Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

1417

A.J. Sommese A. Biancofiore
E.L. Livorni (Eds.)

Algebraic Geometry

Proceedings, L'Aquila 1988

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

1417

A.J. Sommese A. Biancofiore
E.L. Livorni (Eds.)

Algebraic Geometry

Proceedings of the International Conference
held in L'Aquila, Italy, May 30–June 4, 1988

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo Hong Kong

Editors

Andrew John Sommese
Department of Mathematics, University of Notre Dame
Notre Dame, Indiana 46556, USA

Aldo Biancofiore
Elvira Laura Livorni
Dipartimento di Matematica, Università degli Studi di L'Aquila
67100 L'Aquila, Italia

Mathematics Subject Classification (1980): 14J99, 14N05, 14M99, 14C99

ISBN 3-540-52217-4 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-52217-4 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210 – Printed on acid-free paper

Introduction

The question of how the geometry of a projective variety is determined by its hyperplane sections has been an attractive area of algebraic geometry for at least a century. A century ago Picard's study of hyperplane sections led him to his famous theorem on the 'regularity of the adjoint '. This result, which is the Kodaira vanishing theorem in the special case of very ample line bundles on smooth surfaces, has led to many developments to this day. Castelnuovo and Enriques related the first Betti number of a variety and its hyperplane section. This and Picard's work led to the Lefschetz hyperplane section theorem and the modern work on ampleness and connectivity. A large part of the study of hyperplane sections has always been connected with the classification of projective varieties by projective invariants. Recent new methods, such as the adjunction mappings developed to study hyperplane sections, have led to beautiful general results in this classification. The papers in this proceedings of the L'Aquila Conference capture this lively diversity. They will give the reader a good picture of the currently active parts of the field.The papers can only hint at the friendly 'give and take' that punctuated many talks and at the mathematics actively discussed during the conference.

The success of this conference was in large part due to the Scientific and Organizing Committe: Professor Mauro Beltrametti (Genova), Professor Aldo Biancofiore (L'Aquila), Professor Antonio Lanteri (Milano), and Professor Elvira Laura Livorni (L'Aquila). The publication of this proceedings would not have been possible except for the efforts of Professor E.L.Livorni.

Andrew J. Sommese

Table of contents

INFINITESIMAL DEFORMATIONS OF NEGATIVE WEIGHTS AND HYPERPLANE SECTIONS

Lucian Bădescu

Introduction

Consider the following:

Problem. Let (Y,L) be a normal polarized variety over an algebraically closed field k, i.e. a normal projective variety Y over k together with an ample line bundle L on Y. Then one may ask under which conditions the following statement holds:

(#) Every normal projective variety X containing Y as an ample Cartier divisor such that the normal bundle of Y in X is L, is isomorphic to the projective cone over (Y,L), and Y is embedded in X as the infinite section.

Recall that the projective cone over (Y,L) is by definition the projective variety $C(Y,L) = \mathrm{Proj}(S[T])$, where S is the graded k-algebra $S(Y,L) = \bigoplus_{i=o}^{\infty} H^{o}(Y,L^{i})$ associated to (Y,L), and the polynomial S-algebra S[T] (with T an indeterminate) is graded by $\deg(sT^{i}) = \deg(s)+i$ whenever $s \in S$ is homogeneous. The infinite section of C(Y,L) is by definition the subvariety $V_{+}(T)$, and it is isomorphic to Y.

This problem has classical roots (see [3] for some historical hints). In [1], [2], [3] and [4], among other things, we produced several examples of polarized varieties (Y,L) satisfying (#). If Y is smooth of dimension $\geqslant 2$, and if T_Y is the tangent bundle of Y, Fujita subsequently proved in [6] the following general criterion: (Y,L) satisfies (#) if $H^{1}(Y,T_{Y}\otimes L^{i}) = o$ for every $i < o$.

In this paper we prove two main results. The first one (which is in the spirit of [4]) considers the case where Y has singularities, and is a criterion for (Y,L) to satisfy (#). This criterion (see theorem 1 in §1) improves a result of [4] and involves the space of first order infinitesimal deformations of the k-algebra S(Y,L). In §2 we apply it to check that the singular Kummer varieties of dimension $\geqslant 3$ and the symmetric products of certain varieties satisfy (#) with respect to any ample line bundle. In §3 we make a few remarks when Y is smooth and state an open question. It should be noted that in the first two sections the Schlessinger's deformation theory (see [18], [19]) plays an essential role.

The second main result (see theorem 6 in §4) shows that if Y is a P^n-bundle ($n \geqslant 1$) over a smooth projective curve B of positive genus, and if X is a normal singular projective variety containing Y as an ample Cartier divisor, then X is isomorphic to the cone C(Y,L). The case $B = P^1$ was discussed in [3], while the case when X is smooth (and B of arbitrary genus), in [1] and [2]. Putting these results together, we get a complete description of all normal projective varieties containing a P^n-bundle over a curve as an ample Cartier divisor (see theorem 7 in §4).

Unless otherwise specified, the terminology and the notations used are standard.

§1. The first main result

In the set-up and notations of the above problem, the graded k-algebra S = S(Y,L) is finitely generated because L is ample (see e.g. [8], chap. III). Let $a_1,\dots,a_n$ be a minimal system of homogeneous generators of S/k, and denote by $k[T_1,\dots,T_n]$ the polynomial k-algebra in n indeterminates $T_1,\dots,T_n$, graded by the conditions that $\deg(T_i) = \deg(a_i) = q_i$ for every $i = 1,\dots,n$. Then S is isomorphic (as a graded k-algebra) to $k[T_1,\dots,T_n]/I$ in such a way that a_i corresponds to $T_i \bmod I$ for every $i = 1,\dots,n$ (where I is the kernel of the homomorphism mapping T_i to a_i). Let $f_1,\dots,f_r$ be a minimal system of homogeneous generators of I, and set:

$$(1) \qquad d = \max(d_1,\dots,d_r), \text{ where } d_i = \deg(f_i).$$

Theorem 1. In the above notations assume the following:

i) $H^1(Y,L^i) = o$ for every $i \in \mathbb{Z}$, or equivalently, $\operatorname{depth}(S_{S_+}) \geqslant 3$, where S_+ is the irrelevant maximal ideal of S.

ii) $T^1_S(-i) = o$ for every $1 \leqslant i \leqslant d$, where d is given by (1), $T^1_S = T^1(S/k,S)$ is the space of first order infinitesimal deformations of the k-algebra S, and $T^1_S = \bigoplus_{i\in\mathbb{Z}} T^1_S(i)$ is the decomposition arising from the G_m-action of the graded k-algebra S (see [18], [17]).

Then the property (#) holds for (Y,L).

Proof. Let X be a normal projective variety containing Y as an ample Cartier divisor such that $O_X(Y)\otimes O_Y \cong L$. Let $t \in H^\circ(X,O_X(Y))$ be a global equation of Y in X, i.e. $\operatorname{div}_X(t) = Y$. Denote by S' the graded k-algebra $S(X,O_X(Y)) = \bigoplus_{i=o}^{\infty} H^{\circ}(X,O_X(iY))$. Then using the standard exact sequence

$$o \longrightarrow O_X((i-1)Y) \xrightarrow{\ t\ } O_X(iY) \longrightarrow L^i \longrightarrow o,$$

the hypothesis i), and a theorem of Severi-Zariski-Serre saying that $H^1(X,O_X(iY)) = o$ for every $i \ll o$, one immediately sees that $S'/tS' \cong S$ (isomorphism of graded k-algebras, where $\deg(t) = 1$).

Then choose $b_1,\dots,b_n \in S'$ homogeneous elements of degrees $q_1,\dots,q_n$ respectively, such that $b_i \bmod tS' = a_i$, $i = 1,\dots,n$. Then $S' = k[b_1,\dots,b_n,t]$. Denote by P the polynomial k-algebra $k[T_1,\dots,T_n,T]$ in n+1 indeterminates $T_1,\dots,T_n,T$, graded by $\deg(T_i) = q_i$, $i = 1,\dots,n$, and $\deg(T) = 1$. For every $m \geqslant 1$ set $S^m = S'/t^mS'$, and consider the surjective homomorphism $\beta_m : P \longrightarrow S^m$ such that $\beta_m(T_i) = b_i'$, $i = 1,\dots,n$, and $\beta_m(T) = t'$, where for every $b \in S'$ we have denoted by b' the element b mod t^mS'. Let $F_1,\dots,F_s$ be a system of homogeneous generators of the ideal $J = \mathrm{Ker}(\beta_m)$, and put $e_i = \deg(F_i)$, $i = 1,\dots,s$.

Now, according to [18], §1 (or also [14]), we can consider:

- The S^m-module $Ex(S^m/k,S)$ of all isomorphism classes of extentions of S^m over k by the S^m-module $S = S^m/t'S^m$. Recall that an extention of S^m/k by S is a k-algebra E together with a surjective homomorphism of k-algebras $E \longrightarrow S^m$ whose kernel is a square-zero ideal of E, isomorphic as an S^m-module to S.

- The S^m-module $T^1(S^m/k,S)$ defined by the following exact sequence

$$(2) \quad \mathrm{Der}_k(P,S) \xrightarrow{\ u\ } \mathrm{Hom}_{S^m}(J/J^2,S) \longrightarrow T^1(S^m/k,S) \longrightarrow o,$$

where $\mathrm{Der}_k(P,S)$ is the S^m-module of all k-derivations of P in S, and u is defined in the following way: if $D \in \mathrm{Der}_k(P,S)$ then u(D) is the element of $\mathrm{Hom}_{S^m}(J/J^2,S)$ defined by the restriction D/J (which necessarily vanishes on J^2). It turns out that $T^1(S^m/k,S)$ is independent of the choice of the presentation P/J of S^m.

Now, the point is that there is a canonical isomorphism of S^m-modules (see [18], theorem 1, page 12, or also [14], page 41o):

$$(3) \qquad \mu : Ex(S^m/k,S) \xrightarrow{\ \sim\ } T^1(S^m/k,S).$$

Since S^m is a graded k-algebra, $T^1(S^m/k,S)$ has a natural gradation $T^1(S^m/k,S) = \bigoplus_{i\in\mathbb{Z}} T^1(S^m/k,S)(i)$ arising from the G_m-action of S^m (see [17], page 19).

Coming back to our situation, consider the element of $Ex(S^m/k,S)$ given by the exact sequence

$$(a_m) \qquad 0 \longrightarrow S \cong t^m S'/t^{m+1}S' \longrightarrow S^{m+1} \longrightarrow S^m \longrightarrow 0.$$

We need to compute $\mu(a_m) \in T^1(S^m/k,S)$ explicitly. By the definition of the isomorphism μ (see [18]), we need to consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J/J^2 & \longrightarrow & P/J^2 & \longrightarrow & P/J & \longrightarrow & 0 \\
 & & \downarrow v & & \downarrow v' & & \downarrow & & \\
0 & \longrightarrow & t^m S'/t^{m+1}S' \overset{w}{\cong} S & \longrightarrow & S^{m+1} & \longrightarrow & S^m & \longrightarrow & 0
\end{array}$$

where v' is the map deduced from β_m. Thus $v(F_i \bmod J^2) = t^m G_i(b_1,\dots,b_n,t) \bmod t^{m+1}S'$, with $G_i(b_1,\dots,b_n,t) \in S'_{e_i-m}$ homogeneous of degree e_i-m. Then $w \circ v \in \mathrm{Hom}_{S^m}(J/J^2,S)$ corresponds to the vector $(G'_1,\dots,G'_s)$, with $G'_i = w(t^m G_i(b_1,\dots,b_n,t) \bmod t^{m+1}S') = G_i(a_1,\dots,a_n,0)$, and recalling the exact sequence (2) we have $\mu(a_m) =$ class of $w \circ v \in T^1(S^m/k,S)$.

According to the explicit description of the gradation of $T^1(S^m/k,S)$ given in [17], page 19, the elements of $T^1(S^m/k,S)(j)$ of degree j correspond to those elements of $\mathrm{Hom}_{S^m}(J/J^2,S)$ given by vectors $(h_1,\dots,h_s)$ with $h_i \in S_{e_i+j}$ homogeneous of degree e_i+j, $i = 1,\dots,s$. Since $\deg(G'_i) = e_i-m$, the foregoing discussion implies:

$$(4) \qquad \mu(a_m) \in T^1(S^m/k,S)(-m) \quad \text{for every } m \geq 1.$$

Now take $m = 1$. Since $S^1 = S$, it follows that $\mu(a_1) = 0$ by hypothesis ii). But the trivial extension of $\mathrm{Ex}(S/k,S)$ is

$$0 \longrightarrow S \cong TS[T]/(T^2) \longrightarrow S[T]/(T^2) \longrightarrow S[T]/(T) \cong S \longrightarrow 0,$$

and therefore there is an isomorphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S \cong TS[T]/(T^2) & \longrightarrow & S[T]/(T^2) & \longrightarrow & S[T]/(T) \cong S & \longrightarrow & 0 \\
 & & \| & & \downarrow \wr & & \downarrow \wr & & \\
0 & \text{---} & S \cong t'S^2 & \text{---} & S^2 & \text{---} & S & \text{---} & 0
\end{array}$$

such that the vertical isomorphism in the middle maps $T \bmod (T^2)$ into t'.

Assume now that we know that for some m, $2 \leq m \leq d$, there is an isomorphism $S[T]/(T^i) \cong S^i$ for every $1 \leq i \leq m$, which maps $T \bmod (T^i)$ into $t' = t \bmod t^i S'$. Then recall that there is a general exact sequence (see [18])

$$T^1(S^m/S,S) \longrightarrow T^1(S^m/k,S) \longrightarrow T^1(S/k,S),$$

where the maps are homogeneous and the second map corresponds to the inclusion $S \hookrightarrow S^m$ obtained by composing the natural inclusion $S \hookrightarrow S[T]/(T^m)$ with the isomorphism $S[T]/(T^m) \cong S^m$. Using this and hypothesis ii) we infer that the map $T^1(S^m/S,S)(-m) \longrightarrow T^1(S^m/k,S)(-m)$ is surjective, wich together with (4) implies that the extension (a_m) comes from $Ex(S^m/S,S) \cong T^1(S^m/S,S)$. In other words, S^{m+1} is an S-algebra and the canonical surjective map $S^{m+1} \longrightarrow S^m$ is a map of S-algebras. Then we can easily define an isomorphism of extensions

$$\begin{array}{ccccccccc}
o & \longrightarrow & S \cong T^m S[T]/(T^{m+1}) & \longrightarrow & S[T]/(T^{m+1}) & \longrightarrow & S[T]/(T^m) & \longrightarrow & o \\
 & & \| & & \downarrow & & \downarrow \wr & & \\
o & \longrightarrow & S = t^m S'/t^{m+1}S' & \longrightarrow & S^{m+1} & \longrightarrow & S^m & \longrightarrow & o
\end{array}$$

where the middle vertical isomorphism is the homomorphism of S-algebras mapping T mod(T^{m+1}) into $t' = t$ mod $t^{m+1}S'$.

Summing up, we have proved by induction on m that there is an isomorphism of graded k-algebras $S[T]/(T^{d+1}) \cong S^{d+1}$ such that T mod(T^{d+1}) corresponds to t mod $t^{d+1}S'$. In particular, there is a commutative diagram

$$\begin{array}{ccc}
S & \xrightarrow{\;h\;} & S^{d+1} = S'/t^{d+1}S' \\
 & \underset{id}{\searrow\!\!\searrow} \quad \swarrow \text{canonical surjection} & \\
 & S = S^1 &
\end{array}$$

Choose homogeneous elements $c_i \in S'_{q_i}$ such that $h(a_i) = c_i \bmod t^{d+1}S'$, $i = 1,\ldots,n$. Then we claim that

(5) $\qquad f_i(c_1,\ldots,c_n) = o$ for every $i = 1,\ldots,r$.

Indeed, since $f_i(c_1,\ldots,c_n) \bmod t^{d+1}S' = f_i(h(a_1),\ldots,h(a_n)) = h(o) = o$, it follows that $f_i(c_1,\ldots,c_n) \in t^{d+1}S'$ for every $i = 1,\ldots,r$. If for some i we would have $f_i(c_1,\ldots,c_n) \neq o$, it would follow that $d_i = \deg(f_i(c_1,\ldots,c_n)) \geqslant d+1$, a contradiction because $d = \max(d_1,\ldots,d_r)$.

Finally, using (5) we can construct a homomorphism of graded k-algebras $f:S \longrightarrow S'$ by putting $f(a_i) = c_i$, $i = 1,\ldots,n$. The equations (5) show that this definition is correct. Then we get a unique homomorphism of graded k-algebras $g:S[T] \longrightarrow S'$ such that $g/S = f$ and $g(T) = t$. Then it is clear that g is surjective, and hence an isomorphism, because both $S[T]$ and S' are domains of the same dimension. In other words, we have proved that $X \cong C(Y,L)$. Q.E.D.

Remarks. 1) Theorem 1 had been proved in [4] in the stronger hypothesis that $T_S^1 = o$, where we had in mind an application to weighted projective spaces.

2) Unfortunately, the hypothesis i) of theorem 1 is quite restrictive. We do not know whether theorem 1 remains still valid if one drops hypothesis i), even if one assumes for example that char(k) = o and $T^1(-i) = o$ for every $i \geq 1$.

Corollary 1. In the notations of theorem 1, assume that ii) holds. Let X be a normal projective variety containing Y as an ample Cartier divisor such that the normal bundle of Y in X is L and $H^1(X, O_X(iY)) = o$ for every $i \geq o$. Then X is isomorphic to the projective cone C(Y,L) and Y is embedded in X as the infinite section.

Indeed, the exact sequence from the beginning of the proof of theorem 1 together with the hypothesis that $H^1(X, O_X(iY)) = o$ for every $i \geq o$ imply that $S'/tS' \cong S$ (in the proof of theorem 1 the hypothesis i) was used only to deduce this isomorphism).

Another immediate consequence of the proof of theorem 1 is the following purely algebraic result:

Corollary 2. Let $S = k[T_1,\dots,T_n]/I$ be an $\mathbb{N}$-graded k-algebra, where the polynomial k-algebra $k[T_1,\dots,T_n]$ in the indeterminates $T_1,\dots,T_n$ is graded by $\deg(T_i) = q_i > o$, $i = 1,\dots,n$, for some fixed system of weights $(q_1,\dots,q_n)$, and I is the ideal generated by some homogeneous polynomials $f_1,\dots,f_r$ of positive degrees. Let S' be an $\mathbb{N}$-graded k-algebra such that S'/tS' is isomorphic to S as a graded k-algebra for some non-zero divisor $t \in S'$ of degree 1. If $T_S^1(-i) = o$ for every $1 \leq i \leq \max(\deg(f_1),\dots,\deg(f_r))$, then S' is isomorphic (as a graded k-algebra) to the polynomial S-algebra S[T] in such a way that t is mapped into T.

§2. Applications of theorem 1

The tools for verifying hypotheses of type ii) of theorem 1 have been developed by Schlessinger in [19]. The lemma 1 below (which is essentially due to Schlessinger) provides examples of singular normal polarized varieties (Y,L) satisfying the condition ii) of theorem 1.

Start with a smooth projective variety V and a finite group G acting on V. Denote by Y the quotient variety V/G and by $f: V \longrightarrow Y$ the canonical morphism. Let L be an ample line bundle on Y and set M = = f*(L). Since f is a finite morphism, M is also ample. Let S = S(Y,L) and $A = S(V,M)$ be the graded k-algebras associated to (Y,L) and (V,M) respectively.

Lemma 1. In the above notations assume the following:

i) $\mathrm{Dim}(V) \geqslant 3$ and char(k) is either zero, or prime to the order $/G/$ of G.

ii) G acts freely on V ouside some closed G-invariant subset of V of codimension $\geqslant 3$.

iii) $H^1(V,M^{-i}) = o$ for every $i \geqslant 1$ (in characteristic zero this is always fulfilled by Kodaira's vanishing theorem).

iv) $H^1(V,T_V \otimes M^{-i}) = o$ for every $i \geqslant 1$, where T_V is the tangent bundle of V.

Then $T^1_S(-i) = o$ for every $i \geqslant 1$.

Proof. Since lemma 1 is not given in [19] in this form, we include its proof for the convenience of the reader. From ii) we infer that the singular locus of Y, Sing(Y), is of codimension $\geqslant 3$, and that f is étale outside Sing(Y). Using this, the normality of Y and [16], §7, it follows that $f_*(M^i)^G = L^i$ for every $i \geqslant o$. This shows that G acts on A by automorphisms of graded k-algebras and that the k-algebra of invariants A^G coincides with S. Consider the cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(A)-(A_+) = W & \xrightarrow{\quad g \quad} & U = \mathrm{Spec}(S)-(S_+) = W/G \\ \Big\downarrow q & & \Big\downarrow p \\ V & \xrightarrow{\quad f \quad} & Y = V/G \end{array}$$

with q and p the canonical projections of the G_m-bundles W and U respectively (see [8], chap. II, §8). If F is the ramification locus of f, then $q^{-1}(F)$ is the ramification locus of g, and hence g acts freely on W ouside a closed G-invariant subset of W. In particular, the singular locus Z of U is of codimension $\geqslant 3$ in U. Then by [19] and [2o] we get that $T_U = g_*(T_W)^G$, where T_U is the tangent sheaf of U. Taking into account of hypothesis i) we infer that T_U is a direct summand of $g_*(T_W)$, and in particular

(6) $H^1(U,T_U)$ is a direct summand of $H^1(U,g_*(T_W)) \cong H^1(W,T_W)$.

On the other hand, it is well known that there is a canonical exact sequence (see e.g. [14] or [21])

$$o \longrightarrow O_W \longrightarrow T_W \longrightarrow q^*(T_V) \longrightarrow o$$

which yields the exact sequence

(7) $H^1(W,O_W) \longrightarrow H^1(W,T_W) \longrightarrow H^1(W,q^*(T_V))$

One has the natural isomorphisms $H^1(W,O_W) \cong \bigoplus_{i\in\mathbb{Z}} H^1(V,M^i)$ and $H^1(W,q^*(T_V)) \cong \bigoplus_{i\in\mathbb{Z}} H^1(V,T_V\otimes L^i)$, which give natural gradings on $H^1(W,O_W)$ and on $H^1(W,q^*(T_V))$ respectively. On the other hand, the middle space in (7) has also a natural gradation $H^1(W,T_W) = = \bigoplus_{i\in\mathbb{Z}} H^1(W,T_W)(i)$ arising from the G_m-action on W, and all these three gradations are compatible with the maps in (7). Therefore, using hypotheses iii) and iv) we get that $H^1(W,T_W)(i) = o$ for every $i<o$. There is also a natural gradation $H^1(U,T_U) = \bigoplus_{i\in\mathbb{Z}} H^1(U,T_U)(i)$ arising from the G_m-action on U,and this gradation is compatible via (6) with the gradation of $H^1(W,T_W)$. Consequently we get:

$$(8) \qquad H^1(U,T_U)(i) = o \quad \text{for every } i<o$$

Since U has only quotient singularities in codimension $\geqslant 3$, by [19] and [2o] we infer that all the singularities of U are rigid, and in particular, $\mathrm{depth}_Z(T_U)\geqslant 3$. Then the exact sequence of local cohomology shows that the restriction map $H^1(U,T_U) \longrightarrow H^1(U-Z,T_U)$ is an isomorphism.

Finally, since U has only quotient (and hence Cohen-Macaulay) singularities and $\mathrm{codim}_U(Z)\geqslant 3$, by [19] and [2o] we get $T^1_S \cong \cong H^1(U-Z,T_U)$. Recalling (8) and the isomorphism $H^1(U-Z,T_U) \cong H^1(U,T_U)$ we get the conclusion of lemma 1. Q.E.D.

Now we illustrate how theorem 1 can be applied -via lemma 1 - on some examples. First we apply theorem 1 to the singular Kummer varieties of dimension $\geqslant 3$. Recall that a singular Kummer variety Y is a variety of the form V/G, where V is an abelian variety of dimension $d\geqslant 2$ and $G\subset \mathrm{Aut}(V)$ is the subgroup of order 2 generated by the involution $u:V \longrightarrow V$ defined by $u(x) = -x$ for every $x\in V$ (where $-x$ is the inverse of x in the group-law of V). If $\mathrm{char}(k) \neq 2$, there are exactly 2^{2d} points of order 2 on V (see [16]), and hence Y = V/G has exactly 2^{2d} isolated singularities (which are all quotient singularities). Now we have:

<u>Theorem 2. Let Y be a singular Kummer variety of dimension $d\geqslant 3$ and let L be an arbitrary ample line bundle on Y. If char(k) $\neq$ 2 then the property (#) holds for (Y,L).</u>

<u>Proof.</u> We first show that lemma 1 implies that $T^1_S(-i) = o$ for every $i\geqslant 1$, with S = S(Y,L). Indeed, the hypotheses i) and ii) of lemma 1 are clearly satisfied, while iii) and iv) follow using the fact that the tangent bundle of an abelian variety is trivial, together with the fact that the Kodaira's vanishing theorem holds for an abelian variety

in arbitrary characteristic (see [16], §16).

It remains to check that $H^1(Y,L^i) = o$ for every $i \in \mathbb{Z}$ (which is the first hypothesis of theorem 1). If $f:V \longrightarrow Y$ is the canonical morphism, then by [19], L^i is a direct summand of $f_*f^*(L^i)$ because $char(k) \neq 2 = /G/$, and hence $H^1(Y,L^i)$ is a direct summand of $H^1(Y,f_*f^*(L^i)) \cong H^1(V,f^*(L^i))$. By [16], §16 the latter space is zero for every $i \neq o$ because $f^*(L)$ is ample. On the other hand, if $i = o$, according to Schlessinger [19], page 24, we infer that $H^1(Y,O_Y) = H^1(V,O_V)^G$, and G acts on $H^1(V,O_V)$ by $t \longrightarrow -t$. It follows that $H^1(Y,O_Y) = o$. Applying theorem 1 we get the conclusion. Q.E.D.

Further examples of singular normal varieties satisfying (#) with respect to any ample line bundle are the symmetric products of certain smooth projective varieties. Let Z be a smooth projective variety of dimension $d \geq 3$, and let Y be the symmetric product $Z^{(n)} = V/G$, where: $n \geq 2$ is a fixed integer, $V = Z^n$ (the direct product of Z with itself n times), and G is the symmetric group of degree n acting on V by $g\cdot(z_1,\ldots,z_n) = (z_{g(1)},\ldots,z_{g(n)})$ for every $g \in G$ and $(z_1,\ldots,z_n) \in V$. Then the ramification locus of the canonical morphism $f:V \longrightarrow Y$ has codimension $d = \dim(V) \geq 3$ in V.

<u>Theorem 3.</u> <u>Let Z be a smooth projective variety of dimension $d \geq 3$ such that $H^1(Z,M) = o$ for every line bundle M on Z, and let $n \geq 2$ be an integer such that either $char(k) = o$, or $n < char(k)$ if $char(k) > o$. Then for every ample line bundle L on $Y = Z^{(n)}$ the property (#) holds for (Y,L).</u>

<u>Note.</u> The simplest examples of varieties Z satisfying the hypotheses of theorem 3 are all smooth hypersurfaces in P^{d+1} with $d \geq 3$.

<u>Proof of theorem 3.</u> The hypotheses imply in particular that $H^1(Z,O_Z) = o$, and then the see-saw principle (see [16],§5) immediately implies that $f^*(L) \cong p_1^*(L_1)\otimes\ldots\otimes p_n^*(L_n)$, with $L_1,\ldots,L_n \in Pic(Z)$ and $p_1:V \longrightarrow Z$ the projection of V onto the i-th factor. Since L is ample on Y and f is finite, $f^*(L)$ is ample on V, and hence L_i is ample on Z for every $i = 1,\ldots,n$. As in the proof of theorem 2, it will be sufficient to check the following:

$$H^1(V,f^*(L^i)) = o \quad \text{for every } i \in \mathbb{Z}, \text{ and}$$

$$H^1(V,T_V \otimes f^*(L^i)) = o \quad \text{for every } i < o,$$

in order to deduce (via lemma 1) that the hypotheses of theorem 1 are satisfied. But these vanishings are easily checked using the Künneth's formulae, the fact that $T_V = p_1^*(T_Z)\oplus\ldots\oplus p_n^*(T_Z)$, the hypotheses of the theorem, and the fact that L_i is ample for $i = 1,\ldots,n$ (which implies that $H^\circ(Z,L_i^j) = o$ for every $j < o$ and $i = 1,\dots,n$). Then the conclusion of the theorem follows from theorem 1. Q.E.D.

§3. A few remarks when Y is smooth

In this section we shall assume that Y is smooth and char(k) = o. Then it is known that the space $T_S^1(i)$ can be computed in the following way (see [23], page 337 and theorem 3.7). First, there is an exact sequence of vector bundles

$$o \longrightarrow O_Y \longrightarrow M \longrightarrow T_Y \longrightarrow o,$$

which is the dual of the exact sequence

$$o \longrightarrow \Omega_Y^1 \longrightarrow F \longrightarrow O_Y \longrightarrow o$$

corresponding to the image of L in $H^1(Y,\Omega_Y^1)$ via the canonical map $H^1(Y,O_Y^*) \cong \mathrm{Pic}(Y) \longrightarrow H^1(Y,\Omega_Y^1)$ induced by the map $O_Y^* \longrightarrow \Omega_Y^1$ given by $f \longmapsto df/f$. Then it is proved in loc. cit. that

$$(9) \quad T_S^1(i) = \mathrm{Ker}(H^1(Y,M\otimes L^i) \longrightarrow H^1(Y,\bigoplus_{j=1}^{n} L^{i+q_j})) \text{ for every } i\in \mathbb{Z},$$

where $S = S(Y,L)$ and $q_1,\dots,q_n$ have the same meaning as at the beginning of §1.

Using (9), the first exact sequence and the Kodaira's vanishing theorem, it follows that the condition "$T_S^1(-i) = o$ for every $i \geq 1$" is a consequence of the condition "$H^1(Y,T_Y\otimes L^{-i}) = o$ for every $i \geq 1$". If Y is smooth and char(k) = o, one can get rid of the unpleasant hypothesis i) of theorem 1 because of the following:

Theorem 4 (See [6]). Let (Y,L) be a smooth polarized variety of dimension ≥ 2 such that $H^1(Y,T_Y\otimes L^{-i}) = o$ for every $i \geq 1$ and char(k) = = o. Then the property (#) holds for (Y,L).

Theorem 4 is proved in [6]; via a quick argument, it is also a consequence of theorem 2 in [22]. Using theorem 4 and the main result of [22] we prove the following:

Theorem 5. Let (Y,L) be a smooth polarized variety such that: char(k) = o, dim(Y) ≥ 2, $H^1(Y,T_Y\otimes L^{-i}) = o$ for i = 1 and i = 2, and the linear system |L| contains a smooth divisor. Then the property (#) holds for (Y,L).

Proof. By theorem 4 it will be sufficient to show that $H^1(Y,T_Y\otimes L^{-i}) = o$ for every $i \geq 1$. Let $H\in |L|$ be a smooth divisor of |L|. Since dim(Y) ≥ 2, H is also connected. If we denote by L_H the restriction $L\otimes O_H$ and by T_H the tangent bundle of H, we have the canonical exact sequence

$$o \longrightarrow T_H \otimes L_H^{-i} \longrightarrow (T_Y \otimes L^{-i}) \otimes O_H \longrightarrow L_H^{1-i} \longrightarrow o,$$

which yields the exact sequence

$$(10_i) \qquad H^\circ(H, T_H \otimes L_H^{-i}) \longrightarrow H^\circ(H, (T_Y \otimes L^{-i}) \otimes O_H) \longrightarrow H^\circ(H, L_H^{1-i}).$$

For every $i \geqslant 2$ the last space is zero. On the other hand, by the main result of [22] (which extends a theorem of Mori-Sumihiro), the first space could be $\neq o$ only if $(H, L_H) \cong (P^1, O(1))$ (and then $i = 2$), in which case it follows easily that $(Y, L) \cong (P^2, O(1))$, and hence the property (#) holds for (Y, L) in this case. Thus we may assume that $H^\circ(H, T_H \otimes L_H^{-i}) = o$ for every $i \geqslant 2$. Then by (10_i) we get that the space in the middle is zero for every $i \geqslant 2$. Finally, using this and the exact sequence

$$(11_i) \quad o \longrightarrow T_Y \otimes L^{-i-1} \longrightarrow T_Y \otimes L^{-i} \longrightarrow (T_Y \otimes L^{-i}) \otimes O_H \longrightarrow o,$$

we infer that the map $H^1(Y, T_Y \otimes L^{-i-1}) \longrightarrow H^1(Y, T_Y \otimes L^{-i})$ is injective for every $i \geqslant 2$. Therefore $H^1(Y, T_Y \otimes L^{-i}) = o$ for every $i \geqslant 1$. Q.E.D.

Corollary. Let (Y, L) be a smooth polarized variety of dimension $d \geqslant 2$ such that there is a smooth divisor $H \in |L|$ for which the exact sequence

$$(12) \qquad o \longrightarrow T_H \longrightarrow T_Y \otimes O_H \longrightarrow L_H \longrightarrow o$$

is not split (in particular, $H^1(H, T_H \otimes L_H^{-1}) \neq o$). Assume moreover that $\mathrm{char}(k) = o$ and $H^1(Y, T_Y \otimes L^{-1}) = o$. Then the property (#) holds for (Y, L).

Proof. According to the proof of theorem 5, the exact sequence (11_1) shows that it is sufficient to prove that $H^\circ(H, (T_Y \otimes L^{-1}) \otimes O_H) = o$. The exact sequence (10_1) yields the exact sequence

$$(13) \quad H^\circ(H, T_H \otimes L_H^{-1}) \longrightarrow H^\circ(H, (T_Y \otimes L^{-1}) \otimes O_H) \longrightarrow H^\circ(H, O_H) \xrightarrow{\partial} H^1(H, T_H \otimes L_H^{-1}).$$

By [22], the first space could be $\neq o$ only in one of the following cases: either $(H, L_H) \cong (P^{d-1}, O(1))$, or $(H, L_H) \cong (P^1, O(2))$. In the first case $(Y, L) \cong (P^d, O(1))$, and hence (Y, L) has the property (#) in this case; the second case is ruled out because then $H^1(H, T_H \otimes L_H^{-1}) = o$, and hence (12) splits. Therefore we may assume $H^\circ(H, T_H \otimes L_H^{-1}) = o$, and then (13) shows that $H^\circ(H, (T_Y \otimes L^{-1}) \otimes O_H) = o$ if and only if $\partial(1) \neq o$. Since $\partial(1)$ is the obstruction in $H^1(H, T_H \otimes L_H^{-1})$ such that (12) be split, we get the result. Q.E.D.

Remark. In a more special situation, L'vovskii proved in [15] a better result than theorem 5 and its corollary. More precisely, assume that $Y \subset P^n$ is a smooth non-degenerate projective subvariety of P^n of dimension $\geqslant 2$ and degree $\geqslant 3$, such that $H^1(Y,T_Y(-1)) = o$ and char(k)= = o. Let $X \subset P^{n+1}$ be an irreducible subvariety of P^{n+1} such that $X \cap P^n =$ = Y, and X is smooth along Y and transversal to P^n, where P^n is embedded in P^{n+1} as a hyperplane. Then X is a cone over Y. In fact, L'vovskii has an even weaker assumption than $H^1(Y,T_Y(-1)) = o$ (loc. cit.). His proof uses completely different techniques.

Coming back to the above corollary, we may ask the following:

<u>Question.</u> Let (Y,L) be a smooth polarized variety of dimension $d \geqslant 2$ such that L is generated by its global sections. Find sufficient conditions ensuring that there is a smooth divisor $H \in |L|$ such that the corresponding exact sequence (12) is not split. Or, enumerate the situations when (12) is split for $H \in |L|$ general.

A necessary condition such that this question has a positive answer is that $H^1(H,T_H \otimes L_H^{-1}) \neq o$ for $H \in |L|$ general. Is it also sufficient? In the case of surfaces, the pairs (Y,L) for which $H^1(H,T_H \otimes L_H^{-1}) = o$ for $H \in |L|$ general, can be easily enumerated. Indeed, by duality and Riemann-Roch on the curve H one gets that this happens if and only if $(H,L_H) \cong (P^1,O(i))$ with i=1, 2, or 3. And by a well-known classical result, (Y,L) is isomorphic to one of the following: $(P^2,O(1))$, $(P^1 x P^1, O(1,1))$, or any smooth hyperplane section of $P^1 x P^2 \subset$ $\subset P^5$ via the Segre embedding (the latter surfaces are all isomorphic to the projective plane blown up at a point).

§4. <u>P^n-bundles over an irrational curve as hyperplane sections</u>

Let B be a smooth projective curve, and let E be a vector bundle of rank n+1 on B, with $n \geqslant 1$. Denote by Y = P(E) the projective bundle associated to E, and by $p: Y \longrightarrow B$ the canonical projection. The main result of this section is the following:

<u>Theorem 6. In the above notations, assume that the genus of B is positive and char(k) = o. Let X be a singular normal projective variety containing Y = P(E) as an ample Cartier divisor. Then X is isomorphic to the projective cone C(Y,L) and Y is embedded in X as the infinite, where L is the normal bundle of Y in X.</u>

The motivation of theorem 6 lies in the fact that, combining it with some results from [1], [2], and [3], we get the following complete description of all normal projective varieties whose hyperplane sections are P^n-bundles over a curve: