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November 20 - 22, 1994

Santa Fe, New Mexico

Edited by

Shafi Goldwasser

Sponsored by

IEEE Computer Society Technical Committee on Mathematical Foundations of Computing

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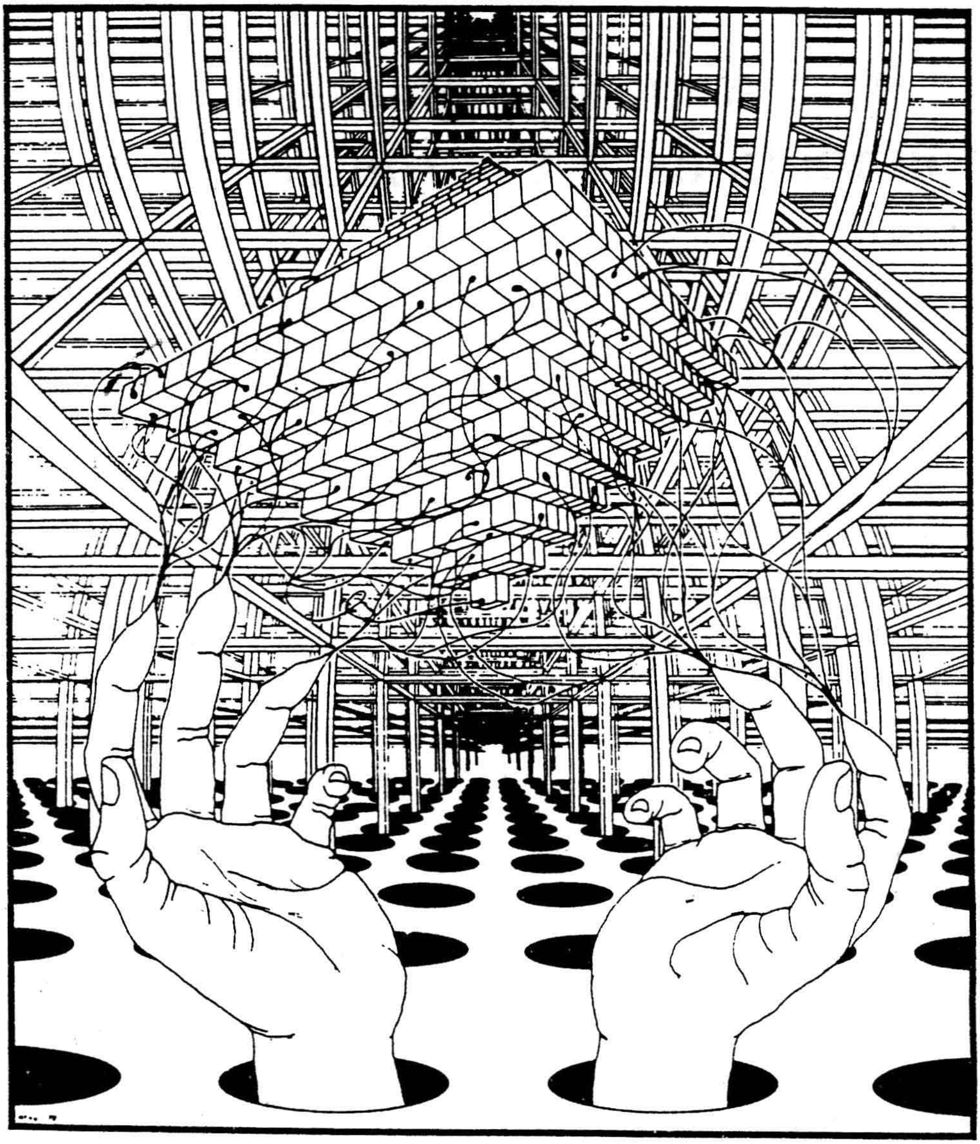

Preface

The papers in this volume were presented at the 1994 IEEE 35th Annual Symposium on Foundations of Computer Science (FOCS'94), sponsored by the IEEE Computer Society Technical Committee on Mathematical Foundations of Computing. The conference was held in Santa Fe, New Mexico from November 20-22, 1994.

The program committee met from June 26-28, 1994, and selected 70 papers from the 235 extended abstracts submitted for review. The submissions were not refereed, and many of them represent reports of continuing research. It is expected that most of these papers will appear in a more complete and polished form in scientific journals in the near future. In addition, Leonard Adleman, Manuel Blum, and Ravi Kannan were invited to give plenary lectures, reprinted in these proceedings.

The committee selected the papers, "Efficient Oblivious Branching Programs for Threshold Functions," by Rakesh Kumar Sinha and Jayram S. Thathacher and, "An Efficient Membership-Query Algorithm for Learning DNF with Respect to the Uniform Distribution," by Jeffrey Jackson, to receive the Machtey Award, given to the best student-authored papers.

The committee wishes to thank all of those who submitted papers for consideration, as well as the individuals who helped with the process of evaluating the extended abstracts. A list of the latter is given on the following page.

The program committee consisted of Avrim Blum, Anne Condon, Oded Goldreich, Shafi Goldwasser, Johan Hästad, Howard Karloff, Laszlo Lovasz, Yishay Mansour, Friedhelm Meyer auf der Heide, Satish Rao, Raimund Seidel, Victor Shoup, Eva Tardos, Moshe Vardi, and Mihalis Yannakakis.

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Table of Contents

Preface xi
Referees xii
Committees xiii

Sunday Session 1A:

Chair: L. Lovasz

Approximate Graph Coloring by Semidefinite Programming 2
D. Karger, R. Motwani, and M. Sudan

Finding Separator Cuts in Planar Graphs Within Twice the Optimal 14
N. Garg, H. Saran, and V.V. Vazirani

Polynomial Time Randomized Approximation Schemes for the Tutte Polynomial of Dense Graphs 24
N. Alon, A. Frieze, and D. Welsh

A Note on the θ Number of Lovász and the Generalized Delsarte Bound 36
M. Szegedy

Sunday Session 1B:

Chair: A. Blum

An Efficient Membership-Query Algorithm for Learning DNF with Respect to the Uniform Distribution 42
J. Jackson

On Learning Discretized Geometric Concepts 54
N.H. Bshouty, Z. Chen, and S. Homer

PAC Learning with Irrelevant Attributes 64
A. Dhagat and L. Hellerstein

The Power of Team Exploration: Two Robots Can Learn Unlabeled Directed Graphs 75
M.A. Bender and D.K. Slonim

Plenary Session

Chair: M. Yannakakis

Algorithmic Number Theory — The Complexity Contribution 88
L.M. Adleman

Sunday Session 2A:

Chair: V. Shoup

On the Power of Quantum Computation 116
D.R. Simon

Algorithms for Quantum Computation: Discrete Logarithms and Factoring 124
P.W. Shor

The Complexity of the Membership Problem for 2-Generated Commutative Semigroups of Rational Matrices 135
J.-Y. Cai, R.J. Lipton, and Y. Zalcstein

Efficient Average-Case Algorithms for the Modular Group 143
J.-Y. Cai, W.H. Fuchs, D. Kozen, and Z. Liu

Sunday Session 2B:

Chair: E. Tardos

Finding the *k* Shortest Paths 154
D. Eppstein

Long Tours and Short Superstrings 166
S.R. Kosaraju, J.K. Park, and C. Stein

Maximum (s,t) — Flows in Planar Networks in $O(|V|\log|V|)$ Time 178
K. Weihe

Estimating the Size of the Transitive Closure in Linear Time 190
E. Cohen

Sunday Session 3A:

Chair: F. Meyer auf der Heide

Rapid Rumor Ramification: Approximating the Minimum Broadcast Time 202
R. Ravi

The Load, Capacity and Availability of Quorum Systems 214
M. Naor and A. Wool

Fast and Lean Self-Stabilizing Asynchronous Protocols 226
G. Itkis and L. Levin

Local Optimization of Global Objectives: Competitive Distributed Deadlock Resolution and Resource Allocation 240
B. Awerbuch and Y. Azar

Sunday Session 3B:

Chair: A. Condon

(De)randomized Construction of Small Sample Spaces in *NC* 252
D.R. Karger and D. Koller

Computing with Very Weak Random Sources 264
A. Srinivasan and D. Zuckerman

Randomness-Efficient Oblivious Sampling 276
M. Bellare and J. Rompel

On the Robustness of Functional Equations 288
R. Rubinfeld

Monday Session 1A:

Chair: J. Hastad

A Lower Bound for the Monotone Depth of Connectivity 302
A.C.-C. Yao

Efficient Oblivious Branching Programs for Threshold Functions 309
R.K. Sinha and J.S. Thathachar

Products and Help Bits in Decision Trees 318
N. Nisan, S. Rudich, and M. Saks

On Rank vs. Communication Complexity 831
N. Nisan and A. Wigderson

Monday Session 1B:

Chair: S. Rao

On the Design of Reliable Boolean Circuits that Contain Partially Unreliable Gates 332
D. Kleitman, T. Leighton, and Y. Ma

Nearly Tight Bounds for Wormhole Routing 347
A. Ranade, S. Schleimer, and D.S. Wilkerson

Scheduling Multithreaded Computations by Work Stealing 356
R.D. Blumofe and C.E. Leiserson

Fast and Feasible Periodic Sorting Networks of Constant Depth 369
M. Kutylowski, B. Oesterdiekhoff, K. Loryś, and R. Wanka

Plenary Session

Chair: S. Goldwasser

Program Result-Checking: A Theory of Testing Meets a Test of Theory 382
M. Blum and H. Wasserman

Monday Session 2A:

Chair: H. Karloff

Beyond Competitive Analysis 394
E. Koutsoupias and C.H. Papadimitriou

A Theory of Competitive Analysis for Distributed Algorithms 401
M. Ajtai, J. Aspnes, C. Dwork, and O. Waarts

On-Line Admission Control and Circuit Routing for High Performance Computing and Communication 412
B. Awerbuch, R. Gawlick, T. Leighton, and Y. Rabani

IP Over Connection-Oriented Networks and Distributional Paging 424
C. Lund, S. Phillips, and N. Reingold

Monday Session 2B:

Chair: O. Goldreich

CS (Computationally-Sound) Proofs 436
S. Micali

On Monotone Formula Closure of SZK 454
A. De Santis, G. Di Crescenzo, G. Persiano, and M. Yung
On the Complexity of Bounded-Interaction and Noninteractive Zero-Knowledge Proofs 466
J. Kilian
Reducibility and Completeness in Multi-Party Private Computations 478
E. Kushilevitz, S. Micali, and R. Ostrovsky

Monday Session 3A:
Chair: M. Yannakakis
"Go With the Winners" Algorithms 492
D. Aldous and U. Vazirani
Randomized Simplex Algorithms on Klee-Minty Cubes 502
B. Gärtner and G.M. Ziegler
Motion Planning on a Graph 511
C.H. Papadimitriou, P. Raghavan, M. Sudan, and H. Tamaki
The Localization Problem for Mobile Robots 521
J.M. Kleinberg

Monday Session 3B:
Chair: V. Shoup
Algebraic Computation Trees in Characteristic $p>0$ 534
M. Ben-Or
An $O(n^{1+\varepsilon} \log b)$ Algorithm for the Complex Roots Problem 540
C.A. Neff and J.H. Reif
Complexity Lower Bounds for Computation Trees with Elementary Transcendental Function Gates 548
D. Grigoriev and N. Vorobjov
On the Computation of Boolean Functions by Analog Circuits of Bounded Fan-In 553
G. Turán and F. Vatan

Tuesday Session 1A:
Chair: L. Lovasz
Expander Codes 566
M. Sipser and D.A. Spielman
The Geometry of Graphs and Some of Its Algorithmic Applications 577
N. Linial, E. London, and Y. Rabinovich
Tail Bounds for Occupancy and the Satisfiability Threshold Conjecture 592
A. Kamath, R. Motwani, K. Palem, and P. Spirakis
Priority Encoding Transmission 604
A. Albanese, J. Blömer, J. Edmonds, M. Luby, and M. Sudan

Tuesday Session 1B:

Chair: M. Vardi

Graph Connectivity and Monadic NP 614
T. Schwentick

A Polynomial-Time Algorithm for Deciding Equivalence of Normed Context-Free Processes 623
Y. Hirshfeld, M. Jerrum, and F. Moller

On the Combinatorial and Algebraic Complexity of Quantifier Elimination 632
S. Basu, R. Pollack, and M.-F. Roy

Set Constraints with Projections are in NEXPTIME 642
W. Charatonik and L. Pacholski

Plenary Session

Chair: L. Lovasz

Markov Chains and Polynomial Time Algorithms .... ~~656~~
R. Kannan

Tuesday Session 2A:

Chair: R. Seidel

A Spectral Approach to Lower Bounds 674
B. Chazelle

Parallel Algorithms for Higher-Dimensional Convex Hulls 683
N.M. Amato, M.T. Goodrich, and E.A. Ramos

More Output-Sensitive Geometric Algorithms 695
K.L. Clarkson

Randomized and Deterministic Algorithms for Geometric Spanners of Small Diameter 703
S. Arya, D.M. Mount, and M. Smid

Tuesday Session 2B:

Chair: S. Rao

A New Efficient Radix Sort 714
A. Andersson and S. Nilsson

Multi-Index Hashing for Information Retrieval 722
D. Greene, M. Parnas, and F. Yao

Optimizing Static Calendar Queues 732
K.B. Erickson, R.E. Ladner, and A. LaMarca

Fully Dynamic Cycle-Equivalence in Graphs 744
M. Rauch Henzinger

Tuesday Session 3A:

Chair: E. Tardos

Maximum Agreement Subtree in a Set of Evolutionary Trees — Metrics and Efficient Algorithms 758
D. Keselman and A. Amir

Optimal Evolutionary Tree Comparison by Sparse Dynamic Programming 770
M. Farach and M. Thorup

Tractability of Parameterized Completion Problems on Chordal and Interval Graphs: Minimum Fill-In and Physical Mapping 780
H. Kaplan, R. Shamir, and R.E. Tarjan

Tuesday Session 3B:
Chair: A. Condon

Lower Bounds on Hilbert's Nullstellensatz and Propositional Proofs 794
P. Beame, R. Impagliazzo, J. Krajíček, T. Pitassi, and P. Pudlák

Measure on Small Complexity Classes, with Applications for BPP 807
E. Allender and M. Strauss

On Syntactic versus Computational Views of Approximability 819
S. Khanna, R. Motwani, M. Sudan, and U. Vazirani

Author Index 837

Sunday
Session 1A

Chair: L. Lovasz

Approximate Graph Coloring by Semidefinite Programming

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Abstract

We consider the problem of coloring k-colorable graphs with the fewest possible colors. We give a randomized polynomial time algorithm which colors a 3-colorable graph on n vertices with $\min\{O(\Delta^{1/3}\log^{4/3}\Delta), O(n^{1/4}\log n)\}$ *colors where* Δ *is the maximum degree of any vertex. Besides giving the best known approximation ratio in terms of n, this marks the first non-trivial approximation result as a function of the maximum degree* Δ. *This result can be generalized to k-colorable graphs to obtain a coloring using* $\min\{\tilde{O}(\Delta^{1-2/k}), \tilde{O}(n^{1-3/(k+1)})\}$ *colors. Our results are inspired by the recent work of Goemans and Williamson who used an algorithm for* semidefinite optimization problems. *which generalize linear programs, to obtain improved approximations for the MAX CUT and MAX 2-SAT problems. An intriguing outcome of our work is a duality relationship established between the value of the optimum solution to our semidefinite program and the Lovász* ϑ*-function. We show lower bounds on the gap between the optimum solution of our semidefinite program and the actual chromatic number; by duality this also demonstrates interesting new facts about the* ϑ*-function.*

1 Introduction

A legal vertex coloring of a graph $G(V,E)$ is an assignment of colors to its vertices such that no two adjacent vertices receive the same color. Equivalently, a legal coloring of G by k colors is a partition of its vertices into k independent sets. The minimum number of colors needed for such a coloring is called the chromatic number of G, and is usually denoted by $\chi(G)$. Determining the chromatic number of a graph is known to be NP-hard (cf. [19]).

Besides its theoretical significance as a canonical NP-hard problem, graph coloring arises naturally in a variety of applications such as register allocation [11, 12, 13] and timetable/examination scheduling [8, 40]. In many applications which can be formulated as graph coloring problems, it suffices to find an *approximately optimum* graph coloring—a coloring of the graph with a small though non-optimum number of colors. This along with the apparent impossibility of an exact solution has led to some interest in the problem of approximate graph coloring. The analysis of approximation algorithms for graph coloring started with the work of Johnson [25] who shows that a version of the greedy algorithm gives an $O(n/\log n)$-approximation algorithm for k-coloring. Wigderson [39] improved this bound by giving an elegant algorithm which uses $O(n^{1-1/(k-1)})$ colors to legally color a k-colorable graph. Subsequently, other polynomial time algorithms were provided by Blum [9] which use $O(n^{3/8}\log^{8/5} n)$ colors to legally color an n-vertex 3-colorable graph. This result generalizes to coloring a k-colorable graph with $O(n^{1-1/(k-4/3)}\log^{8/5} n)$ colors. The best known performance guarantee for general graphs is due to Halldórsson [24] who provided a polynomial time algorithm using a number of colors which is within a factor of $O(n(\log\log n)^2/\log^3 n)$ of the optimum.

Recent results in the hardness of approximations indicate that it may be not possible to substantially improve the results described above. Lund and Yannakakis [33] used the results of Arora, Lund, Motwani, Sudan, and Szegedy [6] and Feige, Goldwasser, Lovász, Safra, and Szegedy [17] to show that there exists a (small) constant $\epsilon > 0$ such that no polynomial time algorithm can approximate the chromatic number of a graph to within a ratio of n^ϵ unless P = NP. Recently, Bellare and Sudan [7] showed that the exponent ϵ in the hardness result can be improved to $1/10$ unless NQP $\neq$ co-RQP, and to $1/13$ unless NP = co-RP. However, none of these hardness results apply to the special case of the problem where the input graph is guaranteed to be k-colorable for some small k. The best hardness result in this direction is due to Khanna, Linial, and Safra [26] who show

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that it is not possible to color a 3-colorable graph with 4 colors in polynomial time unless P = NP.

In this paper we present improvements on the result of Blum. In particular, we provide a randomized polynomial time algorithm which colors a 3-colorable graph of maximum degree Δ with $\min\{\tilde{O}(\Delta^{1/3}), O(n^{1/4}\log n)\}$ colors; moreover, this can be generalized to k-colorable graphs to obtain a coloring using $\tilde{O}(\Delta^{1-2/k})$ or $\tilde{O}(n^{1-3/(k+1)})$ colors. Besides giving the best known approximations in terms of n, our results are the first non-trivial approximations given in terms of Δ. Our results are based on the recent work of Goemans and Williamson [20] who used an algorithm for *semidefinite optimization problems* (cf. [22, 2]) to obtain improved approximations for the MAX CUT and MAX 2-SAT problems. We follow their basic paradigm of using algorithms for semidefinite programming to obtain an optimum solution to a relaxed version of the problem, and a randomized strategy for "rounding" this solution to a feasible but approximate solution to the original problem. Motwani and Naor [35] have shown that the approximate graph coloring problem is closely related to the problem of finding a CUT COVER of the edges of a graph. Our results can be viewed as generalizing the MAX CUT approximation algorithm of Goemans and Williamson to the problem of finding an approximate CUT COVER. In fact, our techniques also lead to improved approximations for the MAX k-CUT problem [18]. We also establish a duality relationship between the value of the optimum solution to our semidefinite program and the Lovász ϑ-function [22, 23, 31]. We show lower bounds on the gap between the optimum solution of our semidefinite program and the actual chromatic number; by duality this also demonstrates interesting new facts about the ϑ-function.

Alon and Kahale [4] use related techniques to devise a polynomial time algorithm for 3-coloring random graphs drawn from a "hard" distribution on the space of all 3-colorable graphs. Recently, Frieze and Jerrum [18] have used a semidefinite programming formulation and randomized rounding strategy essentially the same as ours to obtain improved approximations for the MAX k-CUT problem with large values of k. Their results required a more sophisticated version of our analysis, but for the coloring problem our results are tight up to poly-logarithmic factors and their analysis does not help to improve our bounds.

Semidefinite programming relaxations are an extension of the linear programming relaxation approach to approximately solving NP-complete problems. We thus present our work in the style of the classical LP-relaxation approach. We begin in Section 2 by defining a relaxed version of the coloring problem. Since we use a more complex relaxation than standard linear programming, we must show that the relaxed problem can be solved; this is done in Section 3. We then show relationships between the relaxation and the original problem. In Section 4, we show that (in a sense to be defined later) the value of the relaxation bounds the value of the original problem. Then, in Sections 5, 6, 7, and 8 we show how a solution to the relaxation can be "rounded" to make it a solution to the original problem. Combining the last two arguments shows that we can find a good approximation. Section 3, Section 4, and Sections 5–8 are in fact independent and can be read in any order after the definitions in Section 2. In Section 9, we investigate the relationship between vector colorings and the Lovász ϑ-function, showing that they are in fact dual to one another. We investigate the approximation error inherent in our formulation of the chromatic number via semi-definite programming in Section 10.

2 A Vector Relaxation of Coloring

In this section, we describe the relaxed coloring problem whose solution is in turn used to approximate the solution to the coloring problem. Instead of assigning colors to the vertices of a graph, we consider assigning (n-dimensional) unit *vectors* to the vertices. To capture the property of a coloring, we aim for the vectors of adjacent vertices to be "different" in a natural way. The *vector k-coloring* that we define plays the role that a hypothetical "fractional k-coloring" would play in a classical linear-programming relaxation approach to the problem. Our relaxation is related to the concept of an orthonormal representation of a graph [31, 22].

Definition 2.1 *Given a graph $G = (V, E)$ on n vertices, a* vector k-coloring *of G is an assignment of unit vectors u_i from the space $\Re^n$ to each vertex $i \in V$, such that for any two adjacent vertices i and j the dot product of their vectors satisfies the inequality*

$$\langle u_i, u_j \rangle \leq -\frac{1}{k-1}.$$

The definition of an *orthonormal representation* [31, 22] requires that the given dot products be equal to zero, a weaker requirement than the one above.

3 Solving the Vector Coloring Problem

In this section we show how the vector coloring relaxation can be solved using semidefinite programming. The methods in this section closely mimic those of Goemans and Williamson [20].

To solve the problem, we need the following auxiliary definition.