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Serge Lang

Topics in Cohomology of Groups

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Preface

The Benjamin notes which I published (in French) in 1966 on the cohomology of groups provided missing chapters to the Artin-Tate notes on class field theory, developed by cohomological methods. Both items were out of print for many years, but recently Addison-Wesley has again made available the Artin-Tate notes (which were in English). It seemed therefore appropriate to make my notes on cohomology again available, and I thank Springer-Verlag for publishing them (translated into English) in the Lecture Notes series.

The most basic necessary background on homological algebra is contained in the chapter devoted to this topic in my *Algebra* (derived functors and other material at this basic level). This material is partly based on what have now become routine constructions (Eilenberg-Cartan), and on Grothendieck's influential paper [Gr 59], which appropriately defined and emphasized δ-functors as such.

The main source for the present notes are Tate's private papers, and the unpublished first part of the Artin-Tate notes. The most significant exceptions are: Rim's proof of the Nakayama-Tate theorem, and the treatment of cup products, for which we have used the general notion of multilinear category due to Cartier.

The cohomological approach to class field theory was carried out in the late forties and early fifties, in Hochschild's papers [Ho 50a], [Ho 50b], [HoN 52], Nakayama [Na 41], [Na 52], Shafarevich [Sh 46], Weil's paper [We 51], giving rise to the Weil groups, and seminars of Artin-Tate in 1949-1951, published only years later [ArT 67].

As I stated in the preface to my *Algebraic Number Theory*, there are several approaches to class field theory. None of them makes any other obsolete, and each gives a different insight from the others.

The original Benjamin notes consisted of Chapters I through IX. Subsequently I wrote up Chapter X, which deals with applications to algebraic geometry. It is essentially a transcription of weekly installment letters which I received from Tate during 1958-1959. I take of course full responsibility for any errors which might have crept in, but I have made no effort to make the exposition anything more than a rough sketch of the material. Also the reader should not be surprised if some of the diagrams which have been qualified as being commutative actually have character -1.

The first nine chapters are basically elementary, depending only on standard homological algebra. The Artin-Tate axiomatization of class formations allows for an exposition of the basic properties of class field theory at this elementary level. Proofs that the axioms are satisfied are in the Artin-Tate notes, following Tate's article [Ta 52]. The material of Chapter X is of course at a different level, assuming some knowledge of algebraic geometry, especially some properties of abelian varieties.

I thank Springer Verlag for keeping all this material in print. I also thank Donna Belli and Mel Del Vecchio for setting the manuscript in AMSTeX, in a victory of person over machine.

Serge Lang
New Haven, 1995

CHAPTER I

Existence and Uniqueness

§1. The abstract uniqueness theorem

We suppose the reader is familiar with the terminology of abelian categories. However, we shall deal only with abelian categories which are categories of modules over some ring, or which are obtained from such in some standard ways, such as categories of complexes of modules. We also suppose that the reader is acquainted with the standard procedures constructing cohomological functors by means of resolutions with complexes, as done for instance in my *Algebra* (third edition, Chapter XX). In some cases, we shall summarize such constructions for the convenience of the reader.

Unless otherwise specified, all functors on abelian categories will be assumed additive. What we call a δ-**functor** (following Grothendieck) is sometimes called a **connected sequence of functors**. Such a functor is defined for a consecutive sequence of integers, and transforms an exact sequence

$$0 \to A \to B \to C \to 0$$

into an exact sequence

$$\cdots \to H^p(A) \to H^p(B) \to H^p(C) \xrightarrow{\delta} H^{p+1}(A) \to \cdots$$

functorially. If the functor is defined for all integers p with $-\infty < p < \infty$, then we say that this functor is **cohomological**.

Let H be a δ-functor on an abelian category $\mathfrak{A}$. We say that H is **erasable** by a subset $\mathfrak{M}$ of objects in A if for every A in $\mathfrak{A}$ there exists $M_A \in \mathfrak{M}$ and a monomorphism $\varepsilon_A : A \to M_A$ such that $H(M_A) = 0$. This definition is slightly more restrictive than the usual general definition (*Algebra*, Chapter XX, §7), but its conditions are those which are used in the forthcoming applications. An **erasing functor** for H consists of a functor

$$M : A \to M(A)$$

of $\mathfrak{A}$ into itself, and a monomorphism ε of the identity in M, i.e. for each object A we are given a monomorphism

$$\varepsilon_A : A \to M_A$$

such that, if $u : A \to B$ is a morphism in $\mathfrak{A}$, then there exists a morphism $M(u)$ and a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\varepsilon_A} & M(A) \\ & & {\scriptstyle u}\downarrow & & \downarrow {\scriptstyle M(u)} \\ 0 & \longrightarrow & B & \xrightarrow[\varepsilon_B]{} & M(B) \end{array}$$

such that $M(uv) = M(u)M(v)$ for the composite of two morphisms u, v. In addition, one requires $H(M_A) = 0$ for all $A \in \mathfrak{A}$.

Let $X(A) = X_A$ be the cokernel of ε_A. For each u there is a morphism

$$X(u) : X_A \to X_B$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M_A & \longrightarrow & X_A & \longrightarrow & 0 \\ & & {\scriptstyle u}\downarrow & & \downarrow {\scriptstyle M(u)} & & \downarrow {\scriptstyle X(u)} & & \\ 0 & \longrightarrow & B & \longrightarrow & M_B & \longrightarrow & X_B & \longrightarrow & 0, \end{array}$$

and for the composite of two morphisms u, v we have $X(uv) = X(u)X(v)$. We then call X the **cofunctor** of M.

Let p_0 be an integer, and $H = (H^p)$ a δ-functor defined for some values of p. We say that M is an **erasing functor for** H in dimension $> p_0$ if $H^p(M_A) = 0$ for all $A \in \mathfrak{A}$ and all $p > p_0$.

We have similar notions on the left. Let H be an exact δ-functor on $\mathfrak{A}$. We say that H is **coerasable** by a subset $\mathfrak{M}$ if for each object A there exists an epimorphism

$$\eta_A : M_A \to A$$

with $M_A \in \mathfrak{M}$, such that $H(M_A) = 0$. A **coerasing functor** M for H consists of an epimorphism of M with the identity. If η is such a functor, and $u : A \to B$ is a morphism, then we have a commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_A & \longrightarrow & M_A & \xrightarrow{\eta_A} & A & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle Y(u)} & & \Big\downarrow{\scriptstyle M(u)} & & \Big\downarrow{\scriptstyle u} & & \\ 0 & \longrightarrow & Y_B & \longrightarrow & M_B & \xrightarrow[\eta_B]{} & B & \longrightarrow & 0 \end{array}$$

and Y_A is functorial in A, i.e. $Y(uv) = Y(u)Y(v)$.

Remark. In what follows, erasing functors will have the additional property that the exact sequence associated with each object A will split over $\mathbf{Z}$, and therefore remains exact under tensor products or hom. An erasing functor into an abelian category of abelian groups having this property will be said to be **splitting**.

Theorem 1.1. First uniqueness theorem. *Let $\mathfrak{A}$ be an abelian category. Let H, F be two δ-functors defined in degrees* $0, 1$ (resp. $0, -1$) *with values in the same abelian category. Let (φ_0, φ_1) and $(\varphi_0, \bar{\varphi}_1)$ be δ-morphisms of H into F, coinciding in dimension* 0 (resp. $(\varphi_{-1}, \varphi_0)$ *and* $(\bar{\varphi}_{-1}, \varphi_0)$). *Suppose that H^1 is erasable* (resp. *H^{-1} is coerasable*). *Then we have $\varphi_1 = \bar{\varphi}_1$* (resp. $\varphi_{-1} = \bar{\varphi}_{-1}$).

Proof. The proof being self dual, we give it only for the case of indices $(0, 1)$. For each object $A \in \mathfrak{A}$ we have an exact sequence

$$0 \to A \to M_A \to X_A \to 0$$

and $H^1(M_A) = 0$. There is a commutative diagram

$$\begin{array}{ccccccc} H^0(M_A) & \longrightarrow & H^0(X_A) & \xrightarrow{\delta_H} & H^1(A) & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \varphi_0} & & \Big\downarrow{\scriptstyle \varphi_0} & & \Big\downarrow{\scriptstyle \varphi_1, \bar{\varphi}_1} & & \\ F^0(M_A) & \longrightarrow & F^0(X_A) & \xrightarrow{\delta_F} & F^1(A) & \longrightarrow & 0 \end{array}$$

with horizontal exact sequences, from which it follows that δ_H is surjective. It follows at once that $\varphi_1 = \bar{\varphi}_1$.

In the preceding theorem, φ_1 and $\bar{\varphi}_1$ are given. One can also prove a result which implies their existence.

Theorem 1.2. Second uniqueness theorem. *Let $\mathfrak{A}$ be an abelian category. Let H, F be δ-functors defined in degrees* $(0,1)$ *(*resp. $0,-1$*) with values in the same abelian category. Let $\varphi_0 : H^0 \xrightarrow{\cdot} F^0$ be a morphism. Suppose that H^1 is erasable by injectives (*resp. *H^{-1} is coerasable by projectives). Then there exists a unique morphism*

$$\varphi_1 : H^1 \to F^1 \text{ (resp. } \varphi_{-1} : H^{-1} \to F^{-1})$$

*such that (φ_0, φ_1) (*resp. $(\varphi_0, \varphi_{-1})$*) is also a δ-morphism. The association $\varphi_0 \mapsto \varphi_1$ is functorial in a sense made explicit below.*

Proof. Again the proof is self dual and we give it only in the cases when the indices are $(0,1)$. For each object $A \in \mathfrak{A}$ we have the exact sequence

$$0 \to A \to M_A \to X_A \to 0$$

and $H^1(M_A) = 0$. We have to define a morphism

$$\varphi_1(A) : H^1(A) \to F^1(A)$$

which commutes with the induced morphisms and with δ. We have a commutative diagram

$$\begin{array}{ccccccc}
H^0(M_A) & \longrightarrow & H^0(X_A) & \xrightarrow{\delta_H} & H^1(A) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_0} & & & & \\
F^0(M_A) & \longrightarrow & F^0(X_A) & \xrightarrow[\delta_F]{} & F^1(A) & &
\end{array}$$

with exact horizontal sequences. The right surjectivity is just the erasing hypothesis. The left square commutativity shows that Ker δ_H is contained in the kernel of $\delta_F \varphi_0(X_A)$. Hence there exists a unique morphism

$$\varphi_1(A) : H^1(A) \to F^1(A)$$

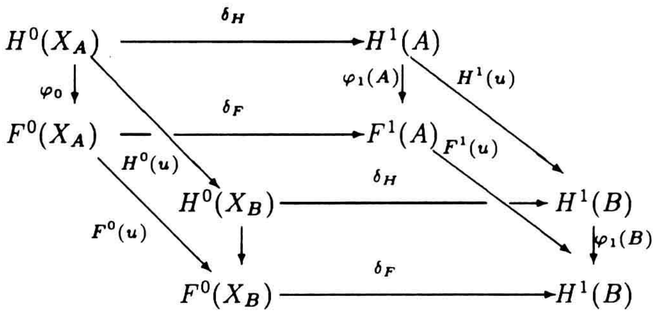
which makes the right square commutative. We shall prove that $\varphi_1(A)$ satisfies the desired conditions.

First, let $u : A \to B$ be a morphism. From the hypotheses, there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & M_A & \longrightarrow & X_A & \longrightarrow & 0 \\
 & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle M(u)} & & \downarrow{\scriptstyle X(u)} & & \\
0 & \longrightarrow & B & \longrightarrow & M_B & \longrightarrow & X_B & \longrightarrow & 0
\end{array}$$

the morphism $M(u)$ being defined because M_A is injective. The morphism $X(u)$ is then defined by making the right square commutative. To simplify notation, we shall write u instead of $M(u)$ and $X(u)$.

We consider the cube:



We have to show that the right face is commutative. We have:

$$\begin{aligned}
\varphi_1(B)H^1(u)\delta_H &= \varphi_1(B)\delta_H H^0(u) \\
&= \delta_F \varphi_0 H^0(u) \\
&= \delta_F F^0(u)\varphi_0 \\
&= F^1(u)\delta_F \varphi_0 \\
&= F^1(u)\varphi_1(A)\delta_H .
\end{aligned}$$

We have used the fact (implied by the hypotheses) that all the faces of the cube are commutative except possibly the right face. Since δ_H is surjective, one gets what we want, namely

$$\varphi_1(B)H^1(u) = F^1(u)\varphi_1(A).$$

The above argument may be expressed in the form of a useful general lemma.

> *If, in a cube, all the faces are commutative except possibly one, and one of the arrows as above is surjective, then this face is also commutative.*

Next we have to show that φ_1 commutes with δ, that is (φ_0, φ_1) is a δ-morphism. Let

$$0 \to A' \to A \to A'' \to 0$$

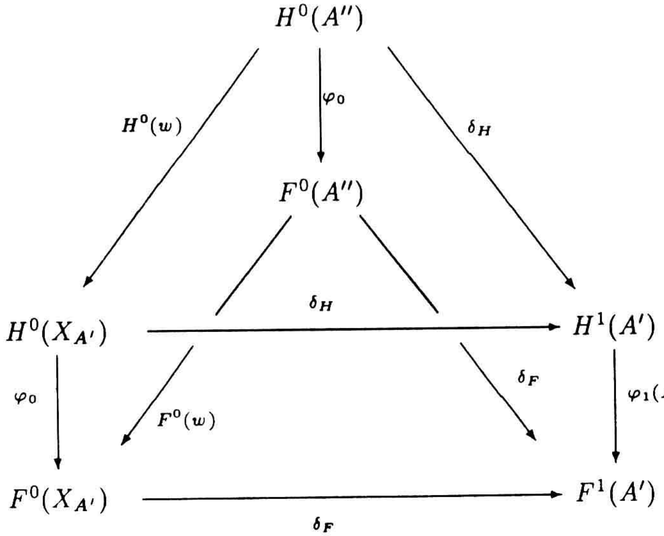
be an exact sequence in $\mathfrak{A}$. Then there exist morphisms

$$v : A \to M_{A'} \qquad \text{and} \qquad w : A'' \to X_{A'}$$

making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & A' & \longrightarrow & M_{A'} & \longrightarrow & X_{A'} & \longrightarrow & 0 \end{array}$$

because $M_{A'}$ is injective. There results the following commutative diagram:



We have to show that the right square is commutative. Note that the top and bottom triangles are commutative by definition of a δ-functor. The left square is commutative by the hypothesis that φ_0 is a morphism of functors. The front square is commutative by definition of $\varphi_1(A')$. We thus find

$$\begin{aligned} \varphi_1(A')\delta_H &= \varphi_1(A')\delta_H H^0(w) && \text{(top triangle)} \\ &= \delta_F \varphi_0 H^0(w) && \text{(front square)} \\ &= \delta_F F^0(w)\varphi_0 && \text{(left square)} \\ &= \delta_F \varphi_0 && \text{(bottom triangle),} \end{aligned}$$

which concludes the proof.

Finally, let us make explicit what we mean by saying that φ_1 depends functorially on φ_0. Suppose we have three functors H, F, E defined in degrees $0, 1$; and suppose given $\varphi_0 : H^0 \longrightarrow F^0$ and $\psi_0 : F^0 \longrightarrow E^0$. Suppose in addition that the erasing functor erases both H^1 and F^1. We can then construct φ_1 and ψ_1 by applying the theorem. On the other hand, the composite

$$\psi_0\varphi_0 = \theta_0 : H^0 \longrightarrow E^0$$

is also a morphism, and the theorem implies the existence of a morphism

$$\theta_1 : H^1 \longrightarrow E^1$$

such that (θ_0, θ_1) is a δ-morphism. By uniqueness, we obtain

$$\theta_1 = \psi_1 \circ \varphi_1.$$

This is what we mean by the assertion that φ_1 depends functorially on φ_0.

§2. Notation, and the uniqueness theorem in Mod(G)

We now come to the cohomology of groups. Let G be a group. As usual, we let $\mathbf{Q}$ and $\mathbf{Z}$ denote the rational numbers and the integers respectively. Let $\mathbf{Z}[G]$ be the group ring over $\mathbf{Z}$. Then

$\mathbf{Z}[G]$ is a free module over $\mathbf{Z}$, the group elements forming a basis over $\mathbf{Z}$. Multiplicatively, we have

$$\left(\sum_{\sigma\in G} a_\sigma \sigma\right)\left(\sum_{\tau\in G} b_\tau \tau\right) = \sum_{\sigma,\tau} a_\sigma b_\tau \sigma\tau,$$

the sums being taken over all elements of G, but only a finite number of a_σ and b_τ being $\neq 0$. Similarly, one defines the group algebra $k[G]$ over an arbitrary commutative ring k.

The group ring is often denoted by $\Gamma = \Gamma_G$. It contains the ideal I_G which is the kernel of the **augmentation homomorphism**

$$\varepsilon : \mathbf{Z}[G] \to \mathbf{Z}$$

defined by $\varepsilon\left(\sum n_\sigma \sigma\right) = \sum n_\sigma$. One sees at once that I_G is $\mathbf{Z}$-free, with a basis consisting of all elements $\sigma - e$, with σ ranging over the elements of G not equal to the unit element. Indeed, if $\sum n_\sigma = 0$, then we may write

$$\sum n_\sigma \sigma = \sum n_\sigma(\sigma - e).$$

Thus we obtain an exact sequence

$$0 \to I_G \to \mathbf{Z}[G] \to \mathbf{Z} \to 0,$$

used constantly in the sequel. The sequence splits, because $\mathbf{Z}[G]$ is a direct sum of I_G and $\mathbf{Z} \cdot e_G$ (identified with $\mathbf{Z}$).

Abelian groups form an abelian category, equal to the category of $\mathbf{Z}$-modules, denoted by $\mathrm{Mod}(\mathbf{Z})$. Similarly, the category of modules over a ring R will be denoted by $\mathrm{Mod}(R)$.

An abelian group A is said to be a G-**module** if one is given an operation (or action) of G on A; in other words, one is given a map

$$G \times A \to A$$

satisfying

$$(\sigma\tau)a = \sigma(\tau a) \qquad e \cdot a = a \qquad \sigma(a + b) = \sigma a + \sigma b$$

for all $\sigma, \tau \in G$ and $a, b \in A$. We let $e = e_G$ be the unit element of G. One extends this operation by linearity to the group ring $\mathbf{Z}[G]$. Similarly, if k is a commutative ring and A is a k-module, one extends the operation of G on A to $k[G]$ whenever the operation of G commutes with the operation of k on A. Then the category of $k[G]$-modules is denoted by $\mathrm{Mod}_k(G)$ or $\mathrm{Mod}(k, G)$.

The G-modules form an abelian category, the morphisms being the G-homomorphisms. More precisely, if $f : A \to B$ is a morphism in $\mathrm{Mod}(\mathbf{Z})$, and if A, B are also G-modules, then G operates on $\mathrm{Hom}(A, B)$ by the formula

$$(\sigma f)(a) = \sigma(f(\sigma^{-1}a)) \quad \text{for} \quad a \in A \quad \text{and} \quad \sigma \in G.$$

If there is any danger of confusion one may write $[\sigma]f$ to denote this operation. If $[\sigma]f = f$, one says that f is a G-**homomorphism**, or a G-**morphism**. The set of G-morphisms from A into B is an abelian group denoted by $\mathrm{Hom}_G(A, B)$. The category consisting of G-modules and G-morphisms is denoted by $\mathrm{Mod}(G)$. It is the same as $\mathrm{Mod}(\Gamma_G)$.

Let $A \in \mathrm{Mod}(G)$. We let A^G denote the submodule of A consisting of all elements $a \in A$ such that $\sigma a = a$ for all $\sigma \in G$. In other words, it is the submodule of fixed elements by G. Then A^G is an abelian group, and the association

$$H_G^0 : A \mapsto A^G$$

is a functor from $\mathrm{Mod}(G)$ into the category of abelian groups, also denoted by **Grab**. This functor is left exact.

We let $\varkappa_G$ denote the canonical map (in the present case the identity) of an element $a \in A^G$ into $H_G^0(A)$.

Theorem 2.1. *Let H_G be a cohomological functor on* $\mathrm{Mod}(G)$ *with values in* $\mathrm{Mod}(\mathbf{Z})$, *and such that H_G^0 is defined as above. Assume that $H_G^r(M) = 0$ if M is injective and $r > 1$. Assume also that $H_G^r(A) = 0$ for $A \in$* $\mathrm{Mod}(G)$ *and $r < 0$. Then two such cohomological functors are isomorphic, by a unique morphism which is the identity on $H_G^0(A)$.*

This theorem is just a special case of the general uniqueness theorem.

Corollary 2.2. *If $G = \{e\}$ then $H_G^r(A) = 0$ for all $r > 0$.*

Proof. Define H_G by letting $H_G^0(A) = A^G$ and $H_G^r(A) = 0$ for $r \neq 0$. Then it is immediately verified that H_G is a cohomological functor, to which we can apply the uniqueness theorem.

Corollary 2.3. *Let $n \in \mathbf{Z}$ and let $n_A : A \to A$ be the morphism $a \mapsto na$ for $a \in A$. Then $H_G^r(n_A) = n_H$ (where H stands for $H_G^r(A)$).*

Proof. Since the coboundary δ is additive, it commutes with multiplication by n, and again we can apply the uniqueness theorem.

The existence of the functor H_G will be proved in the next section.

We say that G **operates trivially** on A if $A = A^G$, that is $\sigma a = a$ for all $a \in A$ and $\sigma \in G$. We always assume that G operates trivially on $\mathbf{Z}, \mathbf{Q}$, and $\mathbf{Q}/\mathbf{Z}$.

We define the abelian group

$$A_G = A/I A_G.$$

This is the factor group of A by the subgroup of elements of the form $(\sigma - e)a$ with $\sigma \in G$ and $a \in A$. The association

$$A \mapsto A_G$$

is a functor from Mod(G) into Grab.

Let U be a subgroup of finite index in G. We may then define the **trace**

$$\mathbf{S}_G^U : A^U \to A^G \qquad \text{by the formula} \qquad \mathbf{S}_G^U(a) = \sum_c \bar{c}a,$$

where $\{c\}$ is the set of left cosets of U in G, and $\bar{c}$ is a representative of c, so that

$$G = \bigcup_c \bar{c}U.$$

If $U = \{e\}$, then G is finite, and in that case the **trace** is written $\mathbf{S}_G$, so

$$\mathbf{S}_G(a) = \sum_{\sigma \in G} \sigma a.$$

For the record, we state the following useful lemma.