Lecture Notes in Mathematics

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Mathematical Theory of Feynman Path Integrals

An Introduction

2nd Edition

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ISBN: 978-3-540-76954-5 e-ISBN: 978-3-540-76956-9
DOI: 10.1007/978-3-540-76956-9

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008921370

Mathematics Subject Classification (2000): 81Q30, 81S40, 28C20, 46Nxx, 46T12, 47D08, 47G30, 60H05, 58D30, 35S30, 41A60, 34E05, 35Q40, 42A38, 81T45

Cover design: WMXDesign GmbH

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface to the Second Edition

This second edition, unfortunately, had to be done without the contribution of Raphael Høegh-Krohn, who died on 28 January 1988. The authors of the present edition hope very much that the result of their efforts would had been appreciated by him. His beloved memory has been a steady inspiration to us. Since the appearance of the first edition many new developments have taken place. The present edition tries to take this into account in several ways, keeping however the basic structure and contents of the first edition. At that time the book was the first rigorous one to appear in the area and was written in a sort of pioneering spirit. In our opinion it is still valid as an introduction to all the work which followed; therefore in this second edition we preserve its form entirely (except for correcting some misprints and slightly improving some formulations). A chapter has been however added, in which many new developments are included. These concern both new mathematical developments in the definition and properties of the integrals, and new exciting applications to areas like low dimensional topology and quantized gauge fields. In addition we have added historical notes to each of the chapters and corrected several misprints of the previous edition. As for references, we have kept all those of the first edition, numbered from 1 to 56 (with the corresponding updating), and added new references (in alphabetic order).

We are very grateful to many coworkers, friends and colleagues, who inspired us in a number of ways. Special thanks are due to Philippe Blanchard, Zdzisław Brzeźniak, Luca Di Persio, Jorge Rezende, Jörg Schäfer, Ambar Sengupta, Ludwig Streit, Aubrey Truman, Luciano Tubaro, and Jean-Claude Zambrini. We also like to remember with gratitude the late Yuri L. Daleckii and Michel Sirugue who gave important contributions to this area of research.

Trento,
June 2005

Sergio A. Albeverio
Sonia Mazzucchi

Preface to the First Edition

In this work we develop a general theory of oscillatory integrals on real Hilbert spaces and apply it to the mathematical foundation of the so-called Feynman path integrals of non-relativistic quantum mechanics, quantum statistical mechanics and quantum field theory. The translation invariant integrals we define provide a natural extension of the theory of finite dimensional oscillatory integrals, which has recently undergone an impressive development, and appear to be a suitable tool in infinite dimensional analysis. For example, on the basis of the present work, we have extended the methods of stationary phase, Lagrange immersions and corresponding asymptotic expansions to the infinite dimensional case, covering in particular the expansions around the classical limit of quantum mechanics. A particular case of the oscillatory integrals studied in the present work are the Feynman path integrals used extensively in physics literature, starting with the basic work on quantum dynamics by Dirac and Feynman, in the 1940s.

In the introduction, we give a brief historical sketch and some references concerning previous work on the problem of the mathematical justification of Feynman's heuristic formulation of the integral. However, our aim with the present publication was not to write a review work, but rather to develop from scratch a self-contained theory of oscillatory integrals in infinite dimensional spaces, in view of the mathematical and physical applications mentioned above.

The structure of the work is briefly as follows. It consists of nine chapters. Chapter 1 is the introduction. Chapters 2 and 4 give the definitions and basic properties of the oscillatory integrals, which we call Fresnel integrals or normalized integrals, for the cases where the phase function is a bounded perturbation of a non-degenerate quadratic form (positive in Chap. 2). Chapters 3 and 5–9 give applications to quantum mechanics, namely N-particle systems with bounded potentials (Chap. 3) and systems of harmonic oscillators with finitely or infinitely many degrees of freedom (Chaps. 5–9), with relativistic quantum fields as a particular case (Chap. 9).

This work appeared first as a Preprint of the Mathematics Institute of Oslo University, in October 1974.

The first named author would like to express his warm thanks to the Institute of Mathematics, Oslo University, for the friendly hospitality. He also gratefully acknowledges the financial support of the Norwegian Research Council for Science and the Humanities. Both authors thank Mrs. S. Cordtsen, Mrs. R. Møller and Mrs. W. Kirkaloff heartily for their patience and skill in typing the manuscript.

Oslo,
March 1976

Sergio A. Albeverio
Raphael J. Høegh-Krohn

Contents

1

Introduction

Feynman path integrals have been introduced by Feynman in his formulation of quantum mechanics [1].[1] Since their inception they have occupied a somewhat ambiguous position in theoretical physics. On one hand they have been widely and profitably used in quantum mechanics, statistical mechanics and quantum field theory, because of their strong intuitive, heuristic and formal appeal. On the other hand most of their uses have not been supported by an adequate mathematical justification. Especially in view of the potentialities of Feynman's approach as an alternative formulation of quantum dynamics, the need for a mathematical foundation has been broadly felt and the mathematical study of Feynman path integrals repeatedly strongly advocated, see e.g. [4]. This is, roughly speaking, a study of oscillating integrals in infinitely many dimensions, hence closely connected with the development of the theory of integration in function spaces, see e.g. [5]. The present work intends to give a mathematical theory of Feynman path integrals and to yield applications to non relativistic quantum mechanics, statistical mechanics and quantum field theory. In order to establish connections with previous work, we shall give in this introduction a short historical sketch of the mathematical foundations of Feynman path integrals. For more details we refer to the references, in particular to the review papers [6].

Let us first briefly sketch the heuristic idea of Feynman path integrals, considering the simple case of a non relativistic particle of mass m, moving in Euclidean space $\mathbb{R}^n$ under the influence of a conservative force given by the potential $V(x)$, which we assume, for simplicity, to be a bounded continuous real valued function on $\mathbb{R}^n$.

The classical Lagrangian, from which the classical Euler–Lagrange equations of motion follow, is

$$L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x). \tag{1.1}$$

[1] A vivid account of the origins of the idea, influenced particularly by remarks of Dirac [2], has been given by Feynman himself in [3].

Hamilton's principle of least action states that the trajectory actually followed by the particle going from the point y, at time zero, to the point x at time t, is the one which makes the classical action, i.e. Hamilton's principal function,

$$S_t(\gamma) = \int_0^t L\left(\gamma(\tau), \frac{\gamma(\tau)}{d\tau}\right) d\tau \tag{1.2}$$

stationary, under variations of the path $\gamma = \{\gamma(\tau)\}$, $0 \leq \tau \leq t$, with $\gamma(0) = y$ and $\gamma(t) = x$, which leave fixed the initial and end points y and x, and the time.

In quantum mechanics the state of the particle at time t is described by a function $\psi(x,t)$ which, for every t, belongs to $L_2(\mathbb{R}^n)$ and satisfies Schrödinger's equation of motion

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) + V(x)\psi(x,t), \tag{1.3}$$

with prescribed Cauchy data at time $t = 0$,

$$\psi(x,0) = \varphi(x), \tag{1.4}$$

where Δ is the Laplacian on $\mathbb{R}^n$ and $\hbar$ is Planck's constant divided by 2π. The operator

$$H = -\frac{\hbar^2}{2m}\Delta + V(x), \tag{1.5}$$

the Hamiltonian of the quantum mechanical particle, is self-adjoint on the natural domain of Δ and therefore $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}tH}$ is a strongly continuous unitary group on $L_2(\mathbb{R}^n)$. The solution of the initial value problem (1.3), (1.4) is

$$\psi(x,t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}tH}\varphi(x). \tag{1.6}$$

From the Lie–Kato–Trotter product formula we have

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}tH} = s - \lim_{k\to\infty}\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{k}V}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{k}H_0}\right)^k, \tag{1.7}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\Delta. \tag{1.8}$$

Assuming now for simplicity that φ is taken in Schwartz space $\mathcal{S}(\mathbb{R}^n)$, we have, on the other hand

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}tH_0}\varphi(x) = \left(2\pi\mathrm{i}\frac{\hbar}{m}t\right)^{-\frac{n}{2}}\int \mathrm{e}^{\mathrm{i}m\frac{(x-y)^2}{2\hbar t}}\varphi(y)dy, \tag{1.9}$$

hence, combining (1.7) and (1.9)

$$e^{-\frac{i}{\hbar}tH}\varphi(x) = s - \lim_{k\to\infty}\left(2\pi i\frac{\hbar}{m}\frac{t}{k}\right)^{-\frac{kn}{2}}\int_{\mathbb{R}^{nk}} e^{-\frac{i}{\hbar}S_t(x_k,\ldots,x_0)}\varphi(x_0)dx_0\ldots dx_{k-1} \tag{1.10}$$

where by definition $x_k = x$ and

$$S_t(x_k,\ldots,x_0) = \sum_{j=1}^{k}\left[\frac{m}{2}\frac{(x_j - x_{j-1})^2}{\left(\frac{t}{k}\right)^2} - V(x_j)\right]\frac{t}{k}. \tag{1.11}$$

The expression (1.10) gives the solution of Schrödinger's equation as a limit of integrals.

Feynman's idea can now be formulated as the attempt to rewrite (1.10) in such a way that it appears, formally at least, as an integral over a space of continuous functions, called paths. Let namely $\gamma(\tau)$ be a real absolutely continuous function on the interval $[0,t]$, such that $\gamma(\tau_j) = x_j$, $j = 0,\ldots,k$, where $\tau_j = \frac{jt}{k}$ and $x_0,\ldots,x_k$ are given points in $\mathbb{R}^n$, with $x_k = x$. Feynman looks upon $S_t(x_k,\ldots,x_0)$ as a Riemann approximation for the classical action $S_t(\gamma)$ along the path γ,

$$S_t(\gamma) = \int_0^t \frac{m}{2}\left(\frac{d\gamma}{d\tau}\right)^2 d\tau - \int_0^t V(\gamma(\tau))\,d\tau. \tag{1.12}$$

Moreover when $k \to \infty$ the measure in (1.10) becomes formally $d\gamma = N\prod_{0\le\tau\le t} d\gamma(\tau)$, N being a normalization, so that (1.10) becomes the heuristic expression

$$\int_{\gamma(\tau)=x} e^{\frac{i}{\hbar}S_t(\gamma)}\varphi(\gamma(0))\,d\gamma, \tag{1.13}$$

where the integration should be over a suitable set of paths ending at time t at the point x. This is Feynman's path integral expression for the solution of Schrödinger's equation and we shall now review some of the work that has been done on its mathematical foundation.[2] Integration theory in spaces of continuous functions was actually available well before the advent of Feynman path integrals, particularly originated by Wiener's work (1923) on the Brownian motion, see e.g. [8]. It was however under the influence of Feynman's work that Kac [9] proved that the solution of the heat equation

$$\frac{\partial}{\partial t}f(x,t) = \sigma\triangle f(x,t) - V(x)f(x,t), \tag{1.14}$$

which is the analogue of Schrödinger's equation when t is replaced by $-it$, σ being diffusion's constant, can be expressed by

$$f(x,t) = \int e^{-\int_0^t V(\gamma(\tau)+x)d\tau}\varphi(\gamma(0)+x)\,dW(\gamma), \tag{1.15}$$

[2] For the physical foundation see the original work of Feynman and the book by Feynman and Hibbs [1]. Also e.g. [7].

where $dW(\gamma)$ is Wiener's measure for the Wiener, i.e. Brownian motion, process with variance $\sigma^2 d\tau$, defined on continuous paths[3] $\gamma(\tau)$, $0 \leq \tau \leq t$, with $\gamma(\tau) = 0$. Hence (1.15) is an expectation with respect to the normal unit distribution indexed by the real Hilbert space of absolutely continuous functions $\gamma(\tau)$, with norm $\|\gamma\|^2 = \int_0^t \left(\frac{d\gamma}{d\tau}\right)^2 d\tau$. From this we see that (1.15) can be formally rewritten as (1.13), with $\frac{i}{\hbar} S_t(\gamma)$ replaced by $-\frac{1}{2}\int_0^t \frac{1}{2\sigma}\left(\frac{d\gamma}{d\tau}\right)^2 d\tau - \int_0^t V(\gamma(\tau))\, d\tau$. Thus (1.15) is a rigorous path integral (Wiener path integral) which plays for the heat equation a similar role as the Feynman path integral for the Schrödinger equation. This fact has been used [10] to provide a "definition by analytic continuation" of the Feynman path integral, in the sense that Feynman's path integral is then understood as the analytic continuation to purely imaginary t of the Wiener integral (1.15). The analogous continuation of the Wiener integral solution of the equation (1.14), with V replaced by iV, which corresponds to Schrödinger's equation with purely imaginary mass m, has been studied by Nelson [10] and allows to cover the case of some singular potentials. These definitions by analytic continuation, as well as the definition by the "sequential limit" (1.10),[4] have the disadvantage of being indirect in as much as they do not exhibit Feynman's solution (1.13) as an integral of the exponential of the action over a space of paths in physical space–time. In particular they are unsuitable for the mathematical realization of the original Dirac's and Feynman's ideas (see e.g. [1, 2])[5] about the approach to the classical limit $\hbar \to 0$, perhaps one of the most beautiful features of the Feynman path integral formalism. Namely (1.13) suggests that a suitable definition of the oscillatory integral should allow for the application of an infinite dimensional version of the method of stationary phase, to obtain, for $\hbar \to 0$, an asymptotic expansion in powers of $\hbar$, with leading term given by the path which makes $S_t(\gamma)$ stationary i.e., according to Hamilton's principle, the trajectory of classical motion. The definition of Feynman path integrals and more general oscillatory integrals in infinitely many dimensions which we give in this work is precisely well suited for this discussion, as shown in [41].[6]

Before we come however to our definition, let us make few remarks on other previous discussions of the mathematical foundations of Feynman path integrals. The attempt to define Feynman integral as a Wiener integral with purely imaginary variance meets the difficulty that the ensuing complex measure has infinite total variation (as first pointed out by Cameron [10], 1) and Daletskii [10], 2), in relation to a remark in [5]) and is thus unsuitable to define integrals like (1.13). For further remarks on this complex measure see [12].

A definition of Feynman path integrals for non relativistic quantum mechanics, not involving analytic continuation as the ones [10] mentioned before,

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3 Actually, Hölder continuous of index less than 1/2, see e.g. [8].
4 For the definition by a "sequential limit", in more general situations, see e.g. [11].
5 See also e.g. the references given in [41] and [42].
6 The results are also briefly announced in [42].

has been given by Ito [13]. We shall describe this definition in Chap. 2. Ito treated potentials $V(x)$ which are either Fourier transforms of bounded complex measures or of the form $c_\alpha x^\alpha$, with $\alpha = 1, 2$, $c_2 > O$. Ito's definition has been further discussed by Tarski [14]. Recently Morette-De Witt [15] has made a proposal for a definition of Feynman path integral, which has some relations with Ito's definition, but is more distributional rather than Hilbert space theoretical in character. The proposal suggests writing the Fourier transform of (1.13) as the "pseudomeasure"[7] $e^{-\frac{i}{\hbar}W}$, looked upon as a distribution acting on the Fourier transform of $e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau}$, provided this exists, where W is the Fourier transform of Wiener's measure with purely imaginary variance. This proposal left open the classes of functions V for which it actually works. Such classes follow however from Chap. 2 of the present work. Despite its, so far, incompleteness as to the class of allowed potentials, let us also mention a general attempt by Garczynski [16] to define Feynman path integrals as averages with respect to certain quantum mechanical Brownian motion processes, which generalize the classical ones. This approach has, incidentally, connections with stochastic mechanics [17], which itself would be worthwhile investigating in relation to the Feynman path formulation of quantum mechanics.[8]

Let us now make a corresponding brief historical sketch about the problem of the mathematical definition of Feynman path integrals in quantum field theory. They were introduced as heuristic tools by Feynman in [1] and applied by him to the derivation of the perturbation expansion in quantum electrodynamics. They have been used widely since then in the physical literature,

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[7] A theory of related pseudomeasures has in-between been developed by Krée. See e.g. [43] and references therein.

[8] Besides the topics touched in this brief historical sketch of the mathematical study of Feynman path integrals of non relativistic quantum mechanics there are others we did not mention, either because they concern problems other than those tackled later in this work or because no clear cut mathematical results are available. Let us mention however three more areas in which Feynman path integrals have been discussed and used, at least heuristically.

(a) Questions of the relation between Feynman's quantization and the usual one: see e.g. [10],6), [31, 35].
(b) Feynman's path integrals on functions defined on manifolds other than Euclidean space, in particular for spin particles. Attempts using the sequential limit and analytic continuation approaches have been discussed to some extent, see e.g. [6],7), [36] and references given therein. For the analytic continuation approach there is available the well developed theory of Wiener integrals on Riemannian manifolds, see e.g. [37].
(c) As mentioned before, an important application of Feynman path integrals is in the discussion of the classical limit, where $\hbar \to 0$. In [41] we tackle this problem and we refer to this paper and [42] also for references (besides e.g. [1, 2, 5, 7, 38]).

see e.g. [18], also under the name of Feynman history integrals. We shall now shortly give their formal expression. For more details see, besides the original papers [1], also e.g. [18]. The classical formal action for the relativistic scalar boson field is $S(\varphi) = S_0(\varphi) + \int_{\mathbb{R}^{n+1}} V(\varphi(\vec{x},t))\, d\vec{x}dt$, with

$$S_0(\varphi) = \frac{1}{2} \int\limits_{\mathbb{R}^{n+1}} \left[\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 - \sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 - m^2\varphi^2 \right] d\vec{x}dt$$

where φ is a function of $\vec{x}$, t, m is a non negative constant, the mass of the field, and V is the interaction. Similarly as in the case of a particle, the classical solutions of the equations of motion is given by Hamilton's principle of least action. The corresponding quantized system is formally characterized by the so called time ordered vacuum expectation values $G(\vec{x}_1, t_1, \ldots, \vec{x}_k, t_k)$, formally given, for $t_1 \le \ldots \le t_k$, $k = 1, 2, \ldots$, by the expectations of the products $\Phi(\vec{x}_1, t_1) \ldots \Phi(\vec{x}_k, t_k)$ in the vacuum state, where $\Phi(\vec{x}, t)$ is the quantum field (see e.g. [18], 7)). An heuristic expression for these quantities in terms of Feynman history integrals is

$$G(\vec{x}_1, t_1, \ldots, \vec{x}_k, t_k) = T\left(\int \mathrm{e}^{\mathrm{i}S(\varphi)} \varphi(\vec{x}_1, t_1) \ldots \varphi(\vec{x}_k, t_k)\, d\varphi \right),$$

where T is the so called time ordering operator and the integrals are thought of as integrals over a suitable subset of real functions φ on $\mathbb{R}^{n+1}$, see e.g. [18], 7). A mathematical justification of this formula, or a related one, would actually provide a solution of the well known problem of the construction of relativistic quantum field theory. Somewhat in connection, in one way or the other, with this problem, a large body of theory on integration in function spaces has been developed since the fifties and we mention in particular the work by Friedrichs [19], Gelfand [20], Gross [21] and Segal [22] and their associates, see also e.g. [23]. With respect to the specific application to quantum field theory, more recently a study of models has been undertaken, see e.g. [24], in which either the relativistic interaction is replaced by an approximate one, with the ultimate goal of removing at a later stage the approximation, or physical space–time is replaced by a lower dimensional one. We find here methods which parallel in a sense those discussed above in relation with Schrödinger's equation and, in a similar way as in that case, we can put these methods in connection with the problem of giving meaning to Feynman path integral, although in this case the connection is even a more indirect one as it was in the non relativistic case. We mention however these methods for their intrinsic interest. The sequential approach based on Lie–Kato–Trotter formula has been used especially in two space–time dimensional models particularly by Glimm, Jaffe and Segal [25]. The analytic continuation approach, in which time is replaced by imaginary time, is at the basis of the so called Euclidean–Markov quantum field theory, pursued vigorously by Symanzik [26] and Nelson [27] and applied particularly successfully, mostly in connection with the fundamental work of

Glimm and Jaffe, for local relativistic models in two space–time dimensions, with polynomial [24] or exponential interactions [28],[9] and in three space–time dimensions with space cut-off [29],[10] respectively in higher dimensions with ultraviolet cut-off interactions [30].[11] Much in the same way as for the heat, Schrödinger and stochastic mechanics equations, there are connections also with stochastic field theory [17], 4) - 7).[12]

Coming now to the Feynman history integrals themselves, it does not seem, to our knowledge, that any work has been done previous to our present work, as to their direct mathematical definition as integrals on a space of paths in physical space–time, except for the free case [14].

We shall now summarize briefly the content of the various sections of our work.

In Chap. 2 we introduce the basic definition for oscillating integrals on a separable real Hilbert space, which we call Fresnel integrals, and we establish their properties. In Chap. 3 this theory is applied to the definition of Feynman path integrals in non relativistic quantum mechanics. We prove that the heuristic Feynman path integral formula (1.13) for the solution of Schrödinger's equation can be interpreted rigorously as a Fresnel integral over a Hilbert space of continuous paths. In addition we derive corresponding formulae also for the wave operators and for the scattering operator.[13] In Chap. 4 we extend, in view of further applications, the definition of Fresnel integrals and give the properties of the new integral, called Fresnel integral relative to a given quadratic form. This theory is applied in Chap. 5 to the definition of Feynman path integrals for the n-dimensional anharmonic oscillator and in Chaps. 6 and 7 to the expression of expectations of functions of dynamical quantities of this anharmonic oscillator with respect to the ground state, respectively the Gibbs states [33] and quasifree states [34] of the correspondent harmonic oscillator.[14] In Chap. 8 we express the time invariant quasifree states on the Weyl algebra of an infinite dimensional harmonic oscillator by

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9 The Wightman axioms for a local relativistic quantum field theory (see e.g. [40]) have been proved, in particular.

10 The space cut-off has now been removed [44].

11 See also [45].

12 We did not mention here other topics which have some relations to Feynman's approach to the quantization of fields, for much the same reason as in the preceding Footnote 8). For discussion of problems in defining Feynman path integrals for spinor fields see e.g. [36] and references given therein. For the problem of the formulation of Feynman path integral in general relativity see e.g. [39, 4],2), and references given there.

13 Similar results hold for a system of N non relativistic quantum mechanical particles, moving each in d-dimensional space, interacting through a superposition of ν-body potentials ($\nu = 1, 2, \ldots$) allowed in particular to be translation invariant.

14 The same results hold for a system of N anharmonic oscillators, with anharmonicities given by superpositions of ν-body potentials, as in the preceding footnote.

Feynman path integrals defined as Fresnel integrals in the sense of Chap. 4, and this also provides a characterization of such states.

Finally, in Chap. 9 we apply the results of Chap. 8 to the study of relativistic quantum field theory. For the ultra-violet cut-off models mentioned above [30] we express certain expectation values, connected with the time ordered vacuum expectation values, in terms of Feynman history integrals, again defined as Fresnel integrals relative to a quadratic form. We also derive the correspondent expressions for the expectations with respect to any invariant quasi-free state, in particular for the Gibbs states of statistical mechanics for quantum fields ([33]3)).

Notes

The introduction appears here unchanged from the one of the first edition which obviously took only into account developments up to the year of appearance (1975). Simultaneously to the appearance of the first edition of this book, a method of stationary phase for Feynman path integrals was developed [87] and Maslov's approach to Feynman path integrals via Poisson processes became known [38]. These and subsequent developments are discussed in Chap. 10. Concerning footnote 2 we might add the following more recent references (articles resp. books on Feynman path integrals and their applications, of general interest, not necessarily concerned with the rigorous approach discussed in the present book): [60, 75, 74, 111, 125, 161, 179, 192, 113, 208, 209, 210, 211, 228, 229, 251, 253, 264, 287, 315, 318, 323, 325, 354, 376, 378, 401, 404, 409, 458, 467, 250, 90].

2

The Fresnel Integral of Functions on a Separable Real Hilbert Space

We consider first the case of the finite dimensional real Hilbert space $\mathbb{R}^n$, with some positive definite scalar product (x, y). We shall use $|x|$ for the Hilbert norm of x, such that $|x|^2 = (x, x)$. Since $e^{\frac{i}{2}|x|^2}$ is a bounded continuous function it has a Fourier transform in the sense of tempered distributions and in fact

$$\int e^{\frac{i}{2}|x|^2} e^{i(x,y)} dx = (2\pi i)^{\frac{n}{2}} e^{-\frac{i}{2}|y|^2}, \tag{2.1}$$

with $dx = dx \dots dx_n$, where $x_i = (e_i, x)$, $e_1, \dots, e_n$ being some orthonormal base in $\mathbb{R}^n$ with respect to the inner product $(\, , \,)$. For a function f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ we shall introduce for convenience the notation

$$\widetilde{\int} f(x) dx = (2\pi i)^{-\frac{n}{2}} \int f(x) dx, \tag{2.2}$$

so that $\widetilde{\int} f(x) dx$ is proportional to the usual integral with a normalization factor that depends on the dimension. We get from (2.1) that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\widetilde{\int} e^{\frac{i}{2}|x|^2} \hat{\varphi}(x) dx = \int e^{-\frac{i}{2}|x|^2} \varphi(x) dx, \tag{2.3}$$

where $\hat{\varphi}(x) = \int e^{i(x,y)} \varphi(y) dy$.

Let now $f(x)$ be the Fourier transform of a bounded complex measure μ, $\|\mu\| < \infty$, $f(x) = \int e^{i(x,y)} d\mu(y)$. We shall denote by $\mathcal{F}(\mathbb{R}^n)$ the linear space of functions which are Fourier transforms of bounded complex measures. Since the space of bounded complex measures $\mathcal{M}(\mathbb{R}^n)$ is a Banach algebra under convolution in the total variation norm $\|\mu\|$, we get that $\mathcal{F}(\mathbb{R}^n)$ is a Banach algebra under multiplication in the norm $\|f\|_0 = \|\mu\|$ for $f(x) = \int e^{i(x,y)} d\mu(y)$. The elements in $\mathcal{F}(\mathbb{R}^n)$ are bounded continuous functions and we have obviously $\|f\|_\infty \leq \|f\|_0$. For any $f \in \mathcal{F}(\mathbb{R}^n)$ of the form

$$f(x) = \int e^{i(x,y)} d\mu(y)$$