Lecture Notes in Mathematics

Martine Queffélec

Substitution Dynamical Systems – Spectral Analysis

1294

Second Edition

Springer

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Preface for the Second Edition

This revised edition initially intended to correct the misprints of the first one. But why does it happen now, while the subject extensively expanded in the past twenty years, and after the publication of two major books (among other ones) devoted to dynamical systems [88] and automatic sequences [14]? Let us try to explain why we got convinced to do this new version. On the one hand, the initial account of the LNM 1294 offered a basis on which much has been built and, for this reason, it is often referred to as a first step. On the other hand, the two previously quoted books consist in impressive and complete compilations on the subject [14, 88]; this was not the spirit of our LNM, almost self-contained and "converging" to the proof of a specific result in spectral theory of dynamical systems. From this point of view, those three books might appear as complementary ones.

This having been said, reproducing the corrected LNM identically would have been unsatisfactory : a lot of contributions have concurred to clarify certain aspects of the subject and to fix notations and definitions; also a great part of the raised questions have now been solved. Mentioning these improvements seemed to us quite necessary. Therefore, we chose to add some material to the first introductory chapters, which of course does not (cannot) reflect the whole progress in the field but some interesting directions. Moreover, two applications of substitutions - more generally of combinatorics of words - to discrete Schrödinger operators and to continued fraction expansions clearly deserved to take place in this new version : two additional appendices summarize the main results in those fields.

The initial bibliography has been inflated to provide a much more up-to-date list of references. This renewed bibliography is still far from being exhaustive and we should refer the interested reader to the two previously cited accounts.

In recent contributions, the terminology has changed, emphasizing on the morphism property. However, we chose to keep to the initial terminology, bearing in mind the fact that this is definitely a second edition.

Lille
December 2009

Martine Queffélec

Preface for the First Edition

Our purpose is a complete and unified description of the spectrum of dynamical systems arising from substitution of constant length (under mild hypotheses). The very attractive feature of this analysis is the link between several domains : combinatorics, ergodic theory and harmonic analysis of measures.

The rather long story of these systems begins perhaps in 1906, with the construction by A. Thue [234] of a sequence with certain non-repetition properties (rediscovered in 1921 by M. Morse [190]):

$$0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\cdots$$

This sequence (called from now on the Thue-Morse sequence) can be obtained by an obvious iteration of the substitution $0 \to 01$, $1 \to 10$, or else, as an infinite block product : $01 \times 01 \times 01 \times \cdots$, where $B \times 01$, for any $0-1$ block B, means : repeat B and then $\tilde{B}$, the block deduced from B by exchanging 0 and 1. Also, if $S_2(n)$ denotes the sum of digits of n in the 2-adic expansion, $u = (u_n)$ with

$$u_n = e^{i\pi S_2(n)}$$

is the ± 1 Thue-Morse sequence.

The Thue-Morse sequence admits a strictly ergodic (= minimal and uniquely ergodic) orbit closure and a simple singular spectrum, as observed by M. Keane [143].

The various definitions of the Thue-Morse sequence lead to various constructions of sequences, and thus, of dynamical systems:

– substitution sequences [55, 63, 68, 104, 188] then [71, 119, 132, 135, 173, 174, 189, 208], ...

– a class of $0-1$ sequences introduced by M. Keane, called generalized Morse sequences [143], admitting in turn extensions [175, 176] then [102, 154–156, 162], ...

– q-multiplicative sequences, with $q = (q_n)$, q_n integer ≥ 2 [59] then [166, 202], ...

In this account, we restrict our attention to the first category of sequences, but, in case of bijective substitutions (chapter 9), we deal with particular G-Morse sequences and q-multiplicative sequences.

Ergodic and topological properties of substitution dynamical systems have been extensively studied; criteria for strict ergodicity [68, 188], zero entropy [68, 209], rational pure point spectrum [68, 173, 174], conditions for presence of mixed spectrum [68] and various mixing properties [71] are main investigations and results in these last years. But, except in some examples ([135, 143], ...), no descriptive spectral analysis of the continuous part of the spectrum has been carried out.

Indeed, not so many dynamical systems lead themselves to a comprehensive computation of spectral invariants. I mean, mainly, maximal spectral type and spectral (global) multiplicity (see [214] for a rather complete historical survey). Of course, transformations with purely discrete spectrum are quite well-known [240], and in this case, the spectrum is simple. In the opposite direction, a countable Lebesgue spectrum occurs in ergodic automorphisms of compact abelian groups as in K-automorphisms (see [61]). A very important class of dynamical systems, with respect to the spectral analysis, consists of gaussian dynamical systems. Guirsanov proved a conjecture of Kolmogorov [110]: the maximal spectral type of a gaussian dynamical system is equivalent to e^{σ}, where σ denotes the spectral measure of the process; and its spectral multiplicity has been shown by Vershik to be either one - with singular spectrum - or infinite ([237, 238], see also [89]). Then arose the question of whether finite multiplicity ≥ 2 (or ≥ 1 for Lebesgue spectrum) was possible, and the last results in multiplicity theory have been mostly constructions of suitable examples. I just quote the last three important ones : Robinson E.A. Jr in [214] exhibits, for every $m \geq 1$, a measure-preserving transformation with singular spectrum and spectral multiplicity m. On the other hand, Mathew and Nadkarni in [177, 178] construct, for every $N \geq 2$, a measure-preserving transformation with a Lebesgue spectrum of multiplicity $N\phi(N)$ (ϕ Euler totient function). In these examples, the transformations are group extensions. Recently, M. Lemanczyk obtained every even Lebesgue multiplicity [160].

Turning back to substitution dynamical systems, we prove the following : for a substitution of length q over the alphabet A (or q-automaton [55]), the spectrum is generated by $k \leq \text{Card } A$ probability measures which are strongly mixing with respect to the q-adic transformation on $\mathbf{T}$; in most examples, these measures are specific generalizations of Riesz products, which is not so surprising because of the self-similarity property inherent in this study. (Note that such Riesz products play a prominent part in distinguishing normal numbers to different bases [136]; see also [50, 198], ...).

Earlier Ledrappier and Y. Meyer already realized classical Riesz products as the maximal spectral type of some dynamical system.

The generating measures of the spectrum of some q-automaton are computable from a matrix of correlation measures, indeed a matrix Riesz product, whose rank gives rise to the spectral multiplicity. For example, the continuous part of the Rudin-Shapiro dynamical system is Lebesgue with multiplicity 2, while, by using the mutual singularity of generalized Riesz products (analyzed in chapter 1), we get various singular spectra with multiplicity 1 or 2, as obtained by Kwiatkowski and Sikorski ([156], see also [101, 102]). For substitutions of nonconstant length, no spectral description seems accessible at present but we state a recent characterization of eigenvalues established by B. Host [119] and list some problems.

We have aimed to a self-contained text, accessible to non-specialists who are not familiar with the topic and its notations. For this reason, we have developed with all details the properties of the main tools such that Riesz products, correlation measures, matrices of measures, nonnegative matrices and even basic notions of spectral theory of unitary operators and dynamical systems, with examples and applications.

More precisely, the text gets gradually more specialized, beginning in chapter 1 with generalities on the algebra $M(\mathbf{T})$ and its Gelfand spectrum Δ. We introduce generalized Riesz products and give a criterion for mutual singularity.

Chapter 2 is devoted to spectral analysis of unitary operators, where all fundamental definitions, notations and properties of spectral objects can be found. We prove the representation theorem and two versions of the spectral decomposition theorem.

We restrict ourselves, in chapter 3, to the unitary operator associated with some measure-preserving transformation and we deduce, from the foregoing chapter, spectral characterizations of ergodicity and of various mixing properties (strong, mild, weak). As an application of D-ergodicity (ergodicity with respect to a group of translations [47]), we discuss spectral properties of some skew products over the irrational rotation [100, 103, 140, 212].

In chapter 4, we investigate shift invariant subsets of the shift space (subshifts), such like the orbit closure of some sequence. Strict ergodicity can be read from the given sequence, if taking values in a finite alphabet. The correlation measure of some sequence - when unique - belongs to the spectral family; hence, from earlier results, we derive spectral properties of the sequence. We give a classical application to uniform distribution modulo 2π (Van der Corput's lemma) and we discuss results around sets of recurrence [25, 35, 93, 219].

From now on we are concerned with substitution sequences. All previously quoted results regarding substitution dynamical systems are proved in chapters 5–6, sometimes with a different point of view and unified notations (strict ergodicity, zero entropy, eigenvalues and mixing properties). We are needing the Perron-Frobenius theorem and, for sake of completeness, we give too a proof of it.

Till the end of the account, the substitution is supposed to have a constant length. We define, in chapter 7, the matrix of correlation measures Σ and we show how to deduce the maximal spectral type from it. Then we prove elementary results about matrices of measures which will be used later.

In chapter 8, we realize Σ as a matrix Riesz product and this fact provides a quite simple way to compute it explicitly. Applying the techniques immediately, we treat the first examples : Morse sequence, Rudin-Shapiro sequence, and a class of sequences arising from commutative substitutions (particular G-Morse sequences), admitting generalized Riesz product as generating measures.

An important class of substitutions is studied in chapter 9 without complete success. It would be interesting in this case to get a more precise estimate of the spectral multiplicity, which is proved to be at least 2 for substitutions over a nonabelian group.

Finally, the main results on spectral invariants in the general case are obtained in chapters 10–11 by using all the foregoing. We have to consider a bigger matrix of correlation measures, involving occurrences of pairs of given letters instead of simple ones, which enjoys the fundamental strong mixing property and provides the maximal spectral type of the initial substitution.

The spectral multiplicity can be read from the matrix Σ, as investigated with the Rudin-Shapiro sequence and some bijective substitution. We obtain in both cases a Lebesgue multiplicity equal to 2, while N-generalized Rudin-Shapiro sequences admit a Lebesgue multiplicity $N\phi(N)$ [203, 211].

In an appendix, we suggest an extension to automatic sequences over a compact nondiscrete alphabet. We give conditions ensuring strict ergodicity of the orbit closure.

As explained before, we preferred to develop topics involving spectral properties of measures and for this reason, the reader will not find in this study a complete survey of substitutions. A lot of relevant contributions have been ignored or perhaps forgotten : we apologize the mathematicians concerned.

Paris
July 1987

Martine Queffélec

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Chapter 1
The Banach Algebra $M(\mathbf{T})$

This first chapter is devoted to the study of the Banach algebra $M(\mathbf{T})$. This study will be brief because we need only little about $M(\mathbf{T})$, and there exist excellent books on the subject, in which all the proofs will be found [123, 141, 218, 232]. We introduce the technics of generalized characters to precise the spectral properties of measures such as generalized Riesz products, which will nicely appear later as maximal spectral type of certain dynamical systems.

1.1 Basic Definitions

We consider $\mathbf{U}$ the multiplicative compact group of complex numbers of modulus one, and $\mathbf{T} = \mathbf{R}\backslash 2\pi\mathbf{Z}$ that we identify with $\mathbf{U}$ by the map $\lambda \to e^{i\lambda}$; $\mathbf{T}$ is equipped with the Haar measure m, identified this way with the normalized Lebesgue measure $\frac{1}{2\pi}dx$ on $[-\pi, \pi]$.

1. The elements of the character group $\Gamma = \hat{\mathbf{T}}$, isomorphic to $\mathbf{Z}$, will be considered sometimes as integers, with addition, sometimes as multiplicative functions on $\mathbf{T}$, and, in this case, we denote by γ_n instead of n the element $t \mapsto e^{int}$.
2. $M(\mathbf{T})$ is the algebra of the regular Borel complex measures on $\mathbf{T}$, equipped with the convolution product of measures, defined by

   $$\mu * \nu(E) = \int_{\mathbf{T}} \mu(E-t)\, d\nu(t)$$

   for $\mu, \nu \in M(\mathbf{T})$ and E any measurable subset of $\mathbf{T}$.
   $M(\mathbf{T})$ is a Banach algebra for the norm

   $$||\mu|| = \int d|\mu|,$$

   $|\mu|$ being the total variation of μ.

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The Fourier coefficients of $\mu \in M(\mathbf{T})$ are, by definition,

$$\hat{\mu}(n) = \int_{\mathbf{T}} e^{int} d\mu(t) = \int \gamma_n d\mu, \quad n \in \mathbf{Z}$$

and satisfy : $||\hat{\mu}||_\infty := \sup_{n\in\mathbf{Z}} |\hat{\mu}(n)| \leq ||\mu||$. The *Fourier spectrum* of μ is the set of integers $n \in \mathbf{Z}$ for which $\hat{\mu}(n) \neq 0$.

3. The measure μ is positive if $\mu(E) \geq 0$ for every measurable set E, and, in this case, the sequence $(\hat{\mu}(n))$ is *positive definite*, namely

$$\sum_{1\leq i,j\leq n} z_i \overline{z_j} \hat{\mu}(i-j) \geq 0$$

for any finite complex sequence $(z_i)_{1\leq i\leq n}$.
Conversely, the *Bochner theorem* asserts that a positive definite sequence $(a_n)_{n\in\mathbf{Z}}$ is the Fourier transform of a positive measure on $\mathbf{T}$.
Positive measures of total mass one are *probability measures*.

4. We recall that μ is a *discrete* measure if $\mu = \sum a_j \delta_{t_j}$, ($\delta_t$ being the unit mass at $t \in \mathbf{T}$) and that μ is a *continuous* measure if $\mu\{t\} = 0$ for all $t \in \mathbf{T}$. $M_d(\mathbf{T})$ is the sub-algebra of discrete measures in $M(\mathbf{T})$ and $M_c(\mathbf{T})$, the convolution-ideal of all continuous measures on $\mathbf{T}$. Every $\mu \in M(\mathbf{T})$ can be uniquely decomposed into a sum

$$\mu = \mu_d + \mu_c$$

where $\mu_d \in M_d(\mathbf{T})$ and $\mu_c \in M_c(\mathbf{T})$ respectively are the discrete part and the continuous part of μ.
There is a necessary and sufficient condition for a measure μ to be continuous, which involves the Fourier transform of μ :

Lemma 1.1 (Wiener). *Let $\mu \in M(\mathbf{T})$. Then :*

$$\mu \in M_c(\mathbf{T}) \Longleftrightarrow \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\hat{\mu}(n)|^2 = 0$$

and in this case, we have $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N+K}^{N+K} |\hat{\mu}(n)|^2 = 0$ *uniformly in K.*

5. Let $\mu, \nu \in M(\mathbf{T})$; we say that μ is *absolutely continuous with respect to* ν and we write $\mu \ll \nu$ if $|\mu|(E) = 0$ as soon as $|\nu|(E) = 0$, for any measurable set E. Then, by the Radon-Nikodym property, $\mu = f \cdot \nu$ where $f \in L^1(\nu)$ is referred to as the density of μ with respect to ν, usually denoted by $d\mu/d\nu$. Let us define

$$L(\nu) = \{\mu \in M(\mathbf{T});\ \mu \ll \nu\} \tag{1.1}$$

So we are allowed to identify $L(\nu)$ with $L^1(\nu)$. The measures μ, ν are said to be *equivalent*, and we write $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.
In the opposite direction, we say that μ and ν are *mutually singular*, and we write $\mu \perp \nu$, if there exists a measurable set E such that

$$|\mu(E)| = ||\mu|| \quad \text{and} \quad |\nu|(E) = 0.$$

The measure μ is said to be *singular* if $\mu \perp m$, m the Haar measure on $\mathbf{T}$.

Every $\mu \in M(\mathbf{T})$ can be uniquely decomposed into a sum

$$\mu = \mu_a + \mu_s$$

where $\mu_a \ll m$ and μ_s is singular (respectively the absolutely continuous part and the singular part of μ).

Affinity between two measures. Let μ and ν be two positive measures in $M(\mathbf{T})$ and $\lambda \in M(\mathbf{T})$ be such that both $\mu \ll \lambda$ and $\nu \ll \lambda$. The affinity $\rho(\mu,\nu)$ of the measures μ and ν is the quantity

$$\rho(\mu,\nu) = \int_{\mathbf{T}} \left(\frac{d\mu}{d\lambda}\right)^{1/2} \left(\frac{d\nu}{d\lambda}\right)^{1/2} d\lambda, \tag{1.2}$$

obviously independent of the choice of λ. Note that

$$\rho(\mu,\nu) = 0 \text{ if and only if } \mu \perp \nu.$$

The measure μ is said to have *independent powers* if $\mu^n \perp \mu^m$ whenever $n \neq m$, $n,m \in \mathbf{N}$, where $\mu^n := \mu * \cdots * \mu$ n times. Such a measure is singular : if not, the absolutely continuous part ν of μ satisfies $0 \neq \nu^n \ll \mu^n$ for all $n \in \mathbf{N}$; thus, if $m \geq n$, $\nu^m \ll \mu^m$, $\nu^m \ll \nu^n \ll \mu^n$ and $\mu^m \not\perp \mu^n$.

It is less obvious to give conditions on the Fourier transform of μ, ensuring the absolute continuity of μ (with respect to m). Of course, it is necessary for μ to satisfy : $\lim_{|n|\to\infty} \hat{\mu}(n) = 0$.

This condition is not sufficient and we shall use the notation $M_0(\mathbf{T})$ for the ideal of all measures μ whose Fourier transform vanishes at infinity. Sometimes, those measures are called *Rajchman measures*, and a nice survey on them appears in [172].

6. $M(\mathbf{T})$ is identified with the dual space $C(\mathbf{T})^*$ of the continuous functions on $\mathbf{T}$. Let (μ_n) and μ in $M(\mathbf{T})$. From Fejer's theorem, μ_n converges to μ in the weak-star topology of $M(\mathbf{T})$, $\sigma(M(\mathbf{T}),C(\mathbf{T}))$, if and only if

$$\hat{\mu}_n(\gamma) \to \hat{\mu}(\gamma) \quad \text{for every} \quad \gamma \in \Gamma.$$

We shall write : $w^* - \lim_{n\to\infty} \mu_n = \mu$. Recall that the unit ball of $M(\mathbf{T})$ is a weak-star compact set.

The following proposition will be used in chapter 4 (see [59]).

Proposition 1.1. *Let (μ_n) and (ν_n) be two sequences of positive measures on $\mathbf{T}$ such that $\mu_n \to \mu$ and $\nu_n \to \nu$ in the weak-star topology of $M(\mathbf{T})$. Then*

$$\limsup_n \rho(\mu_n, \nu_n) \leq \rho(\mu,\nu),$$

where ρ denotes the affinity defined in (1.2)

Proof. The proof involves the Cauchy-Schwarz inequality applied with some suitable partition of unity. We assume, without loss of generality, that μ and ν are probability measures and we fix a probability measure λ dominating both μ and ν. We put $M_0 = \{\frac{d\mu}{d\lambda} = 0\}$, $N_0 = \{\frac{d\nu}{d\lambda} = 0\}\backslash M_0$ and we consider for $j \in \mathbf{Z}$ and some fixed $\varepsilon > 0$

$$U_j = \{x \in \mathbf{T}\backslash(M_0 \cup N_0),\ (1+\varepsilon)^j \frac{d\mu}{d\lambda}(x) < \frac{d\nu}{d\lambda}(x) \leq (1+\varepsilon)^{j+1} \frac{d\mu}{d\lambda}(x)\}. \quad (1.3)$$

Clearly, the sequence (U_j), supplemented by M_0 and N_0, provides an infinite partition of $\mathbf{T}$. In particular, $\sum_j \mu(U_j) < \infty$ and we fix J such that

$$\sum_{|j|\geq J} \mu(U_j) \leq \varepsilon^2. \quad (1.4)$$

From now on, we denote by $V_0, V_1, V_2, \ldots, V_{2J}, V_{2J+1}$ the finite Borel partition

$$M_0, N_0, U_{-J+1}, \ldots, U_{J-1}, \cup_{|j|\geq J} U_j$$

of $\mathbf{T}$. Note that, for every $2 \leq j \leq 2J$,

$$(1+\varepsilon)^{j-1-J} \mu(V_j) \leq \nu(V_j) \leq (1+\varepsilon)^{j-J} \mu(V_j), \quad (1.5)$$

by integrating the inequalities (1.3) on U_j with respect to λ. For each j, $0 \leq j \leq 2J+1$, let us choose by regularity an open set $\omega_j \supset V_j$ such that

$$\mu(\omega_j) \leq (1+\varepsilon)^{1/2} \mu(V_j),\ \nu(\omega_j) \leq (1+\varepsilon)^{1/2} \nu(V_j);$$

let $(f_j)_{0\leq j\leq 2J+1}$ be a continuous partition of unity subordinate to the open covering $(\omega_j)_{0\leq j\leq 2J+1}$. Clearly we have

$$\int_{\mathbf{T}} f_j \, d\mu \leq \mu(\omega_j) \leq (1+\varepsilon)^{1/2} \mu(V_j) \quad (1.6)$$

as well as

$$\int_{\mathbf{T}} f_j \, d\nu \leq \nu(\omega_j) \leq (1+\varepsilon)^{1/2} \nu(V_j). \quad (1.7)$$

We deduce that $\rho(\mu_n, \nu_n) :=$

$$\begin{aligned}\int_{\mathbf{T}} \left(\frac{d\mu_n}{d\lambda}\right)^{1/2} \left(\frac{d\nu_n}{d\lambda}\right)^{1/2} d\lambda &= \sum_{j=0}^{2J+1} \int_{\mathbf{T}} \left(f_j \frac{d\mu_n}{d\lambda}\right)^{1/2} \left(f_j \frac{d\nu_n}{d\lambda}\right)^{1/2} d\lambda \\ &\leq \sum_{j=0}^{2J+1} \left(\int_{\mathbf{T}} f_j \, d\mu_n\right)^{1/2} \left(\int_{\mathbf{T}} f_j \, d\nu_n\right)^{1/2}\end{aligned}$$