University Mathematics

Joseph Blakey

SECOND EDITION

University Mathematics

A TEXTBOOK FOR STUDENTS OF SCIENCE & ENGINEERING

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PREFACE

THIS BOOK IS INTENDED AS A FIRST-YEAR COURSE IN PURE Mathematics at any University and contains practically all the various branches of Mathematics required by students, excluding projective geometry, although the analysis is not intended to be rigorous. It would also be very suitable for advanced Sixth Form students in Grammar Schools, especially those entering for special scholarship examinations.

As practically the whole of the problems are taken from London University Science and Engineering Degree examination papers in Pure Mathematics, and since the book covers the whole of the syllabus for the London General Degree in Science (which covers Subsidiary Mathematics), excluding projective geometry which is optional, it will be found eminently suitable for this course.

My grateful thanks are due to the Senate of the London University for permission to use examples from their final degree examination papers, and also to an old student, Mr. D. Stewart, for his assistance in the preparation of the manuscript.

Even after the most careful checking it is quite possible that some errors have been overlooked, and I shall be greatly indebted to anyone submitting corrections.

Owing to the change in syllabus for the London General Degree, in order to cover the Part I paper in Mathematics, chapters on Spherical Trigonometry and Moments of Inertia have been added. My thanks are due to the Cleaver-Hume Press for permission to use examples on Spherical Trigonometry from my *Degree Mathematics: Formulae and Examples*. Many corrections have also been made, and the book now covers the whole of Mathematics Part I and a portion of the Part II syllabus.

JOSEPH BLAKEY.

Sunderland Technical College,
1958.

CONTENTS

CHAPTER I

Revision

It has been found that the facility with which many students can solve problems has been greatly impaired by lack of knowledge of fundamental formulæ, and it has been deemed necessary to devote this initial chapter to a statement of essential formulæ.

ALGEBRAIC

1. Quadratic equation $ax^2 + bx + c = 0$, roots α and β with $\alpha > \beta$.

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

$$\alpha + \beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Roots real if $b^2 > 4ac$, coincident if $b^2 = 4ac$, imaginary if $b^2 < 4ac$, and rational if $b^2 - 4ac$ is a complete square.

2. Quadratic expression $E \equiv ax^2 + bx + c$.

If α and β are the roots of $ax^2 + bx + c = 0$ as above, then:

(i) When a is positive,

E is positive if $x > \alpha$ or $< \beta$;
E is negative if $\beta < x < \alpha$.

(ii) When a is negative,

E is positive if $\beta < x < \alpha$;
E is negative if $x > \alpha$ or $< \beta$.

3. For the **arithmetical progression** (A.P.)

$$a,\ a + d,\ a + 2d,\ \ldots$$

(i) The nth term is $a + (n - 1)d$;

(ii) **Sum to** n **terms** $= \frac{n}{2}\{2a + (n - 1)d\} = \frac{n}{2}(a + l)$, **where** l **is the last term.**

4. For the geometrical progression (G.P.)

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

(i) The nth term is ar^{n-1};

(ii) Sum to n terms $= \dfrac{a(1-r^n)}{1-r}$.

If the numerical value of r ($|r|$ or "modulus of r") be less than unity, then the sum to infinity $= \dfrac{a}{1-r}$.

5. The first n natural numbers are $1, 2, 3, \ldots, n$.

If $S_1 =$ their sum, $S_2 =$ sum of their squares, and $S_3 =$ sum of their cubes,

$$S_1 = \frac{n}{2}(n+1);\quad S_2 = \frac{n(n+1)(2n+1)}{6};\quad S_3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

6. The binomial theorem states that, for all rational values of n,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots;$$

and when $a = 1$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots.$$

TRIGONOMETRIC

7.

Angle	$90° - \theta$	$90° + \theta$	$180° - \theta$	$180° + \theta$
sin	$+\cos\theta$	$+\cos\theta$	$+\sin\theta$	$-\sin\theta$
cos	$+\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
tan	$+\cot\theta$	$-\cot\theta$	$-\tan\theta$	$+\tan\theta$

Angle	$270° - \theta$	$270° + \theta$	$360° - \theta$ or $-\theta$
sin	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$
cos	$-\sin\theta$	$+\sin\theta$	$+\cos\theta$
tan	$+\cot\theta$	$-\cot\theta$	$-\tan\theta$

8.

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$\pm\infty$

9. If $\alpha = \sin^{-1} a$, $\beta = \cos^{-1} b$, $\gamma = \tan^{-1} c$, where α is the angle between -90° and $+90^\circ$ whose sine has the value a, β is the angle between 0° and 180° whose cosine has the value b, and γ is the angle between -90° and $+90^\circ$ whose tangent has the value c, then the general solutions of

$$\sin\theta = a;\ \cos\theta = b;\ \tan\theta = c$$

are given by

$$\theta = n\,.\,180^\circ + (-1)^n\alpha,$$
$$\theta = n\,.\,360^\circ \pm \beta,$$
$$\theta = n\,.\,180^\circ + \gamma$$

respectively, where n is any integer.

10. For all values of x,

$$\sin^2 x + \cos^2 x = 1;\ \tan^2 x + 1 = \sec^2 x;\ 1 + \cot^2 x = \operatorname{cosec}^2 x.$$

11. $\sin(A+B) = \sin A\cos B + \cos A\sin B.$

$\sin(A-B) = \sin A\cos B - \cos A\sin B.$

$\cos(A+B) = \cos A\cos B - \sin A\sin B.$

$\cos(A-B) = \cos A\cos B + \sin A\sin B.$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

12. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}.$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}.$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta);\ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta).$$

13.
$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$
$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$
$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.$$

14.
$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2};$$
$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}.$$
$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2};$$
$$\cos x - \cos y = 2 \sin \frac{x+y}{2} \sin \frac{y-x}{2}.$$
$$2 \sin \theta \cos \varphi = \sin (\theta + \varphi) + \sin (\theta - \varphi);$$
$$2 \cos \theta \sin \varphi = \sin (\theta + \varphi) - \sin (\theta - \varphi);$$
$$2 \cos \theta \cos \varphi = \cos (\theta + \varphi) + \cos (\theta - \varphi);$$
$$2 \sin \theta \sin \varphi = \cos (\theta - \varphi) - \cos (\theta + \varphi).$$

15. With standard notation for a triangle ABC:

Sine rule. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Cosine rule. $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \text{ etc.}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}, \text{ etc.}$$
$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ etc.}$$
$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \text{ etc.}$$
$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ etc.}$$
$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}, \text{ etc.}$$

Area of triangle ABC $= \Delta = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$
$= \frac{1}{2}ab \sin C = \sqrt{s(s-a)(s-b)(s-c)}$.

Radius of circumcircle $= R = \dfrac{a}{2 \sin A} = \dfrac{b}{2 \sin B} = \dfrac{c}{2 \sin C} = \dfrac{abc}{4\Delta}$.

$r = (s-a) \tan A/2 = (s-b) \tan B/2 = (s-c) \tan C/2 = \Delta/s$.

$r_1 = s \tan A/2 = (s-c) \cot B/2 = (s-b) \cot C/2 = \Delta/(s-a)$.

$r_2 = (s-c) \cot A/2 = s \tan B/2 = (s-a) \cot C/2 = \Delta/(s-b)$.

$r_3 = (s-b) \cot A/2 = (s-a) \cot B/2 = s \tan C/2 = \Delta/(s-c)$.

Co-ordinate Geometry

16. Distance between the points (x_1, y_1) **and** (x_2, y_2) **is given by**

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

Point $(\bar{x}, \bar{y})$ **dividing the join of** (x_1, y_1), (x_2, y_2) **in the ratio** $(\lambda_2 : \lambda_1)$ **is given by**

$$\bar{x} = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \quad \bar{y} = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2},$$

where λ_1 and λ_2 have the same sign for internal division and opposite signs for external division.

Area of triangle joining the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$.

17. Equations of a straight line are:

General equation: $ax+by+c=0$. a, b, c are constants.

Slope equation: $y = mx + c$. m is the slope and c the intercept on OY.

Intercept equation: $\dfrac{x}{a}+\dfrac{y}{b}=1$. a and b are the intercepts on OX and OY respectively.

Perpendicular equation: $x \cos \alpha + y \sin \alpha = p$, where p is the perpendicular from O (always positive) and makes an angle α with OX.

Equation of line joining (x_1, y_1), (x_2, y_2):

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}.$$

Equation of line, slope m, **through** (x_1, y_1):

$$y - y_1 = m(x - x_1).$$

18. Angle θ between lines of slopes m_1, m_2 is given by

$$\theta = \tan^{-1} \pm \frac{m_1 - m_2}{1 + m_1 m_2}.$$

The lines are parallel if $m_1 = m_2$, and perpendicular if $m_1 m_2 = -1$.

19. The perpendicular from (x_1, y_1) on the line $x \cos \alpha + y \sin \alpha = p$ is of length

$$p - x_1 \cos \alpha - y_1 \sin \alpha.$$

The perpendicular from (x_1, y_1) on the line $ax + by + c = 0$ is of length

$$\pm \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}},$$

where the + sign is taken if c is +ve, and the negative sign if c is --ve.

The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{(a_2^2 + b_2^2)}}.$$

20. The equation of the circle, centre (x_1, y_1), radius r, is

$$(x - x_1)^2 + (y - y_1)^2 = r^2;$$

and if the origin O be its centre, the equation is

$$x^2 + y^2 = r^2.$$

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

its centre is $(-g, -f)$, and radius $\sqrt{g^2 + f^2 - c}$.

The equation of the circle on the join of (x_1, y_1), (x_2, y_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

The equation of the tangent at (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

The equation of the tangent of slope m to the circle $x^2 + y^2 = r^2$ is

$$y = mx \pm r\sqrt{1 + m^2}.$$

The length (t) of the tangent from (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is given by}$$
$$t^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

21. Roots of the general equation of the nth degree. Any equation of the nth degree in x must have n roots.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0,$$

then this equation must be identical with

$$a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) = 0,$$

i.e.
$$a_0\{x^n - x^{n-1}\sum_1^n \alpha_r + x^{n-2}\sum_1^n \alpha_r\alpha_p - \ldots\} = 0.$$

Hence, comparing coefficients in the two equations

$$\sum_1^n \alpha_r = -\frac{a_1}{a_0}; \quad \sum_1^n \alpha_r\alpha_p = +\frac{a_2}{a_0} \qquad (r \neq p);$$

$$\sum_1^n \alpha_r\alpha_p\alpha_q = -a_3/a_0 \qquad (r \neq p \neq q), \text{ and so on.}$$

Example 1.—Find the general value of θ satisfying the equation $\cos 4\theta = \frac{1}{2}$, and hence find the roots of the equation $16x^4 - 16x^2 + 1 = 0$. Hence, show that

(i) $\sec\frac{\pi}{12}\sec\frac{5\pi}{12}\sec\frac{7\pi}{12}\sec\frac{11\pi}{12} = 16;$

(ii) $\sec\frac{\pi}{12}\sec\frac{5\pi}{12}\sec\frac{7\pi}{12} + \sec\frac{\pi}{12}\sec\frac{5\pi}{12}\sec\frac{11\pi}{12} + \sec\frac{\pi}{12}\sec\frac{7\pi}{12}\sec\frac{11\pi}{12}$
$+ \sec\frac{5\pi}{12}\sec\frac{7\pi}{12}\sec\frac{11\pi}{12} = 0.$

Since
$$\cos 4\theta = \tfrac{1}{2},$$
$$\therefore\ 4\theta = 2n\pi \pm \frac{\pi}{3},$$
and
$$\theta = n\frac{\pi}{2} \pm \frac{\pi}{12},$$
where n is any integer.

When $n = 0$, $\theta = \pi/12$ (using + sign);

$n = 1$, $\theta = \frac{5\pi}{12}$ or $\frac{7\pi}{12}$;

$n = 2$, $\theta = \frac{11\pi}{12}$ (using − sign).

$\therefore$ 4 consecutive roots of the equation are

$$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}. \qquad \ldots\ldots\ldots \quad (1)$$

Now $\cos 4\theta = 2\cos^2 2\theta - 1$,

$$= 2\{2\cos^2\theta - 1\}^2 - 1 = 8\cos^4\theta - 8\cos^2\theta + 1.$$

$\therefore$ equation $\cos 4\theta = \frac{1}{2}$ can be written

$$8\cos^4\theta - 8\cos^2\theta + 1 = \tfrac{1}{2},$$

i.e.
$$16\cos^4\theta - 16\cos^2\theta + 1 = 0.$$

Let $\cos\theta = x$ and this equation becomes

$$16x^4 - 16x^2 + 1 = 0, \qquad \text{(ii)}$$

and its roots will be given by $\cos\theta$, where θ takes the values in (i), since the other possible values of θ merely repeat the values of $\cos\theta$ given by

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}.$$

$\therefore$ the roots of $16x^4 - 16x^2 + 1 = 0$ are

$$\cos 15°, \cos 75°, \cos 105°, \text{ and } \cos 165°,$$

i.e.
$$x = \pm 0{\cdot}2588, \pm 0{\cdot}9659 \text{ to 4 decimal places.}$$

The equation (ii) can be written $\dfrac{1}{x^4} - \dfrac{16}{x^2} + 16 = 0$, where $x = \cos\theta$, and θ is given by (i),

i.e.
$$y^4 - 16y^2 + 16 = 0, \qquad \text{(iii)}$$

where $y = \sec\theta$, and θ is given by (i).

The product of the roots of this equation is 16 (coefficient of unity in (iii)).

$$\therefore \sec\frac{\pi}{12}\sec\frac{5\pi}{12}\sec\frac{7\pi}{12}\sec\frac{11\pi}{12} = 16.$$

Also the sum of the products taken three at a time must equal the coefficient of y in (iii), i.e. zero.

Hence,
$$\sec\frac{\pi}{12}\sec\frac{5\pi}{12}\sec\frac{7\pi}{12} + \sec\frac{\pi}{12}\sec\frac{5\pi}{12}\sec\frac{11\pi}{12} + \sec\frac{5\pi}{12}\sec\frac{7\pi}{12}\sec\frac{11\pi}{12}$$
$$+ \sec\frac{\pi}{12}\sec\frac{7\pi}{12}\sec\frac{11\pi}{12} = 0.$$

EXAMPLES ON CHAPTER I

The following examples are taken from London University examination papers:

1. The equation $(x-1)^2 = \lambda(x-2\mu)(x-4)$ has equal roots. Show that, μ having any given value, this statement is true only for one value of λ other than zero.

Show also that, λ having any given value other than zero, the statement is true for two values of μ, which are real only if the value of λ is less than unity.

If the value of λ is -15, find the two sets of equal roots of the equation.

2. (i) Find the ranges of values of x for which the expression $x^2 - 6x + 7$ lies in value between ± 1.

(ii) Find the least value of k so that the expression

$$3x^2 + 12xy + 7y^2 + k(x^2 + y^2)$$

shall be greater than zero for all real values of x and y.

3. Show that, if a and b have opposite signs, the expression $ax + b/x$ can assume all real values; but that if a and b are of the same sign, it cannot assume any value lying between $\pm 2\sqrt{(ab)}$.

Express $\dfrac{x^2+1}{x(x-1)}$ as a function of y, where $x - 1 = xy$, and hence, or otherwise, show that the expression can only assume values which do not lie between $-2 \pm 2\sqrt{2}$.

4. Find the condition that the expression $ax^2 + bx + c$ should have the same sign for all values of x.

Show that, for real values of x, the expression $(ax^2 + bx + c)/(cx^2 + bx + a)$ will be capable of all real values if $b^2 > (a + c)^2$.

Show also that there will be two values between which it cannot lie if $4ac < b^2 < (a + c)^2$, and two values between which it must lie if $b^2 < 4ac$.

5. Show that, if $y = (2x^2 - 4)/(x^2 - x - 2)$, y can assume all real values for real values of x, and find the ranges of values of x for which $y > 1$.

Draw a rough graph of the function $(2x^2 - 4)/(x^2 - x - 2)$.

6. Find the conditions that the expression $ax^2 + bx + c$ should be positive for all real values of x.

Show that the expression $(x^2 - 3ax + 2a^2)/(x^2 - 3x + 2)$, where $a \neq 1$, can assume any real value for real values of x only if $\frac{1}{2} \leqslant a \leqslant 2$.

Show that, if $a = 0$, there will be two extreme values between which the expression cannot lie, and determine these values.

7. Prove that, if $a + b + c = 0$ and $bc + ca + ab + 3m = 0$, then the expression E, where $E = (x^2 + ax + m)(x^2 + bx + m)(x^2 + cx + m)$, will contain no powers of x except those whose index is a multiple of three.

Given that the expression $x^6 + 16x^3 + 64$ has a factor of the form $x^2 - 2x + m$, resolve it into three quadratic factors similar to E, and deduce all the roots of the equation $x^6 + 16x^3 + 64 = 0$.

8. (i) If α, β are the roots of the equation $ax^2 + 2x + b = 0$, evaluate $(a^2\alpha^4 - b^2)(a^2\beta^4 - b^2)$.

(ii) Prove that, whatever real values x may take, the value of $\dfrac{5x^2+8x+5}{4x^2+10x+4}$ cannot lie between ± 1.

9. (i) Show that, if $\dfrac{ax^2+2bx+c}{cx^2+2bx+a}$, where a, b, c are positive, can assume all possible real values, then $(a + c)^2 < 4b^2$.

(ii) Draw a rough graph of the function $\dfrac{4x^2+6x+1}{x^2+3x+2}$ showing clearly the form of the graph when x is numerically large.

Find the range of values of x for which the value of the function is greater than 4.

10. Sum to n terms the series whose rth term is $(2r+1)3^r$. [*Hint.*—Series is an arithmetico-geometric one, and the method of finding the sum is the same as for developing the formula for a G.P.]

11. Determine the coefficients A, B, C, so that, if $f(x)$ denotes the polynomial $Ax^5 + Bx^3 + Cx$, then $f(x) - f(x-1) = (2x-1)^4$, for all values of x.

Find the sum of the fourth powers of the first n *odd* integers (positive).

12. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where n is a positive integer, find the value of $a_0^2 + a_1^2 + \ldots + a_n^2$, and prove that

$$a_1^2 + 2a_2^2 + 3a_3^2 + \ldots + na_n^2 = \frac{n}{2}(a_0^2 + a_1^2 + \ldots + a_n^2).$$

Show also that $a_1 + 2a_2 + 3a_3 + \ldots + na_n = n \cdot 2^{n-1}$.

13. If $A + B + C = \pi$, prove that

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1.$$

A hexagon, two of whose sides are of length $2x$, two of length $2y$, and two of length $2z$, is inscribed in a circle. Prove that the radius r of this circle is given by the equation $r^3 - (x^2 + y^2 + z^2)r - 2xyz = 0$.

14. A, B, C are three points in order on a straight line so that AB $= 2a$, BC $= 2b$. Semicircles are drawn on AB, BC, CA as diameters, on the same side of the line ABC. Show that the radius of the circle drawn to touch all three semicircles is $ab(a+b)/(a^2+ab+b^2)$.

15. A triangle, two of whose sides are a and b $(a > b)$, is inscribed in a circle of diameter d. Show that

(i) $2d^2\Delta = ab(a\sqrt{a^2 - b^2} \pm b\sqrt{b^2 - a^2})$.

(ii) $\angle\text{BCA} = \cos^{-1}\left\{\dfrac{ab \pm (\sqrt{d^2 - a^2})(d^2 - b^2)}{d^2}\right\}$.

16. If AD, BE, CF are the altitudes of a given acute-angled triangle ABC, find the angles and lengths of the triangle DEF.

Prove that, with the usual standard notation,

(i) area of ΔDEF $= 2\Delta\cos A\cos B\cos C$.

(ii) perimeter of ΔDEF $= abc/2R^2$.

17. I_1, I_2, I_3 are the centres of the escribed circles of the triangle ABC, opposite to A, B, C respectively. Show that

(i) The centre of the inscribed circle of the triangle ABC is the orthocentre of the triangle $I_1I_2I_3$.

(ii) $I_1I_2 = 4R\cos\frac{1}{2}C$, where R is the radius of the circle circumscribing triangle ABC.

(iii) The ratio of the area of the triangle ABC to the area of the triangle $I_1I_2I_3$ is $2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ to 1.

18. (i) If the median through the vertex A of the triangle ABC makes angles θ, ϕ respectively with the sides AB and AC, prove that

$$c\sin\theta = b\sin\phi, \text{ and } \tan\tfrac{1}{2}(\phi - \theta) = \frac{c-b}{c+b}\tan\tfrac{1}{2}A.$$

(ii) The inscribed circle of a triangle ABC touches the sides BC, CA, AB in X, Y, Z respectively. I is the centre of the circle. If XI produced meets ZY in L, prove that AL is a median of the triangle.

19. A and B are two fixed marks on one bank of a river at a known distance a apart; C, D are two points on the opposite bank also at a distance a apart, such that CD is parallel to AB and A, B, C, D are in the same horizontal plane. If AB subtends angles α and β at C and D respectively, show that the width of the river is $2a(\cot\alpha + \cot\beta)/\{4 + (\cot\alpha - \cot\beta)^2\}$.

20. Find the general value of θ satisfying the equation $2\cos 3\theta = 1$, and hence find the roots of the equation $8x^3 - 6x - 1 = 0$.

Show that (i) $\sec\dfrac{\pi}{9} + \sec\dfrac{5\pi}{9} + \sec\dfrac{7\pi}{9} = -6$;

(ii) $\sec^2\dfrac{\pi}{9} + \sec^2\dfrac{5\pi}{9} + \sec^2\dfrac{7\pi}{9} = 36$.

21. R is the circumradius of the triangle ABC and r_1, r_2, r_3 are the radii of the escribed circles.

If the distances between the centres of the escribed circles are α, β, γ, prove that

(i) $\Delta ABC = r_1r_2r_3/\sqrt{r_2r_3 + r_3r_1 + r_1r_2}$.

(ii) $$8R = \frac{\alpha\beta\gamma}{\sqrt{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)}}, \text{ where } \sigma = \tfrac{1}{2}(\alpha+\beta+\gamma).$$

22. ABC is an acute-angled triangle; the perpendiculars to BC, CA, AB through A, B, C, respectively, meet the sides at D, E, F, and they are concurrent at H.

Prove that, if R be the radius of the circumscribing circle of the triangle ABC,

(i) $\Delta BHC : \Delta CHA : \Delta AHB = \tan A : \tan B : \tan C$;

(ii) the radius of the circumcircle of ΔDEF is $\frac{1}{2}R$;

(iii) the radius of the inscribed circle of triangle DEF is $2R\cos A\cos B\cos C$.

23. If O, H are the circumcentre and orthocentre respectively of the triangle ABC, prove that $OH^2 = R^2(1 - 8\cos A\cos B\cos C)$, where R is the radius of the circumscribing circle.

ABC is an actute-angled triangle and X, Y, Z are the mid-points of the minor arcs BC, CA, AB of the circumscribing circle. Find the angles of the triangle XYZ and prove that I, the centre of the inscribed circle of triangle ABC, is the orthocentre of the triangle XYZ.

Hence deduce that $OI^2 = R^2(1 - 8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)$.

24. If PN, the perpendicular from $P \equiv (p, q)$ on to the line $lx + my + n = 0$, is produced to P′ so that NP′ = PN, find the co-ordinates of P′, the reflection of the point P in the line.

Find the equation of the reflection of the line $x + y = 1$ in the line $x + 2y = 3$.

25. ABCD is a cyclic quadrilateral. The equations of the sides AB, BC, DA are $x - 3y = 0$, $4x - 3y - 9 = 0$, $x + 4y + 4 = 0$ respectively. If the side BC is of length 5/3 units, find the equations of the two lines along which the side CD can lie.

26. Find the equation of the circle of which the points (x_1, y_1), (x_2, y_2) are the extremities of a diameter.

Find the co-ordinates of the extremities of the diameter perpendicular to the one above.

27. Write down the equation of the perpendicular bisector of the line joining (x_1, y_1), (x_2, y_2).

The equations of the perpendicular bisectors of the sides AB, AC of a triangle ABC are $x + y = 0$ and $x - 2y = 0$, and the side BC passes through the point (1, 2). Show that the locus of A is the circle $x^2 + y^2 - x + 7y = 0$.

28. Show that, for all values of the constants p and q, the circle whose equation is $(x - a)(x - a + p) + (y - b)(y - b + q) = r^2$ bisects the circumference of the circle whose equation is $(x - a)^2 + (y - b)^2 = r^2$.

Find the equation of the circle that bisects the circumference of the circle $x^2 + y^2 + 2y = 3$ and touches the line $(x - y) = 0$ at the origin.

29. The line $lx + my = 0$ bisects at right angles the line joining the points $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$.

Show that $$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{-2(lx_1 + my_1)}{l^2 + m^2}.$$

Show that the locus of a point, which is such that its reflections in two straight lines $lx + my = 0$ and $lx - my = 0$ are collinear with a fixed point (h, k), is the circle whose equation is $(l^2 - m^2)(x^2 + y^2) + (l^2 + m^2)(hx - ky) = 0$.

30. If the equation $x^3 + 3ax^2 + 3bx + c = 0$ has a repeated root, show that this root also satisfies the quadratic equation $x^2 + 2ax + b = 0$; hence show that the value of the repeated root is $\dfrac{c - ab}{2(a^2 - ab)}$.

Solve the equation $4x^3 - 12x^2 - 15x - 4 = 0$.

31. Find by non-graphical methods, the values of x for which the inequalities following hold:

$$\text{(i)}\ \tfrac{1}{6}(x^3 - x^2) > x^2 + 9x + 12.$$

$$\text{(ii)}\ \frac{1}{2 - x} < \frac{1}{x - 3}.$$

Illustrate your solutions, in case (ii) with a rough sketch.

32. Given that α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$; express $\alpha^3 + \beta^3 + \gamma^3$ in terms of p, q, r and show that

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} = \frac{1}{r^3}(3pqr - q^3 - 3r^2).$$

33. If I is the incentre of the triangle ABC and α, β, γ are respectively the angles BIC, CIA, AIB, prove that

$$\frac{a\,.\,IA}{\sin\alpha} = \frac{b\,.\,IB}{\sin\beta} = \frac{c\,.\,IC}{\sin\gamma}.$$

Show that $PA^2 \sin A + PB^2 \sin B + PC^2 \sin C$ takes the same value for all points P on the incircle of the triangle ABC, and find the value of this expression when ABC is an equilateral triangle of side l.