Convex and Starlike Mappings in Several Complex Variables

Sheng Gong

SCIENCE PRESS
Beijing

Sheng Gong

Convex and Starlike Mappings in Several Complex Variables

SCIENCE PRESS
Beijing

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China
Second Printed in 2006

Printed in Beijing

ISBN 7-03-006525-5 (Science Press, Beijing)

PREFACE

This interesting book deals with the theory of convex and starlike biholomorphic mappings in several complex variables. The underlying theme is the extension to several complex variables of geometric aspects of the classical theory of univalent functions. Because the author's introduction provides an excellent overview of the content of the book, I will not duplicate the effort here. Rather, I will place the book into historical context.

The theory of univalent functions long has been an important part of the study of holomorphic functions of one complex variable. The roots of the subject go back to the famous Riemann Mapping Theorem which asserts that a simply connected region Ω which is a proper subset of the complex plane C is biholomorphically equivalent to the open unit disk Δ. That is, there is a univalent function (holomorphic bijection) $f : \Delta \to \Omega$. In the early part of this century work began to focus on the class S of normalized ($f(0) = 0$ and $f'(0) = 1$) univalent functions defined on the unit disk. The restriction to univalent functions defined on the unit disk is justified by the Riemann Mapping Theorem. The subject contains many beautiful results that were obtained by fundamental techniques developed by many mathematicians, including Koebe, Bieberbach, Loewner, Goluzin, Grunsky, and Schiffer. The best-known aspect of univalent function theory is the so-called Bieberbach conjecture which was proved by de Branges in 1984.

A particularly elegant branch of complex analysis is geometric function theory in which the objective is to understand the relationship between geometric properties of the range of a holomorphic function and analytic properties of the function. Geometric function theory combines with univalent function theory in the study of geometrically defined subsets of the class S. Two important subsets of S are the families of convex and starlike functions. A function f in S is called convex if the image $f(\Delta)$ is a convex subset of C. A function f in S is starlike (with respect to the origin) if for each point $w \in f(\Delta)$ the straight line segment between 0 and w is contained in $f(\Delta)$. A good understanding of these two families was achieved by exploiting the geometry of the image.

It is natural to inquire whether the theory of univalent functions can extend to several complex variables. In the context of several complex variables one speaks of biholomorphic mappings rather than univalent functions as in one variable. This issue was explicitly raised by Henri Cartan in 1931. In several complex variables there is no analog of the Riemann Mapping Theorem, so there is not a single standard region like Δ in which to consider biholomorphic mappings. Cartan pointed out that some classical results of univalent function theory did not have analogs in several complex variables. At the same time he suggested that one should investigate the important geometrically defined subclasses of convex and starlike biholomorphic mappings. Even though parts of the classical theory of univalent functions have no analogs in several complex variables, Cartan suggested that geometric restrictions on the range of biholomorphic mappings might lead to an interesting theory.

As a matter of fact, there was little work in the geometric directions suggested by Cartan until the 1970's when a number of results dealing with the convex and starlike biholomorphic mappings appeared. At the time there was no systematic development of the subject. The situation changed dramatically in the late 1980's with a burst of new activity. A number of elegant parallels with the classical theory of starlike and convex univalent functions have been emerged. Even though the field is still evolving, the theory of convex and starlike biholomorphic mappings has now advanced to the stage that a unified presentation of the known results is needed. This book gathers together and presents in a unified manner the current state of affairs for convex and starlike biholomorphic mappings in several complex variables.

Because Professor Sheng Gong has been and continues to be one of the main contributors to the development of the theory of convex and starlike biholomorphic mappings, it is natural that he should prepare a monograph on the subject. In fact, the majority of the results presented are due to Professor Gong, his co-workers and his students. At the same time, open problems remain and the interested reader will find suggestions for future research.

Unfortunately, there has been a sharp division between researchers in one complex variable and those in several complex variables. The theory of convex and starlike mappings is an ideal meeting ground for the two groups. It is an area of several complex variables which contains many parallels with the classical theory while requiring the tools of several complex variables. This book provides a bridge between the two groups. It will serve as an excellent introduction to geometric funcion theory in several complex variables to workers in univalent function theory who have limited background in several complex variables. For instance, even though a result may hold in more generality, Professor Sheng Gong often first presents a proof in the special case of the unit ball or polydisk. In this special setting the proofs are more accessiable. Then either a more general result is established, or the relevant literature is cited. The presentation is essentially self-contained. Professor Sheng Gong deserves the thanks of the geometric function theory community for writing an informative, up-to-date monograph that will certainly foster more work in the area.

David Minda
Cincinnati, OH
1997

Contents

Introduction

§0.1 Introduction

Geometrical function theory of one complex variable has a long history and obtained a large number of important and interesting results. However there are counterexamples to show that many of these results are not true in several complex variables.

Perhaps H. Cartan was the first mathematician to systematically extend geometrical function theory from one variable to several variables. In 1933, in P. Montel's book on univalent function theory, Henri Cartan[1] wrote an appendix entitled *Sur la possibilite d'extendre aux fonctions de plusieurs variables complexes la theorie des fonctions univalents* in which he called for a number of generalizations of properties of univalent functions in one variable to biholomorphic mappings in several complex variables. He pointed out that there does not exist a corresponding Bieberbach conjecture in the case of several complex variables even in the simplest situation and that the boundedness of the modulus of the second coefficient of the Taylor expansions of the normalized univalent functions on the unit disc is not true in several complex variables. He also demonstrated that the corresponding growth and covering theorems fail in the case of several complex variables. He indicated particular interest in the properties of the determinant of the complex Jacobians of biholomorphic mappings in several complex variables. (The square of the magnitude of the determinant of the complex Jacobian is the infinitesimal magnification factor of volume in $\mathbb{C}^n$.) He stated a *theoreme presumé* that the magnitude of the determinant of the Jacobian of a normalized biholomorphic mapping would have a finite upper and positive lower bound depending only on $|z| = r < 1$. He also illustrated the significance and merits of determining these bounds. That his conjecture does not hold has been known for some time.

For demonstration, we exhibit the following counter-example.

For any positive integer k, let $f(z) = (f_1(z), f_2(z))$, $z = (z_1, z_2)$, with

$$\begin{cases} f_1(z) = z_1, \\ f_2(z) = z_2(1 - z_1)^{-k} = z_2 + kz_1z_2 + \cdots. \end{cases}$$

Then f is a normalized biholomorphic mapping on the unit ball B^2 in $\mathbb{C}^2$, that is, $f(0) = 0$, the Jacobian J_f of f at $z = 0$ is identity matrix and J_f is given by

$$J_f = \begin{pmatrix} 1 & 0 \\ \dfrac{kz_2}{(1-z_1)^{k+1}} & \dfrac{1}{(1-z_1)^k} \end{pmatrix}.$$

Thus $|\det J_f| = |1 - z_1|^{-k}$, which yields

$$\max_{|z|\le r} |\det J_f| = (1 - r)^{-k} \to \infty \quad as \quad k \to \infty$$

and

$$\min_{|z|\le r} |\det J_f| = (1 + r)^{-k} \to 0 \quad as \quad k \to \infty.$$

We cannot expect $|\det J_f|$ to be bounded from above or below if we assume only that $f(z)$ is biholomorphic.

In the same paper H. Cartan suggested the study of starlike mappings and convex mappings in several complex variables. Since then, many mathematicians have worked on this field and derived many significant results.

Since 1988, there have been many attempts to extend the geometric function theory of one complex variable to several complex variables; in particular, we studied the mappings on the ball, on Reinhardt domains, on bounded symmetric domains, on bounded convex circular domains and on bounded starlike circular domains. In this small book, we will systematically state and prove the results obtained by the author, relevant researchers and other mathematicians. There are still many interesting open problems.

In the next section, we will state and prove one interesting counterexample, which was given by FitzGerald[1]. We previously mentioned that the boundedness of the modulus of the second coefficient of the Taylor expansion of normalized univalent functions on the unit disc is not true in the several complex variables case. But there are many coefficients of the same order in the Taylor expansion of a biholomorphic mapping of several complex variables. We may ask the following question: Is one coefficient of the Taylor expansion unbounded, can we make a combinations of many coefficients that is bounded for biholomorphic mappings from certain domains in $\mathbb{C}^n$? But FitzGerald's counterexample tells us that there is no such combination. Actually, for any combination of the coefficients of a Taylor expansion of a biholomorphic mapping in any domain in $\mathbb{C}^n$, the modulus of the combination is unbounded. This counterexample strongly suggests that if you try to extend certain results of geometric function theory of one complex variable to that of several complex variables and expect to obtain some affirmative conclusions, it is reasonable to add some other condition, such as convexity or starlikeness, on the biholomorphic mappings. This is one reason why we studied convex and starlike mappings in several complex variables.

In Chapter I, the necessary and sufficient conditions for starlikeness of holomorphic mappings on bounded starlike circular domains, r-domains and Carathéodory complete domains are given. In Chapter II, we will consider the criteria of convexity for holomorphic mappings on the unit ball. For the polydisc, Suffridge gave the necessary and sufficient condition for a holomorphic mapping to be convex. We extend his Theorem with two different ways. In Chapter III, two different kinds of growth theorems for normalized starlike biholomorphic mappings on some Reinhardt domains, classical domains, and more general, bounded starlike circular domains are given, and we prove that these growth theorems are equivalent. In Chapter IV, growth theorems for normalized convex biholomorphic mappings on the unit ball are given. For bounded convex circular domains, the precise upper and lower bound estimations of the modulus of the normalized convex biholomorphic mappings are given. Using the Harish-Chandra representation theory of the symmetric space, we study the distortion theorem for the linear-invariant family on symmetric spaces in Chapter V. We give the distortion theorem for normalized convex biholomorphic mappings on the unit ball at the begining of

Chapter VI. The upper and lower bounds of $J_f(z)\overline{J_f(z)}'(J_f(z)$ is the Jacobian of f) for a normalized convex biholomorphic mapping f on the unit ball are estimated using some matrices which are related to the Bergman metric of the unit ball B^n. We also extend this result to bounded convex circular domain. Using the results from Chapter V, another distortion theorem for biholomorphic convex mappings on unit ball, and the distortion theorem for locally biholomorphic convex mappings and starlike mappings on symmetric domains are given in Chapter VI. In the last chapter, four geometric properties of normalized convex mappings on the unit ball are given. These include the estimation of the main curvature of the image of the hypersphere with radius $r(0 < r < 1)$, the volume of the hypersphere with radius r, the estimation of the Bloch constant and the two-point distortion theorem.

I would like to express my sincere thanks to the Department of Mathematics at the University of California, San Diego for their hospitality in providing me with a stimulating environment in which some of this research was carried out. I am greatly indebted to my friends, Professor Carl FitzGerald, Professor Taishun Liu, Professor Shikun Wang, Professor Qihuang Yu, and Professor Xuean Zheng for their continuous cooperation concerning this topic and their support through difficult situations, especially Professor Carl FitzGerald , who has helped me to organize the material of this monograph and had many fruitful discussions with me, and Professor Xuean Zheng, who read the manuscript and gave me many very important suggestions and comments to improve this small book. Also I am deeply indebted to Professor David Minda for writing a wonderful preface and giving many very important suggestions. It is a pleasure to thank Dr. Carolyn Thomas who made useful suggestions for mathematics and for improving the English throughout the text. Finally, I wish to thank Miss Hong Ge for her hard work in typing the manuscript.

§0.2 Counterexamples

There are many counterexamples to show that some results of geometric function theory of one complex variables fail if we attempt to extend them to several complex variables. In this section, we present one counterexample which was given by FitzGerald[1]. This counterexample tell us that it is impossible to make any combination of coefficients of the Taylor expansion of normalized biholomorphic mappings such that the modulus of the combination is bounded.

In geometric function theory of one variable, we consider the class **S** of normalized analytic functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{0.2.1}$$

which are defined and one to one in the unit disc $\{z : |z| < 1\}$. In 1907, Koebe showed that $|a_2|$ is bounded for all functions in **S**. In 1916, the precise bound, $|a_2| \leq 2$, was proved by Bieberbach. This result leads to bounds on the growth of $|f(z)|$ and on $|f'(z)|$ and the well-known Bieberbach conjecture. These results also show that **S** is a normal family.

In one variable theory, the only normalized univalent analytic function on the plane is z. But there are many normalized biholomorphic mappings taking the space $\mathbb{C}^n$ into itself. (cf. Rosay and Rudin[1])

Let $F = (f_1, f_2, \cdots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a normalized holomorphic mapping on $\mathbb{C}^n$, i.e., $F(0) = 0$ and $J_F(0) = I$ (I is the unit matrix). Then each component of F is a holomorphic function of several complex variables $z = (z_1, z_2, \cdots, z_n)$, and can be written as follows:

$$f_k(z_1, z_2, \cdots, z_n) = z_k + \sum d^{(k)}_{(j_1, j_2, \cdots, j_n)} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}, \tag{0.2.2}$$

where $k = 1, 2, \cdots, n$ and each j_m, $m = 1, 2, \cdots, n$ is a non-negative integer, and $j_1 + j_2 + \cdots + j_n \geq 2$.

The following two examples are biholomorphic mappings on $\mathbb{C}^n$.

Example 1 Let $b = (b_1, b_2, \cdots, b_n) \in \mathbb{C}^n, c = (c_1, c_2, \cdots, c_n) \in \mathbb{C}^n$, and define $b \cdot c = \sum\limits_{i=1}^{n} b_i c_i$. Let $v \in \mathbb{C}^n$, and $v \neq 0$. Assume that $A, B, C, \cdots$ are vectors from $\mathbb{C}^n$ such that $A \cdot v = 0$, $B \cdot v = 0$, $C \cdot v = 0$, $\cdots$. Let a be a complex number. Consider the normalized polynomial mapping

$$w = z + av(A \cdot z)(B \cdot z)(C \cdot z) \cdots. \tag{0.2.3}$$

The product is finite and has at least two factors that involve the product of $(A \cdot z)$ and $(B \cdot z)$. Then this mapping is biholomorphic.

To prove the claim, it is sufficient to obtain the inverse of the mapping. Dot the equation with A. Since $A \cdot v = 0$, the equation is $A \cdot w = A \cdot z$. Similarly $B \cdot w = B \cdot z$, $C \cdot w = C \cdot z$, $\cdots$. Hence $z = w - av(A \cdot w)(B \cdot w)(C \cdot w)\cdots$, and the mapping is inverted.

Example 2 Let a be a nonzero complex number and define a biholomorphic mapping of $\mathbb{C}^n$ into $\mathbb{C}^n$ by its coordinate functions: $w_1 = z_1 \exp(az_2)$, and $w_k = z_k$ for $k = 2, 3, \cdots, n$.

Clearly $z_k = w_k$ for $k = 2, 3, \cdots, n$, and $\exp(az_2)$ is known and is nonzero. Thus $z_1 = w_1 \exp(-aw_2)$. We obtain the inverse of the mapping.

Applying a permutation of the independent variables and the same permutation of the dependent variables, it is possible to create other normalized biholomorphic mappings of $\mathbb{C}^n$ into $\mathbb{C}^n$ from the previous examples. For the discussion here, it is important to see how the lower order terms behave under the composition of these examples.

Let $m \geq 1$ be an integer. If $w = z + P_m(z) + O(|z|^{m+1})$ and $w = z + Q_m(z) + O(|z|^{m+1})$ are two mappings where P_m and Q_m are vectors in $\mathbb{C}^n$ with each coordinate function being a homogeneous polynomial of degree m and all other terms of high order indicated by the expression $O(|z|^{m+1})$, then the composition of these mappings is given by

$$w = z + P_m(z) + Q_m(z) + O(|z|^{m+1}). \tag{0.2.4}$$

Examples 1 and 2 show that there are many biholomorphic mappings of $\mathbb{C}^n$ into $\mathbb{C}^n$. There are many coefficients of second order terms. For each coordinate function there are $\dfrac{n(n+1)}{2}$ coefficients; for the full mapping, there are $\dfrac{n^2(n+1)}{2}$ coefficients of the second order terms. We already know that the magnitude of each coefficient is unbounded. But it is still possible that the magnitude of some combination of coefficients is bounded. The striking fact is that there is no limitation on choice of the coefficients of the second order terms! Given a set of complex numbers for the respective coefficients of the second order expressions, there is a way to extend the multivariable power series such that the resulting map is defined and biholomorphic on $\mathbb{C}^n$.

Theorem 0.2.1 (FitzGerald) *Let* $\{P_1, P_2, \cdots, P_n\}$ *be a sequence of* n *homogeneous polynomials of the second order in* n *variables* $(n \geq 2)$. *Then, for each* $k = 1, 2, \cdots, n$, *there exists a function*

$$f_k(z_1, z_2, \cdots, z_n) = z_k + P_k(z_1, z_2, \cdots, n) + O(|z|^3)$$

such that $F = (f_1, f_2, \cdots, z_n)$ *is a biholomorphic mapping of* $\mathbb{C}^n$ *into* $\mathbb{C}^n$.

Proof. To generate all the second order terms, we need only to consider the following four biholomorphic mappings of $\mathbb{C}^n$ into $\mathbb{C}^n$, where we write the expansion only up to the second order.

$$\begin{array}{llll} (1) & w_1 = z_1 + az_1^2, & (2) & w_1 = z_1 + az_1z_2, \\ & w_2 = z_2, & & w_2 = z_2, \\ & \cdots, & & \cdots, \\ & w_n = z_n. & & w_n = z_n. \\ (3) & w_1 = z_1 + az_2^2, & (4) & w_1 = z_1 + az_2z_3, \\ & w_2 = z_2, & & w_2 = z_2, \\ & \cdots, & & \cdots, \\ & w_n = z_n. & & w_n = z_n. \end{array}$$

Consider a permutation on the set $\{2, 3, \cdots, n\}$. Apply the same permutation to the indices of both the independent and dependent variables. Each the second order term for the first coordinate function can be obtained in this way. By permutating $\{1, 2, \cdots, n\}$, every the second order term in any coordinate function can be obtained, from (1) through (4).

These four initial segments of mappings would generate all possible segments up to the second order using permutations of both the independent and dependent variables and by compositions. It suffices to show that these four initial segments are indeed the initial segments of normalized biholomorphic mappings of $\mathbb{C}^n$ into $\mathbb{C}^n$. In cases (3) and (4), these are such biholomorphic mappings. In the case of (2), this is the initial segment of example 2.

It remains only to find an appropriate type of mapping which has (1) for its initial segment. In Example 1, we consider $v = (1, 1, 0, \cdots, 0)$ and $A = B = (1, -1, 0, \cdots, 0)$. The mapping is

$$\begin{aligned} w_1 &= z_1 + a(z_1 - z_2)^2 = z_1 + az_1^2 - 2az_1z_2 + az_2^2, \\ w_2 &= z_2 + a(z_1 - z_2)^2 = z_2 + az_1^2 - 2az_1z_2 + az_2^2, \\ w_3 &= z_3, \\ &\cdots \\ w_n &= z_n. \end{aligned} \tag{0.2.5}$$

Consider a mapping with initial segment (3) with a replaced by $-a$.

$$\begin{aligned} w_1 &= z_1 - az_2^2, \\ w_2 &= z_2, \\ &\cdots \\ w_n &= z_n. \end{aligned} \tag{0.2.6}$$

The composition of (0.2.5) and (0.2.6) is the following biholomorphic mapping:

$$\begin{aligned} w_1 &= z_1 + az_1^2 - 2az_1z_2, \\ w_2 &= z_2 + az_1^2 - 2az_1z_2 + az_2^2, \\ w_3 &= z_3, \\ &\cdots \\ w_n &= z_n. \end{aligned} \tag{0.2.7}$$

Again consider a mapping with initial segment (3) with a replaced by $-a$. Now exchange indices 1 and 2 in the subscripts of the independent and dependent variables.

$$\begin{aligned} w_1 &= z_1, \\ w_2 &= z_2 - az_1^2, \\ w_3 &= z_3, \\ &\cdots \\ w_n &= z_n. \end{aligned} \tag{0.2.8}$$