# Recent Progress in Theory and Applications of Modern Complex Analysis

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（现代复分析理论和应用的新进展）

SCIENCE PRESS
Beijing

# Preface

This book consists of select works of the author, which include most important results about complex analytic theory, methods and applications obtained by the author in recent 25 years, mainly properties of solutions and various boundary value problems for nonlinear elliptic equations and systems, parabolic equations and systems, hyperbolic and mixed complex equations with parabolic degeneracy. In other words, a large portion of the works is devoted to boundary value problems for general elliptic complex equations of first and second order, initial-boundary value problems for nonlinear parabolic complex equations and systems of second order including some equations and systems in higher dimensional domains, and properties of solutions for hyperbolic complex equations of second order. Moreover, some results about second order complex equations of mixed (elliptic-hyperbolic) type are introduced. Applications of nonlinear complex analysis to continuum mechanics, and approximate methods of elliptic complex equations are also investigated.

In Chapter 1, we introduce the foundational theorems of nonlinear quasiconformal mappings, mainly the existence and uniqueness of solutions for some nonlinear elliptic complex equations from the $N+1$-connected domain onto an $N+1$-connected circular domain are proved by using a new method. In addition, we consider general quasiconformal shift theorems in some multiply connected domains.

In Chapter 2, we first discuss the Riemann-Hilbert problem for general nonlinear elliptic complex equations of first order with non-smooth boundary. Afterwards the solvability of the oblique derivative problems for nonlinear uniformly elliptic systems of second order equations in multiply connected domains is discussed. For this sake, we propose a modified Riemann-Hilbert boundary value problem for elliptic systems of first order equations, and establish an integral expression together with a priori estimate of solutions for the modified boundary value problem. Finally by using the above estimates and the Leray-Schauder theorem, the solvability of the above problem with some conditions is verified.

In Chapter 3, we first give the general formulas of solutions of discontinuous Riemann-Hilbert boundary value problem for analytic functions in the upper half-plane and unit-disk, which include the Keldych-Sedov formula in the upper half-plane as a special case. Moreover, we discuss the discontinuous irregular oblique derivative problem for nonlinear elliptic equations of second order in multiply connected domains. It is known that the discontinuous Riemann-Hilbert boundary value problems possess the important applications to mechanics and physics.

In Chapter 4, we introduce two approximate methods for solving boundary value problems for elliptic equations and systems, i.e., the Newton imbedding method and variation-difference method, in which we give the approximate solutions of the corresponding boundary value problems for nonlinear elliptic complex equations of first order and second order, and their error estimates.

The Tricomi problem for the Chaplygin equation $K(y)u_{xx}+u_{yy}=0$ is a famous problem in subsonic and transonic gas dynamics. L. Bers in 1958 posed the Tricomi and Frankl problem for the Chaplygin equation in multiply connected domains, but up to 2007 one has not seen that the problems are completely solved. Chapter 5 deals with the oblique derivative problem for second order nonlinear equations of mixed type in multiply connected domains, which includes the Tricomi and Frankl problems of Chaplygin equation as its special cases. We get the representation of solutions of the boundary value problems for the equations, and then prove the uniqueness and existence of solutions for the problems by a new complex analytic method.

In Chapter 6, we introduce the applications of complex analysis to continuum mechanics, mainly some free boundary value problems in gas dynamics and filtration theory are handled, where some parts of the boundaries of the domains are unknown and thus called free boundaries. By using the method of quasiconformal mappings, hodograph and other methods, the free boundary value problems can be reduced to discontinuous boundary value problems for linear or nonlinear elliptic complex equations in fixed domains. Then the above free boundary value problems can be solved. In final four sections, the inverse problems for quasilinear elliptic systems of first order equations with Riemann-Hilbert type map and elliptic equations of second order from Dirichlet to Neumann Map are discussed, which possess important application in mechanics and physics.

Chapter 7 concerns some boundary value problems for some functions of several complex variables in the polycylinder and the Clifford analysis.

The contents of Chapter 8 are generalizations of some contents of elliptic equations and systems in the plane to nonlinear elliptic, parabolic equations and systems of second order in higher dimensional domains. Firstly, a priori estimates of solutions for the above boundary value problems with some conditions are given, and then by using the estimates of above solutions and fixed-point theorem, the existence of solutions for the above problems is proved.

Two special features are presented in this book: one is that elliptic and parabolic complex equations are almost discussed in general and nonlinear cases, and several boundary value problems are studied in multiply connected domains, and the other is that several complex analytic methods are used to investigate various problems on elliptic, parabolic, hyperbolic equations and systems, as well as equations of mixed type.

The contents in this book originate in the author and his cooperative colleagues, and great majority of the results is firstly obtained by them in the world. Many questions and problems investigated in this book deserve further investigations. The author sincerely hope that the reader will enjoy reading the book.

Finally, the preparation of this book was supported by the National Natural Science Foundation of China, its support has provided a wonderful environment to obtain many results reported in this book. In the meantime the author would like to acknowledge the editorial staff of Science Press for making it possible to publish this book.

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# Contents

# Chapter 1

# Foundational Theorems of Nonlinear Quasiconformal Mappings and Quasiconformal Shift Theorems

In this chapter, we first introduce the foundational theorems of nonlinear quasiconformal mappings, mainly the existence and uniqueness of homeomorphic solutions for some nonlinear elliptic complex equations from the general $N+1$-connected domain onto an $N+1$-connected circular domain are proved by a new method. Next, we consider the nonlinear quasiconformal shift theorems in some multiply connected domains.

## 1.1 Existence Theorems of Nonlinear Quasiconformal Mappings in Multiply Connected Domains

We discuss the nonlinear uniformly elliptic complex equation in the form

$$w_{\bar{z}}=F(z,w,w_z),\ F(z,w,w_z)=Q(z,w,w_z)w_z \text{ in } D,$$
$$Q(z,w,w_z)=\begin{cases} Q_1(z,w,w_z)+Q_2(z,w,w_z)\overline{w}_{\bar{z}}/w_z, & \text{for } w_z\neq 0 \text{ in } D,\\ 0, \text{ for } w_z=0 \text{ in } D, \text{ or } z\notin D, \end{cases} \tag{1.1.1}$$

where $D$ is an $N+1$-connected bounded domain in the complex plane $\mathbf{C}$ with the $N+1$ boundary components $\Gamma_j\in C_\mu\,(0<\mu<1, j=0,1,\ldots,N,\ \Gamma_0=\Gamma_{N+1})$, and $\Gamma_j\,(j=1,\ldots,N)$ are located in the domain $D_0$ bounded by $\Gamma_0$. In particular, the linear complex equation

$$w_{\bar{z}}=Q(z)w_z \text{ in } D, \tag{1.1.2}$$

is the so-called the Beltrami equation. Suppose that the complex equation (1.1.1) satisfies **Condition C**, namely

(1) $Q_j(z,w,U)\,(j=1,2)$ are measurable in $z\in D$ for any continuous function $w(z)$ in $\overline{D}$ and any measurable function $U(z)\in L_{p_0}(D)$, and continuous in $w\in\mathbf{C}$ for almost every point $z\in D, U\in\mathbf{C}$, where $p_0,p\,(2<p_0\leqslant p)$ are positive constants.

(2) The complex equation (1.1.1) satisfies the uniform ellipticity condition

$$|F(z,w,U_1)-F(z,w,U_2)|\leqslant q_0\,|U_1-U_2|, \tag{1.1.3}$$

for almost every point $z\in D$, in which $w,U_1,U_2\in\mathbf{C}$ and $q_0\,(<1)$ is a non-negative constant. It is clear that

$$w_{\bar z}=F(z,w,w_z),\ F(z,w,w_z)=\begin{cases} q_0w_z^2/2, & \text{for } |w_z|\leqslant 1,\\ q_0w_z^{-2}/2, & \text{for } |w_z|>1,\end{cases}$$

satisfies Condition C.

There is no harm in assuming that the domain $D$ is an $N+1$-connected circular domain with the boundary $\Gamma=\cup_{j=0}^{N}\Gamma_j\ (\Gamma_j=|z-z_j|=r_j,\ j=0,1,\ldots,N,z_0=0,r_0=1)$, because otherwise, through a conformal mapping $\zeta(z)$ from $D$ onto the circular domain $\Omega$ such that $0\in\Omega$ and $1\in\partial\Omega$, the complex equation (1.1.1) is reduced to

$$w_{\bar\zeta}=Q[z,w,w_\zeta\zeta'(z)][\zeta'(z)/\overline{\zeta'(z)}]w_\zeta,$$

obviously the above equation satisfies Condition C still.

**Theorem 1.1.1** *Suppose that the complex equation* (1.1.1) *satisfies Condition* C. *Then there exists a homeomorphic solution* $w=w(z)$ *of* (1.1.1) *in* $\overline{D}$, *which quasiconformally maps from the* $N+1$*-connected domain* $D$ *onto an* $N+1$*-connected circular domain* $G$, *such that* $w(0)=0$ *and* $w(1)=1$.

**Proof** If $w(z)$ is a homeomorphic solution of equation (1.1.1), and substitute it into the coefficient $Q(z)=Q(z,w,w_z)$ of (1.1.1), then $w(z)$ is the solution of the linear Beltrami equation (1.1.2), which can be expressed the form

$$w(z)=\Phi[\chi(z)],\ \chi(z)=z+T\omega-T_0\omega,\ \omega(z)\in L_{p_0}(\overline{D}), \tag{1.1.4}$$

where $\chi(z)=z+T\omega\ (T\omega=-\frac{1}{\pi}\iint\frac{\omega(\zeta)}{\zeta-z}d\sigma_\zeta,\ T_0\omega=T\omega|_{z=0})$ is a complete homeomorphism of Beltrami equation (1.1.2) in $\mathbf{C}$ (see [46]). Let the solution $w(z)$ be substituted into (1.1.1), we have

$$\begin{aligned}&\omega(z)=F[z,w(z),\Phi'(\chi)(1+\Pi\omega)]/\Phi'(\chi),\ \text{i.e.}\\ &\omega(z)=F[z,w(z),e^{W(z)}(1+\Pi\omega)]e^{-W(z)},\end{aligned} \tag{1.1.5}$$

in which $W(z)=\ln\Phi'(\chi)$ and $0\leqslant\arg\Phi'(\chi(0))<2\pi$ is selected. From Theorems 1.1 and 1.3, Chapter III, [48]9), we can get that $w(z)$, $W(z)=\ln\Phi'(\chi)$ satisfy the estimates

$$C_\alpha[w(z),\overline{D}]\leqslant M_1,\ L_{p_0}[|w_{\bar z}|+|w_z|,\overline{D}]\leqslant M_2,\ C_\alpha[\ln\Phi'[\chi(z)],D_*]\leqslant M_3, \tag{1.1.6}$$

where $\alpha = 1-2/p_0\,(p_0 > 2)$, $M_j = M_j(q_0, p_0, D)$, $j = 1, 2$, $M_3 = M_3(q_0, p_0, D, D_*)$, in which $D_*$ is a closed subdomain in $D$.

On the basis of the above discussion, we can first suppose that the coefficient $Q(z, w, w_z) = 0$ in $\overline{D}\backslash D_*$, and introduce a subset $B_M$ in the Banach space $B$ : $C(D_*) \times C(D_*)$, whose elements are the systems of functions: $g = [w(z), W(z)]$ with the norm $||g|| = C[w(z), D_*] + C[W(z), D_*]$ satisfying the conditions

$$C[w(z), D_*] \leqslant M_1,\ C[W(z), D_*] \leqslant M_3, \tag{1.1.7}$$

where $M_1, M_3$ are as stated in (1.1.6). It is easy to see that $B_M$ is a closed, bounded and convex set in $B$.

We arbitrarily choose a system of functions $g = [w(z), W(z)] \in B_M$, and substitute $w(z), W(z)$ in the integral equation

$$\omega^*(z) = f(z, w, W, \Pi\omega^*),\ f = F[z, w, \mathrm{e}^W(1 + \Pi\omega^*)]\mathrm{e}^{-W}. \tag{1.1.8}$$

Noting Condition C, we get

$$\begin{aligned} L_{p_0}[f(z, w, W, \Pi\omega^*), D_*] &\leqslant q_0 L_{p_0}[f(z, w, W, \Pi\omega^*), D_*] \\ &\leqslant q_0[\varLambda_{p_0} L_{p_0}[\omega^*, D_*] + \pi^{1/p_0}], \end{aligned} \tag{1.1.9}$$

$$L_{p_0}[f(z, w, W, \Pi\omega_1^*) - f(z, w, W, \Pi\omega_2^*), D_*] \leqslant q_0 \varLambda_{p_0} L_{p_0}[\omega_1^* - \omega_2^*, D_*], \tag{1.1.10}$$

where $p_0\,(> 2)$ is a positive number, such that $q_0\varLambda_{p_0} < 1$ (see Lemmas 1.1 and 1.5, Chapter II, [48]9), or Theorem 3.4.1, [52]). Applying the principle of contraction, a unique solution $\omega^*(z)$ of (1.1.5) can be obtained, and

$$L_{p_0}[\omega^*, D_*] \leqslant \frac{q_0}{1 - q_0\varLambda_{p_0}} \pi^{1/p_0} = M_4 < \infty. \tag{1.1.11}$$

Moreover, taking into account

$$|\omega^*(z)| \leqslant q_0\,|1 + \Pi\omega^*| \leqslant q_0 + q_0|\Pi\omega^*|, \tag{1.1.12}$$

and setting

$$Q(z) = \begin{cases} \omega^*/(1+\Pi\omega^*), \text{ for } z \in D_* \text{ and } 1+\Pi\omega^* \neq 0, \\ 0, \text{ for } z \notin D_* \text{ or } 1 + \Pi\omega^* = 0, \end{cases} \tag{1.1.13}$$

we know that $\chi^*(z) = z + T\omega^* - T\omega_0^*$ is a complete homeomorphism of the linear Beltrami equation

$$w_{\bar z} - Q(z)w_z = 0,\ |Q(z)| \leqslant q_0 < 1, \tag{1.1.14}$$

such that $\chi^*(0) = 0$, $\chi^*(\infty) = \infty$, and there exists a unique univalent function $\Phi^*(\chi)$, which maps the domain $\chi(D)$ onto the $N+1$-connected circular domain $G$ with the conditions $\Phi^*(0) = 0$, $\Phi^*[\chi^*(1)] = 1$, hence

$$w^*(z) = \Phi^*[\chi^*(z)] \tag{1.1.15}$$

is a homeomorphic solution of (1.1.14), which quasiconformally maps the domain $D$ onto the domain $G$, such that $w^*(0) = 0$ and $w^*(1) = 1$. Moreover $w^*(z)$, $\ln \Phi'[\chi^*(z)]$ $(0 < \arg \Phi^{*\prime}(0) < 2\pi)$ satisfy the estimates in (1.1.6), this shows $g^* = [w^*(z), W^*(z)] \in B_M$, we denote by $g^* = S(g)$ the mapping from $g \in B_M$ into $g^* \in B_M$.

In the following, we shall verify a fact: $g^* = S(g)$ is a continuous mapping from $B_M$ into a compact subset of itself.

It is free to select $g_n = [w_n(z), W_n(z)] \in B_M$ $(n = 0, 1, 2, \ldots)$ with the condition $||g_n - g_0|| \to 0$ as $n \to \infty$. Denote

$$g_n^* = S(g_n) = [w_n^*(z), W_n^*(z)] \in B_M \; (n = 0, 1, 2, \ldots),$$

where $w_n^*(z) = \Phi_n^*[\chi_n^*(z)]$, $\chi_n^*(z) = z + T\omega_n^* - T_0\omega_n^*$, $W_n^*(z) = \ln \Phi_n^{*\prime}[\chi_n^*(z)]$, $T_0\omega_n^* = T\omega_n^*|_{z=0}$, and

$$\omega_n^*(z) = f(z, w_n, W_n, \Pi\omega_n^*),\; n = 0, 1, 2, \ldots. \tag{1.1.16}$$

From the above equations, we get

$$\begin{aligned} \omega_n^*(z) - \omega_0^*(z) &= f(z, w_n, W_n, \Pi\omega_n^*) - f(z, w_n, W_n, \Pi\omega_0^*) + c_n(z), \\ c_n(z) &= f(z, w_n, W_n, \Pi\omega_0^*) - f(z, w_0, W_0, \Pi\omega_0^*). \end{aligned} \tag{1.1.17}$$

Noting Condition C, we have

$$\begin{aligned} |c_n(z)| \leqslant & |f(z, w_n, W_n, \Pi\omega_0^*) - f(z, w_n, W_0, \Pi\omega_0^*)| \\ & + |f(z, w_n, W_0, \Pi\omega_0^*) - f(z, w_0, W_0, \Pi\omega_0^*)| \\ \leqslant & 2q_0|e^{W_0 - W_n} - 1| \cdot |1 + \Pi\omega_0^*| \\ & + |f(z, w_n, W_0, \Pi\omega_0^*) - f(z, w_0, W_0, \Pi\omega_0^*)| \\ = & h_n(z). \end{aligned} \tag{1.1.18}$$

Because $h_n(z)$ converges to 0 for almost every point in $D_*$, as stated in the proof of Lemma 1.2, Chapter III, [48]9) or the formula (2.4.18), Chapter 2 below, $L_{p_0}[c_n, D_*] \to 0$ as $n \to \infty$ can be verified. Furthermore from (1.1.17), we have

$$L_{p_0}[\omega_n^*(z) - \omega_0^*(z), D_*] \leqslant \frac{1}{1 - q_0 \Lambda_{p_0}} L_{p_0}[c_n, D_*], \tag{1.1.19}$$

therefore $L_{p_0}[\omega_n^*(z) - \omega_0^*(z), D_*] \to 0$ as $n \to \infty$. On the basis of Lemma 1.1, Chapter II, [48]9), we can derive $C_\alpha[\chi_n^*(z) - \chi_0^*(z), D_*] \to 0$ as $n \to \infty$, where $\alpha = 1 - 2/p_0$. By the convergent theorem of sequence of domains (see [48]12)), we can prove that $w_n^*(z) = \Phi_n^*[\chi_n^*(z)]$, $\Phi_n^{*\prime}[\chi_n^*(z)]$ in $D_*$ uniformly converge $w_0^*(z) = \Phi_0^*[\chi_0^*(z)]$, $\Phi_0^{*\prime}[\chi_0^*(z)]$ ($\neq 0$) respectively, consequently

$$||g_n^* - g_0^*|| = C[w_n^*(z) - w_0^*(z), D_*] + C[W_n^*(z) - W_0^*(z), D_*] \to 0 \text{ as } n \to \infty.$$

This shows that $g^* = S(g)$ is a continuous mapping in $B_M$.

Next, we arbitrarily choose $g_n = [w_n(z), W_n(z)] \in B_M$ $(n = 1, 2, \ldots)$. Denote $g_n^* = S(g_n) = [w_n^*(z), W_n^*(z)] \in B_M$ $(n = 1, 2, \ldots)$, it is clear that $w_n^*(z)$, $W_n^*(z) = \ln \Phi_n^{*\prime}[\chi_n^*(z)]$ satisfy the estimations in (1.1.6), hence we can choose their subsequences in $Q_*$, which uniformly converge to $w_0^*(z)$, $W_0^*(z)$ respectively. Moreover $||g_n^* - g_0^*|| \to 0$ as $n \to \infty$ can be derived. This shows that $g^* = S(g)$ is a mapping from $B_M$ into a compact of itself.

By using the Schauder fixed point theorem, there exists a system of functions $g = [w(z), W(z)] \in B_M$, such that

$$g = S(g) = [w(z), W(z)],\ w(z) = \Phi[\chi(z)],$$
$$W(z) = \ln \Phi'[\chi(z)],\ \chi(z) = z + T\omega - T_0\omega,$$

and $\omega(z)$ satisfies

$$\begin{aligned} \omega(z) &= f[z, w, W, \Pi\omega) = F[z, w(z), \mathrm{e}^{W(z)}(1 + \Pi\omega)]\mathrm{e}^{-W(z)} \\ &= F[z, w(z), \Phi'(\chi)(1 + \Pi\omega)]/\Phi'(\chi), \text{ i.e.} \\ \Phi'(\chi)\omega(z) &= F[z, w(z), \Phi'(\chi)(1 + \Pi\omega)], \end{aligned} \tag{1.1.20}$$

for almost every point in $D_*$. Taking into account $w_{\bar{z}} = \Phi'(\chi)\omega(z)$, $w_z = \Phi'(\chi)(1 + \Pi\omega)$, we see that $w(z) = \Phi[\chi(z)]$ is just the homeomorphic solution of the nonlinear uniformly elliptic complex equation (1.1.1).

If we multiply the coefficient $Q(z, w, w_z)$ of complex equation (1.1.1) by the function

$$\sigma_n(z) = \begin{cases} 1, & z \in D_n = \{z|\mathrm{dist}(z, \Gamma \cup \{0\}) \geqslant 1/n\}, \\ 0, & z \notin D_n, \end{cases} \tag{1.1.21}$$

where $n$ is a positive integer, we obtain the complex equation

$$w_{\bar{z}} = F_n(z, w, w_z),\ F_n = \sigma_n(z)F(z, w, w_z). \tag{1.1.22}$$

Obviously, (1.1.22) is also satisfied Condition C and its coefficient is equal to 0 in the neighborhood of the boundary $\Gamma$ and $z = 0$. As stated before, we have proved that there exists a homeomorphic solution

$$w_n(z) = \Phi_n[\chi_n(z)],\ \chi_n(z) = z + T\omega_n - T_0\omega_n, \tag{1.1.23}$$

of equation (1.1.22), and $\{\chi_n(z)\}$, $\{w_n(z)\}$ satisfy the estimations in (1.1.6), hence we can choose their subsequences, which uniformly converge to $\chi_0(z)$, $w_0(z)$, and $w_0(z) = \Phi_0[\chi_0(z)]$, $\chi_0(z)$ is a homeomorphism in $\overline{D}$, and $\Phi_0(\chi)$ is a univalent analytic function in $\chi_0(D)$, such that $w_0(0) = 0$, $w_0(1) = 1$. Denote by $\omega_0(z)$ the solution of the integral equation

$$\begin{gathered}\omega_0(z) = f[z, w_0(z), \Phi_0'[\chi_0(z)], \Pi\omega_0],\\ f = F[z, w_0(z), \Phi_0'[\chi_0(z)](1+\Pi\omega_0)]/\Phi_0'[\chi_0(z)].\end{gathered} \tag{1.1.24}$$

Setting $\omega_n(z) = f[z, w_n(z), \Phi_n'[\chi_n(z)], \Pi\omega_n]$, as stated in (1.1.18), we can prove that for any closed subdomain $D_*$ of $D$, provided that $n$ is large enough, we must have

$$\begin{aligned}&|f[z, w_n(z), \Phi_n'[\chi_n(z)], \Pi\omega_0] - f[z, w_0(z), \Phi_0'[\chi_0(z)], \Pi\omega_0]|\\ &\leqslant 2q_0\left|\frac{\Phi_0'[\chi_0(z)]}{\Phi_n'[\chi_n(z)]} - 1\right| \cdot |1+\Pi\omega_0| + |f[z, w_n(z), \Phi_0'[\chi_0(z)], \Pi\omega_0]\\ &\quad - f[z, w_0(z), \Phi_0'[\chi_0(z)], \Pi\omega_0]| = h_n(z),\end{aligned} \tag{1.1.25}$$

from the above formula, we see that $h_n(z)$ converge to 0 for almost every point in $D$, as $n \to \infty$. Thus as stated before, we can obtain

$$L_{p_0}[|\omega_n(z) - \omega_0(z)| + |\Pi(\omega_n(z) - \omega_0(z))|, \overline{D}] \to 0, \text{ as } n \to \infty,$$

and $\chi_0(z) = z + T\omega_0 - T_0\omega_0$, this shows that $w_0(z) = \Phi_0[\chi_0(z)]$ is a required solution of equation (1.1.1).

Besides by the similar method, we can prove the foundational theorems of quasiconformal mappings from the $N$+1-connected domain onto an $N$+1-connected rectilinear slit domain with $N$ parallel rectilinear slits, and $N$+1-connected circular slit domain with $N$ circular slits. In [31], the author obtained the corresponding result of Theorem 1.1.1 for the quasilinear complex equation (1.1.1) with $Q_j = Q_j(z, w)$ $(j = 1, 2)$ by using another method.

Finally, we mention that if the coefficient $Q(z)$ of the linear Beltrami complex equation

$$w_{\bar{z}} = Q(z)w_z \text{ in } D \tag{1.1.26}$$

is measurable and satisfying the non-uniformly elliptic condition $|Q(z)| < 1$ in $D$, then the foundational theorem of quasiconformal mappings is not must held. For instance, for the unit disc $D = \{|z| < 1\}$ and the coefficient

$$Q(z) = \frac{nz^2|z|^{2n-2}}{1 + (n-1)|z|^{2n}} \text{ in } D, \tag{1.1.27}$$

in which $n$ is any positive integer, it is easy to see that $|Q(z)| \to 1$ as $|z| \to 1$, i.e. the complex equation (1.1.26) is non-uniform ellipticity in $D$ with the degenerate boundary $\Gamma = \{|z| = 1\}$, and the complex equation (1.1.26) with the coefficient as stated in (1.1.27) possesses the homeomorphic solution

$$\zeta(z) = \frac{1}{1 - |z|^{2n}} \text{ in } D, \tag{1.1.28}$$

which maps the unit disc $D$ onto the whole $\zeta$ plane $\mathbf{C}$. It is not difficult to see that any homeomorphic solution of (1.1.26), (1.1.27) can be expressed as the form

$$w(z) = \Phi[\zeta(z)] \text{ in } D$$

(for example, see Theorem 3.30, [46]), in which $\zeta(z)$ possesses the form (1.1.28), and $\Phi(\zeta)$ is a univalent entire function, namely

$$\Phi(\zeta) = a\zeta + b \ (a \neq 0),$$

where $a, b$ are complex constants. This shows that the homeomorphic solution of (1.1.26), (1.1.27) maps the unit disc $D$ onto the whole $w$ plane, hence in this case the Riemann mapping theorem cannot hold.

## 1.2 Uniqueness Theorems of Nonlinear Quasiconformal Mappings in Multiply Connected Domains

In order to prove the uniqueness of quasiconformal mappings from the general $N+1$-connected domain onto an $N+1$-connected circular domain, we need to add a condition, i.e. $F(z, w, U)$ in equation (1.1.1) satisfies the inequality:

$$|F(z, w_1, U) - F(z, w_2, U)| \leqslant R(z, w_1, w_2, U)|w_1 - w_2|, \tag{1.2.1}$$

where $w_j = w_j(z)\,(j = 1, 2)$ are any continuous functions in $D\backslash\{0\}$ and the real-valued function $R(z, w_1, w_2, U) \in Lp_0(\overline{D})\,(2 < p_0 < p)$.

**Theorem 1.2.1** *Suppose that equation* (1.1.1) *satisfies Condition* C *and* (1.2.1). *Then the homeomorphic solution $w(z)$ of* (1.1.1) *from the $N+1$-connected domain $D$ onto an $N+1$-connected circular domain $G$ is unique, provided the solution $w(z)$ satisfies one of the following two conditions:*

1) *Three points on the boundary $\Gamma$ remain to be not variable.*

2) *One boundary point and one inner point of the domain $D$ remain to be invariable.*

*Besides, we require that the solution $w(z)$ possesses the expression*

$$w(z) = \Phi[\zeta(z)] \text{ in } D_\varepsilon = \{z|\text{dist}(z, \Gamma) < \varepsilon\,(\varepsilon > 0)\}, \tag{1.2.2}$$

*in which $\zeta(z)$ is a fixed homeomorphism in $D_\varepsilon$ from $\Gamma_j\,(j = 1, \ldots, N)$ onto the circles $L_j\,(j = 1, \ldots, N)$, and $\Phi(\zeta)$ on $\zeta(\Gamma)\,(j = 1, \ldots, N)$ is an analytic function. Note that if $Q(z, w, w_z) = Q(z)$ in $D_\varepsilon$, then the above condition* (1.2.2) *is true.*

Here we mention that for proving uniqueness of quasiconformal mappings from the $N+1$-connected domain onto an $N+1$-connected rectilinear slit domain with $N$ parallel rectilinear slits, or $N+1$-connected circular slit domain with $N$ circular slits, the condition (1.2.2) should be canceled.

**Proof** There is no harm in assuming that the domain $D$ is an $N+1$-connected circular domain as stated in Section 1.1. If $w_1(z)$, $w_2(z)$ are two homeomorphic solutions as stated in the theorem and $w_1(z) \not\equiv w_2(z)$. There is no harm in assuming that

$$|w_1(z) - w_j| = |w_2(z) - w_j| = \rho_j \text{ on } \Gamma_j\ (j = 0, 1, \ldots, m)$$

and

$$|w_1(z)-w_j|=\rho_j^{(1)} \neq |w_2(z)-w_j|=\rho_j^{(2)} \text{ on } \Gamma_j, j=m+1, \ldots, m',$$
$$w_j^1 \neq w_j^2,\ j = m' + 1, \ldots, N,$$

where $w_j$ is the center of the circles $w_1(\Gamma_j)$, $w_2(\Gamma_j)$ $(j = 0, 1, \ldots, m')$, and $w_j^1 \neq w_j^2$, in which $w_j^1$, $w_j^2$ are the centers of the circles $w_1(\Gamma_j)$, $w_2(\Gamma_j)$ $(j = m' + 1, \ldots, N)$, obviously $m \leqslant m' \leqslant N$.

We can symmetrically extend the function $w_k(z)$ $(k = 1, 2)$ onto the outside of the domain $D$ with respect to the circles $\Gamma_j$ $(0 \leqslant j \leqslant m)$, namely

$$W_k(z) = \begin{cases} w_k(z), & z \in \overline{D}, \\ \dfrac{\rho_j^2}{\overline{w_k(r_j^2/\overline{z-z_j}+z_j)-w_j}}+w_j, & z \in D_j\ (j=0, 1, \ldots, m), \end{cases} \tag{1.2.3}$$

where $D_j$ $(j = 0, 1, \ldots, m)$ are the symmetrical domains of $D$ with respect to the circles $\Gamma_j$ $(0 \leqslant j \leqslant m)$. The function $W(z) = W_1(z) - W_2(z)$ satisfies the uniformly elliptic complex equation in the form

$$W_{\bar{z}} = Q^*(z)W_z + A(z)W, \tag{1.2.4}$$

for almost every point in $D_\eta^* = \overline{D} \cup \{1 < |z| < 1 + \eta\} \cup \{r_1 - \eta \leqslant |z - z_1| < r_1\} \cup \cdots \cup \{r_m - \eta \leqslant |z - z_m| < r_m\}$, in which $|Q^*| \leqslant q_0$ in $D_\eta^*$. Provided $\eta$ is sufficiently small, $W(z)$ has not any zero point in $\{\mathbf{C}\backslash\overline{D}\} \cap D_\eta^* \cup \tilde{D}_\eta^*$, where $\tilde{D}_\eta^* = \{1/(1+\eta) < |z| < 1\} \cup \{r_1 < |z - z_1| < r_1^2/(r_1 - \eta)\} \cup \cdots \cup \{r_m < |z - z_m| < r_m^2/(r_m - \eta)\}$, and $W(z)$ can be expressed as

$$W(z) = \Phi[\chi(z)]\mathrm{e}^{\phi(z)}. \tag{1.2.5}$$

Denote $\Gamma_{0\eta} = \{|z| = 1+\eta\}$, $\tilde{\Gamma}_{0\eta} = \{|z| = 1/(1+\eta)\}$, $\Gamma_{j\eta} = \{|z-z_j| = r_j-\eta\}$, $\tilde{\Gamma}_{j\eta} = \{|z - z_j| = r_j^2/(r_j - \eta)\}, j = 1, \ldots, m$, thus we have

$$W(z)=\begin{cases}\dfrac{1}{\overline{w_1(1/\bar{z})}}-\dfrac{1}{\overline{w_2(1/\bar{z})}}=-\dfrac{\overline{W(\zeta)}}{w_1(\zeta)w_2(\zeta)} \text{ for } |z|=|\dfrac{1}{\zeta}|=1+\eta,\\ -\dfrac{\rho_j^2\overline{W(\zeta)}}{[w_1(\zeta)-w_j][w_2(\zeta)-w_j]} \text{ for } |z-z_j|=\dfrac{r_j^2}{|\zeta-z_j|}=r_j-\eta,\\ \qquad j=1,\ldots,m.\end{cases} \tag{1.2.6}$$

Hence we get

$$\begin{aligned}\frac{1}{2\pi}\Delta_{\Gamma_{0\eta}}\arg W(z)&=\frac{1}{2\pi}[\Delta_{\tilde{\Gamma}_{0\eta}}\arg w_1(\zeta)+\Delta_{\tilde{\Gamma}_{0\eta}}\arg w_2(\zeta)-\Delta_{\tilde{\Gamma}_{0\eta}}\arg W(\zeta)]\\&=2-\frac{1}{2\pi}\Delta_{\tilde{\Gamma}_{0\eta}}\arg W(\zeta),\\ \frac{1}{2\pi}\Delta_{\Gamma_{j\eta}}\arg W(z)&=\frac{1}{2\pi}[\Delta_{\tilde{\Gamma}_{j\eta}}\arg(w_1(\zeta)-w_j)\\&\quad+\Delta_{\tilde{\Gamma}_{j\eta}}\arg(w_2(\zeta)-w_j)-\Delta_{\tilde{\Gamma}_{j\eta}}\arg W(\zeta)]\\&=-2-\frac{1}{2\pi}\Delta_{\tilde{\Gamma}_{j\eta}}\arg W(\zeta),\ j=1,\ldots,m,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi}\Delta_{\Gamma_j}\arg W(z)&=\frac{1}{2\pi}\Delta_{\Gamma_j}\arg[(w_1(\zeta)-w_j)-(w_2(\zeta)-w_j)]\\&=\frac{1}{2\pi}\left\{\Delta_{\Gamma_j}\arg[w_1(\zeta)-w_j]+\Delta_{\Gamma_j}\left[1-\frac{w_2(\zeta)-w_j}{w_1(\zeta)-w_j}\right]\right\}\\&=-1\,(j=m+1,\ldots,m')\end{aligned}$$

where $m'\leqslant N$. Noting that $w(z)=w_1(z)-w_2(z)$ possesses the expression as in (1.2.2), where $\Phi(\zeta)$ is an analytic function on $\zeta(\Gamma_j)\,(j=m'+1,\ldots,N)$, we can verify

$$\frac{1}{2\pi}\Delta_{\Gamma_j}\arg W(z)\leqslant 0,\ j=m'+1,\ldots,N. \tag{1.2.7}$$

Setting that

$$\Gamma_\eta=\Gamma_{0\eta}\cup\Gamma_{1\eta}\cup\cdots\cup\Gamma_{m\eta}\cup\Gamma_{m+1}\cup\cdots\cup\Gamma_N,$$
$$\tilde{\Gamma}_\eta=\tilde{\Gamma}_{0\eta}\cup\tilde{\Gamma}_{1\eta}\cup\cdots\cup\tilde{\Gamma}_{m\eta}\cup\Gamma_{m+1}\cup\cdots\cup\Gamma_N,$$

from the above formulas, we get

$$\begin{aligned}\frac{1}{2\pi}\Delta_{\Gamma_\eta}\arg W(z)&=N_D+N_\Gamma\\&=2(1-m)-\frac{1}{2\pi}\Delta_{\tilde{\Gamma}_\eta}\arg W(z)+\frac{1}{\pi}\sum_{j=m+1}^{N}\Delta_{\Gamma_j}\arg W(z)\\&\leqslant 2-N_D,\ \ 2N_D+N_\Gamma\leqslant 2.\end{aligned} \tag{1.2.8}$$

where denote by $N_D$, $N_\Gamma$ the totals of zero points in $D$ and $\Gamma$, which contracts the condition 1) or 2). Hence $W(z)\equiv 0$ in $D$, i.e. $w_1(z)\equiv w_2(z)$ in $D$.

## 1.3 General Quasiconformal Shift Theorems in Multiply Connected Domains

Let $\Gamma_j(j=0,1,\ldots,N)$, $L_j(j=1,\ldots,M)$ be the boundary contours of an $N+M+1$-connected domain $D$ in $\mathbf{C}$, where $\Gamma_j\,(j=1,\ldots,N)$, $L_j\,(j=1,\ldots,M)$ are situated inside $\Gamma_0$. In $D$, there are some mutually exclusive contours $\gamma_j\,(j=1,\ldots,n)$, $l_j\,(j=1,\ldots,m)$. We assume that

$$\Gamma=\cup_{j=0}^{N}\Gamma_j,\ L=\cup_{j=1}^{M}L_j,\ l=\cup_{j=1}^{m}l_j,\ \gamma=\cup_{j=1}^{n}\gamma_j\in C_\mu^1\,(0<\mu<1),$$

and denote

$$D_\gamma^-=\cup_{j=1}^{n}D_{\gamma_j},\ D_l^-=\cup_{j=1}^{m}D_{l_j},\ D^-=(D_\gamma^-\cup D_l^-)\cap D,$$
$$D^+=D\backslash\overline{D^-},\ \tilde{D}_l^-=D^-\cap D_l^-,\ \tilde{D}_l^+=(D^+\cap D^-)\backslash\overline{D_l^-},$$

where $D_{\gamma_j}$ and $D_{l_j}$ are the domains surrounded by $\gamma_j$ and $l_j$, respectively. For convenience, there is no harm in assuming that $D$ is a circular domain, and $\Gamma_0=\{|z|=1\}$, and $z=0\in D^+$. We deal with the nonlinear uniformly elliptic complex equation of the first order

$$w_{\bar z}=F(z,w,w_z),\ F(z,w,w_z)=Q(z,w,w_z)w_z\ \text{ in } D, \tag{1.3.1}$$

and suppose that (1.3.1) satisfies **Condition** C$'$:

(1) $Q(z,w,U)$ is continuous in $w\in\mathbf{C}$ for almost every point $z\in D$, $U\in\mathbf{C}$, and is measurable in $z\in D$ for all continuous functions $w(z)$ and all measurable functions $U(z)\in D^+\cup D^-\backslash\{0\}$.

(2) The complex equation (1.3.1) satisfies the uniform ellipticity condition

$$|F(z,w,U_1)-F(z,w,U_2)|\leqslant q_0|U_1-U_2|, \tag{1.3.2}$$

for almost every point $z\in D$, in which $w,U_1,U_2\in\mathbf{C}$ and $q_0\,(<1)$ is a non-negative constant.

The quasiconformal shift theorem for the nonlinear complex equation (1.3.1) in $D^\pm$ requires assuming the continuity of a homeomorphic solution $w(z)$ in $\overline{D^\pm}\backslash\{0\}$ with the condition $w(0)=\infty$, which also has to satisfy the shift conditions

$$\begin{aligned}&w^+[\alpha(t)]=w^-(z),\ t\in\gamma,\ w^+[\alpha(t)]=\overline{w^-(z)},\ t\in l,\\&w^+[\alpha(t)]=w^-(z),\ t\in L,\ w^+[\alpha(t)]=\overline{w^-(z)}+\mathrm{i}h(t),\ t\in\Gamma,\end{aligned} \tag{1.3.3}$$

where $\alpha(t)$ maps each of $\gamma_j$, $l_j$, $L_j$, and $\Gamma_j$ topologically onto themselves. They give positive shifts on $\gamma\cup\Gamma$ and inverse shifts on $l\cup L$, in which $\alpha[\alpha(t)]=t,\ t\in L\cup\Gamma$, $\alpha(t)$ has the fixed points $a_j\in\Gamma_j\,(j=0,1,\ldots,N)$ and

$$C_\mu^1[\alpha(t),\partial D^\pm]\leqslant d<+\infty,\ |\alpha'(t)|\geqslant 1/d>0, \tag{1.3.4}$$