves her.
(11) This sentence is false.
(12) There is life outside our solar system.

1.2 On both the sides of a piece of paper it is written, 'The sentence on the other side is false.' Are the two sentences so written statements? Why? What if on one side 'The sentence on the other side is true' is written and on the other side 'The sentence on the other side is false'?

1.3 A barber in a village makes an announcement, 'I shave those (and only those) persons in this village who do not shave themselves.' Is this announcement a statement? Why?

2. Negation, Conjunction, Disjunction and Truth Tables

There are three ways of manufacturing new statements from given ones. Any complicated statement can be shown to be obtained from some very elementary or simple statements (such as 'it rains' or 'John is intelligent'). Let us study one by one the three ways of generating new statements.

(i) Negation: To negate a statement is to make another statement which will be opposite to the original one in terms of truth value. That is, we want that the negation be true precisely when the original statement is false, and vice versa. The simplest way to achieve this would be to merely prefix the phrase 'it is not the case that' before the original statement. For example the negation of 'John is intelligent' would be 'It is not the case that John is intelligent'. However, this is a very mechanical way and sentences formed like this tend to be linguistically clumsy. It is much better to say that 'John is not intelligent' which conveys exactly the same meaning. We can even go further and abbreviate 'not intelligent' to 'unintelligent', or to, say, 'dumb', provided we agree that 'intelligent' and 'dumb' are antonyms, that is, words whose meanings are opposite to each other. Similarly, 'X is rich' may be negated as 'X is poor' and so on.

In mathematical statements symbols are often used for brevity. The negation of such statements is expressed by putting a slash (/) over that symbol which incorporates the principal verb in the statement. Thus '$x = y$' (read 'x is equal to y') is negated as '$x \neq y$' (read 'x is not equal to y'). Similarly, '$x \notin A$' is the negation of '$x \in A$', (read 'x belongs to A').

Things are not so simple when we come to more complicated sentences. For example suppose the original statement is 'John is very intelligent'. Of course one way to negate it is simply to say 'It is not the case that John is very intelligent'. But this is hardly satisfying. A much more reasonable answer is 'John is not very intelligent'. In practice we often regard such a statement as a polite way of saying that John is dumb (in fact very dumb). In mathematics this is not so. The original statement is about the degree of John's intelligence. There are many degrees of intelligence ranging from very high to very low. Just because John lacks a very high degree does not automatically mean that the level of his intelligence is very low. It could be that he is just average. In this case, therefore, the correct logical negation is merely 'John is not very intelligent' and not 'John is very dumb'. A similar remark applies in all cases where there are more than two possibilities. For example to say that a book is not red does not necessarily mean that it must be of some other specific colour, say, green (although they are complementary colours). Hence, the logical negation of 'The book is red' is simply 'The book is not red' and not 'The book is green' or 'The book is blue'.

Even greater care is necessary when we come to statements such as 'Every man is mortal'. A layman is most apt to negate this statement either as 'Every man is immortal' or 'No man is mortal' or perhaps even as 'Every woman is mortal'. A logician will however negate it as 'There exists a man who is not mortal' or as 'There is an immortal man'. Recall that the statement in question is about a whole class, namely, the class of all men. To say it is false simply means that it is false in at least one instance. This easily leads to the correct negation. Note that the negation of a statement beginning with 'every' is an existence statement. We can also state the negation as 'Not every man is mortal'. But it is better to avoid saying 'Every man is not mortal' as it is likely to be confused with 'Every man is immortal'.

When we come to the negation of existence statements, the tables are turned around. As expected, the negation will be a statement about a class, asserting that every member of that class fails to be something whose existence is asserted by the original statement. For example the logical negation of 'There exists a rich man' would be that 'Every man is poor' or 'No man is rich', and not 'There exists a poor man'. The negation of 'There exists a woman such that no man loves her' is 'For every woman there exists a man who loves her'. The reader is strongly urged to master these types of sentences thoroughly, that is, to interpret them precisely as well as to negate them correctly as mathematics is full of such statements.

Sometimes statements are represented by symbols p, q, r, etc. just as in algebra one uses the symbol x, y, z, . . . , etc. to denote some variable quantities. With this notation there is a special symbol, $\neg$ (read as 'not'), for negation. For example if p stands for the statement 'John is strong' then $\neg p$ is read as 'not p' and denotes the statement 'John is not strong'. Note that, by the very definition, the truth values of p and $\neg p$ are always opposite of each other, no matter what p stands for. Although $\neg p$ is standard, the notations $\sim p$ or p' are also commonly used for the negation of p.

(ii) Conjunction: When two statements are joined together by the word 'and', the resulting statement is called their conjunction. For example the conjunction of 'John is rich' and 'Bob is weak' is 'John is rich and Bob is weak'. Where possible, we can paraphrase the conjunction so that it looks better English. For example the conjunction of 'John is intelligent' and 'John is rich' can be stated less mechanically as 'John is both intelligent and rich'. Also, instead of 'and' we can use 'but' when we want to stress an implied contrast between the two statements. For example we may say 'Bombay is big but Delhi is beautiful'. Mathematically, it means the same as 'Bombay is big and Delhi is beautiful'.

The standard notation for conjunction is $\wedge$, read as 'and'. Symbolically, if p and q are statements then their conjunction is denoted by $p \wedge q$ and is read as 'p and q'. The reason for introducing the special symbol $\wedge$ instead of simply writing the word 'and' is that we use the word 'and' in ordinary language so often that its use as a symbol for conjunction is sometimes likely to cause confusion. Note that we can form the conjunction of any two statements whatsoever. There need not be any semantic connection between the two, although in practice such a connection is usually understood. That is why a statement such as 'Missiles are costly and grandma chews gum' sounds ridiculous, although mathematically it is a perfectly valid statement obtained by the process of conjunction.

When we come to the truth value of a conjunction of two statements, it should be obvious to anyone that it is true when both the statements are true. If either one of them or both of them are false, then the conjunction would be false. As there is no such thing as 'half true' in mathematics, we must accept the conjunction as false even when one of its constituents is true and the other false.

(iii) Disjunction: The third way of forming new statements out of old ones is by taking the disjunction of two statements. For this, merely put the word 'or' between the statements. Thus the disjunction of 'John is intelligent' and 'There is life on Mars' is 'John is intelligent or there is life on Mars'. Sometimes we put the word 'either' before the first statement to make the disjunction sound nice, but it is not necessary to do so, so far as a logician is concerned. The symbolic notation for disjunction is $\vee$ (read as 'wedge' or 'or'). If p, q are two statements their disjunction is denoted by $p \vee q$.

A word of warning regarding the use of 'or' is very important. In practice, when one says 'I shall spend my vacation either in Bombay or in Pune' it is generally implied that the person will not spend it both in Bombay and in Pune. Indeed, often the very nature of the two statements is such that they cannot hold simultaneously, for example the statement, 'This book is either red or green'. Occasionally, both the possibilities are also implied together. In such a case the words 'or both' are added to emphasize it as in the statement 'A person with such a handwriting must be either a doctor or a crook or both'. In logic or in mathematics, however, it is not necessary to specify 'or both'. The word 'or' as it is used in mathematics always implies either one or both the two alternatives. This is known as the *inclusive sense* of the word 'or'. When only one of the possibilities is intended and the simultaneous holding of both of them is not automatically precluded by the