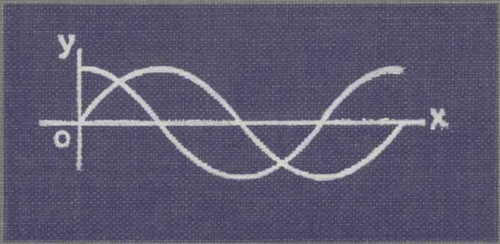

ALGEBRA AND TRIGONOMETRY

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PREFACE

The purpose of this book is to give a modern treatment of algebra and trigonometry that exhibits the logical structure of these disciplines and includes those topics essential for subsequent mathematical study.

Chapter 1 illustrates how the manipulative maneuvers of elementary algebra are based upon the field axioms, which are stated for the real number system. The chapter further provides rigorous proofs of simple theorems. Although the same degree of formalism is not followed throughout the book, it is hoped that students will learn to appreciate what is involved in a proof and acquire some skill in formulating correct proofs. The order axioms for the real numbers are added in Chapter 4 and used extensively in work with inequalities of the type needed in calculus.

Sets and functions are introduced in Chapter 2, and these concepts are used throughout the book.

In the work on equations and systems of equations emphasis is placed upon equivalent equations and equivalent systems. Reduction to echelon form is employed in the solution of systems of linear equations. Matrices and determinants are introduced and applied to linear systems. The algebra of matrices is presented as an example of an algebraic system that possesses many but not all of the properties of the algebra of real numbers.

The treatment of mathematical induction, the binomial theorem, progressions, exponents, and logarithms is for the most part conventional. Some or all of these topics can be omitted by students who studied them in high school.

Complex numbers are introduced as ordered pairs of real numbers, with the usual notation given later. The relations connecting complex numbers, vectors, and polar and Cartesian coordinates are described, and applications of the trigonometric form are offered. The topics discussed in the chapter on the theory of equations are those needed for later work in mathematics.

The work on trigonometry is concentrated in Chapters 9 and 10, except for the applications to complex numbers. Chapter 9 deals with trigonometric functions of angles and their applications to indirect measure and vectors. This chapter can be omitted or covered rapidly by students who have had a traditional trigonometry course in high school. Chapter 10 treats circular functions of real numbers and the really important aspects of trigonometry. The relation between circular and trigonometric functions, which permits a dual interpretation of identities and equations, is stressed. The role that the periodic character of the circular functions plays in applications to problems in physical science is illustrated by examples from harmonic motion and sound.

Throughout the book, both in algebra and trigonometry, the function concept is emphasized, in the belief that practice in thinking in terms of functions will be of great value to the student. The axiomatic approach in Chapters 1, 4, and 11 will give the student some indication of the nature of modern algebra. A serious attempt has been made to maintain as high a level of rigor as is feasible at this stage in the student's development.

The book is adaptable to courses of different lengths. It can be covered completely in a year course. A three-hour semester course for students with fair training in high school algebra and some trigonometry can be based upon Chapters 1, 2, 3, 4, 5, 6, 10, 11, and 12. Courses of four and five semester hours are also possible.

The author is grateful for the permission given him by the late Professor Edward T. Browne to use some of the material in their *College Algebra*. He also expresses his appreciation to Professors Burton W. Jones and Gaylord Merriman who read portions of the manuscript and made many valuable suggestions.

E. A. C.

Chapel Hill, North Carolina
January 1960

CONTENTS

I

NUMBERS

1. Introduction

Algebra is concerned with numbers and with objects which behave like numbers. Two of the most important number systems are the **integers** (positive, negative, and zero whole numbers) and the **rational numbers** (integers and fractions). The word *rational* used in this connection comes from *ratio,* because of the fact that any rational number can be expressed in the form $\frac{a}{b}$, as the ratio of the integers a and b. The rules for adding, subtracting, multiplying, and dividing rational numbers are assumed to be familiar. For most practical problems involving measure, the rational numbers suffice.

There are, however, problems in mathematics for which the rational numbers are inadequate. For example, there is no rational number which expresses exactly the length of the hypotenuse of a right triangle with legs each of length 1 (see Fig. 1–1). This interesting fact is reputed to have been discovered by the Greek mathematician, Pythagoras, in the fifth century B.C. We sketch a proof of it here.

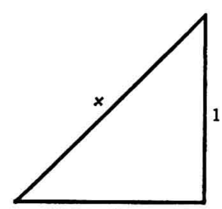

Fig. 1–1

If we denote the length of the hypotenuse of the triangle in Fig. 1–1 by x, the famous Pythagorean theorem in geometry yields

$$x^2 = 1^2 + 1^2, \quad x^2 = 2.$$

Hence, the proof reduces to showing that there is no rational number whose square is 2. To show this we use the indirect method of proof. That is, we assume the opposite of what we want to prove and demonstrate that this leads to a contradiction. Hence, we conclude that the original proposition must be true.

The next step then is to assume that there *is* a rational number $\frac{a}{b}$ whose square is 2. A rational number can always be reduced to lowest terms —

that is, to a form in which numerator and denominator have no common integral divisor greater than 1. We may assume that this reduction has been done and that a and b have *no common factor greater than* 1. We have

$$\left(\frac{a}{b}\right)^2 = 2, \quad \frac{a^2}{b^2} = 2, \quad a^2 = 2b^2.$$

The right member, $2b^2$, of the last equation is an even integer, and hence the left member, a^2, is also an even integer. This means that the integer a itself must be even. For if a were odd, its square, a^2, would also be odd (see Exercise 2 at end of Section 2). Since a is even, we may write $a = 2c$, where c is an integer. Substituting $2c$ for a in the equation above, we get

$$(2c)^2 = 2b^2, \quad 4c^2 = 2b^2, \quad 2c^2 = b^2.$$

The same type of argument used to show that a is even can be applied to the equation $2c^2 = b^2$ to show that b is even. Since a and b are both even, they have 2 *as a common factor*. This contradicts the fact that *a and b have no common factor greater than* 1. Thus, the assumption that there exists a rational number $\frac{a}{b}$ in lowest terms such that $\left(\frac{a}{b}\right)^2 = 2$ leads to a contradiction. We conclude that there is no rational number whose square is 2.

It would be a highly unsatisfactory state of affairs to be able to construct a line segment and have no number to represent its length. To remedy this and other deficiencies the **irrational** numbers were devised. Included among the irrationals are $\sqrt{2}$ and the number π which denotes the ratio of the circumference to the diameter of any circle. We cannot adequately define an irrational number here. Instead, in the next section we shall give a partial characterization of the system consisting of both the *rational* and the *irrational* numbers. This system is called the **real number system,** and is the one with which we shall be most concerned in this book.

The composition of the system of real numbers is indicated in the following diagram:

Real number system	Rational numbers	Integers $(\cdots, -1, 0, 1, 2, \cdots)$
		Fractions $(\frac{1}{2}, \frac{4}{5}, -\frac{2}{3}, \cdots)$
	Irrational numbers	$(\sqrt{3}, \sqrt[5]{11}, \pi, -\sqrt[3]{4}, \cdots)$

2. Algebraic Properties of the Real Number System

We now state some of the fundamental properties of the real number system, involving the relation of equality (=) and the operations of addition (+) and multiplication (·). (Also, juxtaposition of two letters, such as ab, indicates multiplication and is the notation usually followed.) Throughout the chapter the letters a, b, and c denote real numbers.

PROPERTIES OF EQUALITY

E_1. $a = a$. **(Reflexive law)**

E_2. If $a = b$, then $b = a$. **(Symmetric law)**

E_3. If $a = b$ and $b = c$, then $a = c$. **(Transitive law)**

E_4. If $a = a'$ and $b = b'$, then $a + b = a' + b'$.

E_5. If $a = a'$ and $b = b'$, then $ab = a'b'$.

Properties E_1, E_2, and E_3 are called the three **equivalence laws.**

Properties E_4 and E_5, called the **well-defined properties** of addition and multiplication, permit the substitution of a letter symbol for its equal in any expression formed by taking sums and products. For example, suppose $x = u$ and $y = v$, and consider the expression $x(2x + 3y)$. By applying E_4 to $x = u$ and $x = u$, we get $2x = 2u$; and from $y = v$, by applying E_4 twice, we get $3y = 3v$. Then from $2x = 2u$ and $3y = 3v$, by E_4, we have $2x + 3y = 2u + 3v$. Finally, E_5 applied to $x = u$ and $2x + 3y = 2u + 3v$ yields

$$x(2x + 3y) = u(2u + 3v).$$

PROPERTIES OF ADDITION

A_1. The set of real numbers is **closed under addition.** (That is, the sum $a + b$ of two real numbers a and b is a real number.)

A_2. The **associative law** holds for the addition of real numbers. (That is, $(a + b) + c = a + (b + c)$ for all real numbers a, b, and c.)

A_3. There is a real number **zero** (0) such that

$$a + 0 = 0 + a = a \qquad \text{for every real number } a.$$

(Zero is called an **identity element** for addition.)

A_4. Corresponding to each real number a there is a real number $-a$, called the **negative** of a, such that

$$a + (-a) = (-a) + a = 0.$$

($-a$ is also called the **additive inverse** of a.)

A_5. The **commutative law** holds for the addition of real numbers. (That is, $a + b = b + a$ for all real numbers a and b.)

Properties of multiplication

M_1. The set of real numbers is **closed under multiplication.** (That is, the product ab of two real numbers a and b is a real number.)

M_2. The **associative law** holds for the multiplication of real numbers. (That is, $(ab)c = a(bc)$ for all real numbers, a, b, and c.)

M_3. There is a real number **one** (1), such that $1 \neq 0$, and

$$a \cdot 1 = 1 \cdot a = a \quad \text{for each real number } a.$$

(1 is called an **identity element** for multiplication.)

M_4. Corresponding to each *nonzero* real number a, there is a real number $\frac{1}{a}$, called the **reciprocal** of a, such that

$$a\frac{1}{a} = \frac{1}{a}a = 1.$$

$\left(\frac{1}{a}\right.$ is also called the **multiplicative inverse** of $\left.a.\right)$

M_5. The **commutative law** holds for the multiplication of real numbers. (That is, $ab = ba$ for all real numbers a and b.)

The (left) distributive law

D. If a, b, and c are real numbers, then $a(b + c) = ab + ac$.

The properties stated here for real numbers are no doubt familiar to the reader. The significance of this set of properties is that the manipulative maneuvers of elementary algebra can be justified on the basis of these properties. For example, the various devices for factoring depend upon the distributive law. Thus, the subject of elementary algebra can be made into a logical structure (similar in form to Euclid's geometry) if one takes a set of statements such as $E_1 - E_5$, $A_1 - A_5$, $M_1 - M_5$, and D as axioms and proves the rules and theorems of algebra from them. We shall shortly illustrate this procedure by a few examples.

First we remark that any mathematical system having properties $E_1 - E_5$, $A_1 - A_5$, $M_1 - M_5$, and D is called a **field.** Thus, the real numbers form a field. Drawing upon his experience, the reader can convince himself that the set of rational numbers also possesses all of the properties mentioned above and thus is another example of a field. The complex numbers, to be studied in Chapter 11, also form a field. The study of fields in general is an important part of advanced modern algebra.

The following examples illustrate the use of the basic properties of the real numbers to perform some algebraic manipulations. The reason justifying each step is listed in the column at the right. Recall that E_4 and E_5 permit the substitution of a letter symbol for its equal in an expression formed by taking sums and products. No reason will be listed when such substitutions are made.

Example 1

Prove: $(a + b)c = ac + bc$. (This property is called the **right distributive law.**)

PROOF:

(1)	$(a + b)c = c(a + b)$	M_5
(2)	$c(a + b) = ca + cb$	D
(3)	$(a + b)c = ca + cb$	(1), (2), and E_3
(4)	$ca + cb = ac + bc$	M_5
(5)	Hence, $(a + b)c = ac + bc$	(3), (4), and E_3.

Example 2

Prove: $a(b + c) + bc = ab + (a + b)c$.

PROOF:

(1)	$a(b + c) + bc = (ab + ac) + bc$	D
(2)	$(ab + ac) + bc = ab + (ac + bc)$	A_2
(3)	$a(b + c) + bc = ab + (ac + bc)$	(1), (2), and E_3
(4)	$ab + (ac + bc) = ab + (a + b)c$	Right distributive law
(5)	$a(b + c) + bc = ab + (a + b)c$	(3), (4), and E_3.

EXERCISES

1. An *even* integer is an integer of the form $2n$ and an *odd* integer is one of form $2n + 1$, where n is an integer. Show that the set of even integers is closed under addition. (That is, show that the sum of two even integers is an even integer.) Is the set of odd integers closed under addition?
2. Prove: The set of even integers is closed under multiplication and the set of odd integers is closed under multiplication. (See Ex. 1.)
3. Construct a proof for the statement that there is no rational number whose square is 3.

   Hint: Study the proof given in Section 1 and use the fact that if $a^2 = 3b^2$, a and b integers, then a must have 3 as a factor.

4. Which of the properties $A_1 - A_5$, $M_1 - M_5$, D hold for the set of all integers? The set of proper fractions (numerators less than denominators)?
5. Use E_2 and E_3 to prove: If $a = c$ and $b = c$, then $a = b$.

In making the proofs for Exercises 6–14 use only the axioms stated in Section 2 and previously proved exercises.

6. Prove: If $a = b$ and $a = a'$, $b = b'$, then $a' = b'$.
7. Prove: $(a + b) + c = (a + c) + b.$
8. Prove: $(ab + cd) + ac = a(b + c) + cd.$
9. Prove: $(ab + cd) + ac = ab + c(a + d).$
10. Prove: $ab + (a + b)c = (a + c)b + ac.$
11. Prove: $b(a + 1) + [a + (-b)] = a(b + 1).$
12. Prove: $(a + b)(c + d) = (ac + bc) + (ad + bd).$
13. Prove: $(x + a)(x + a) = x^2 + 2ax + a^2.$
14. Prove: $(x + a)(x + b) = x^2 + (a + b)x + ab.$

3. Some Further Properties of Real Numbers

In Section 2 some properties ($E_1 - E_5$, $A_1 - A_5$, $M_1 - M_5$, D) of the real numbers were stated and applied to a few simple exercises. These properties form an axiomatic foundation for most of elementary algebra. That is, most of the processes and theorems of elementary algebra can be justified or proved by deductive reasoning on the basis of these fundamental properties. In this section we shall prove a few theorems to illustrate how the subject of elementary algebra can be developed logically from a set of axioms — that is, from a set of statements assumed to be true. The structure of any branch of mathematics must ultimately rest upon a set of statements whose truth is assumed.

The formulation of proofs in algebra is probably somewhat unfamiliar to the reader, and he may become impatient with the details involved in writing out a proof — especially when he has known the conclusion for a long time. However, proof is an indispensable part of mathematics, and the practice secured here will help to acquaint the reader with the true nature of mathematics as a logical structure. We invite him to enter into the spirit of what is being attempted.

At first, in writing out proofs, we take only one step at a time and list the reason for each step, as in the examples in Section 2. As one develops skill, several steps may be combined and the reasons not written down; but there must be reasons, and one must be prepared to give them if called upon.