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by

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Dedicated to Professor Shiing-Shen Chern

for his 92nd Birthday

Preface

> *The problem of distinguishing prime numbers from composite, and of resolving composite numbers into their prime factors, is one of the most important and useful in all arithmetic. ... The dignity of science seems to demand that every aid to the solution of such an elegant and celebrated problem be zealously cultivated.*
>
> C. F. GAUSS (1777–1855)

Primality testing and integer factorization, as identified by Gauss in his *Disquisitiones Arithmeticae*, Article 329, in 1801, are the two most fundamental problems, as well as two most important research fields in number theory, particularly in *computational* number theory[1]. With the advent of digital computers, they have also been found unexpected and surprising applications in computing and particularly in cryptography and information security. In this book, we shall introduce various methods/algorithms for primality testing and integer factorization, and their applications in public-key cryptography and information security. More specifically, we shall first review some basic concepts and results in number theory in Chapter 1. Then in Chapter 2 we shall discuss various algorithms for primality testing and prime number generation, with an emphasis on the Miller-Rabin probabilistic test, the Goldwasser-Kilian and Atkin-Morain elliptic curve tests, and the Agrawal-Kayal-Saxena deterministic test. There is also an introduction to large prime number generation in Chapter 2. In Chapter 3 we shall introduce various algorithms, particularly the Elliptic Curve Method (ECM), the Quadratic Sieve (QS) and the Number Field Sieve (NFS) for integer factorization. Also in Chapter 3 we shall discuss some other computational problems that are related to factoring, such as the square root problem, the discrete logarithm problem and the quadratic residuosity problem. In Chapter 4, we shall discuss

[1] Of course, the primality testing problem (PTP) has *now* been solved, thanks to Agrawal, Kayal and Saxena [5]. That is, the PTP can now be solved in $\mathcal{P}$ (deterministic polynomial-time). However, the integer factorization problem (IFP) is still open. That is, we still do not have an efficient (i.e., deterministic polynomial-time) algorithm for IFP; in the author's opinion, the IFP may indeed be an $\mathcal{NP}$-hard problem, although no proof can be given yet at present.

some of the most widely used cryptographic systems based on the computationally intractable problems such as integer factorization, square roots, quadratic residuosity, discrete logarithms, and elliptic curve discrete logarithms.

We have tried to make this book as self-contained as possible, so that it can be used either as a textbook suitable for a course for final-year undergraduate or first-year postgraduate students, or as a basic reference in the field.

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Coventry, September 2003 S. Y. Y.

Notation

All notation should be as simple as the nature of the operations to which it is applied.

CHARLES BABBAGE (1791–1871)

Notation	Explanation
$\mathbb{N}$	set of natural numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$
$\mathbb{Z}$	set of integers (whole numbers): $\mathbb{Z} = \{0, \ \pm n : n \in \mathbb{N}\}$
$\mathbb{Z}^+$	set of positive integers: $\mathbb{Z}^+ = \mathbb{N}$
$\mathbb{Z}_{>1}$	set of positive integers greater than 1: $\mathbb{Z}_{>1} = \{n : n \in \mathbb{Z} \text{ and } n > 1\}$
$\mathbb{Q}$	set of rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$
$\mathbb{R}$	set of real numbers: $\mathbb{R} = \{n + 0.d_1d_2d_3\cdots : n \in \mathbb{Z},\ d_i \in \{0, 1, \cdots, 9\}$ and no infinite sequence of 9's appears$\}$
$\mathbb{C}$	set of complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$
$\mathbb{Z}/n\mathbb{Z}$	also denoted by $\mathbb{Z}_n$, residue classes modulo n; ring of integers modulo n; field if n is prime
$(\mathbb{Z}/n\mathbb{Z})^*$	multiplicative group; the elements of this group are the elements in $\mathbb{Z}/n\mathbb{Z}$ that are relatively prime to n: $(\mathbb{Z}/n\mathbb{Z})^* = \{[a]_n \in \mathbb{Z}/n\mathbb{Z} : \ \gcd(a, n) = 1\}$
$\#((\mathbb{Z}/n\mathbb{Z})^*)$	also denoted by $\|(\mathbb{Z}/n\mathbb{Z})^*\|$, order of the group $(\mathbb{Z}/n\mathbb{Z})^*$, i.e., the number of elements in the group
$\mathbb{F}_p$	finite field with p elements, where p is a prime number
$\mathbb{F}_q$	finite field with $q = p^k$ a prime power

$\mathbb{Z}[x]$	set of polynomials with integer coefficients
$\mathbb{Z}_n[x]$	set of polynomials with coefficients from $\mathbb{Z}_n$
$\mathbb{Z}[x]/h(x)$	set of polynomials modulo polynomial $h(x)$, with integer coefficients
$\mathbb{Z}_p[x]/h(x)$	also denoted by $\mathbb{F}_p[x]/h(x)$; set of polynomials modulo polynomial $h(x)$, with coefficients from $\mathbb{Z}_p$
G	group
$\|G\|$	also denoted by $\#(G)$, order of group G
R	ring
K	(arbitrary) field
E	elliptic curve $y^2 = x^3 + ax + b$
$E/\mathbb{Q}$	elliptic curve over $\mathbb{Q}$
$E/\mathbb{Z}_n$	elliptic curve over $\mathbb{Z}_n$
$E/\mathbb{F}_p$	elliptic curve over $\mathbb{F}_p$
$\mathcal{O}_E$	point at infinity on E
$E(\mathbb{Q})$	elliptic curve group formed by points on $E/\mathbb{Q}$
$\|E(\mathbb{Q})\|$	number of points in $E(\mathbb{Q})$
$\Delta(E)$	discriminant of E, $\Delta(E) = -16(4a^3 + 27b^2) \neq 0$
F_n	Fermat numbers: $F_n = 2^{2^n} + 1,\ n \geq 0$
$\mathcal{P}$	class of problems solvable in deterministic polynomial time
$\mathcal{NP}$	class of problems solvable in non-deterministic polynomial time
$\mathcal{RP}$	class of problems solvable in random polynomial time with one-sided errors
$\mathcal{ZPP}$	class of problems solvable in random polynomial time with zero errors
IFP	Integer Factorization Problem
DLP	Discrete Logarithm Problem
ECDLP	Elliptic Curve Discrete Logarithm Problem
SQRTP	SQuare RooT Problem
QRP	Quadratic Residuosity Problem
CFRAC	Continued FRACtion method (for factoring)
ECM	Elliptic Curve Method

NFS	Number Field Sieve
QS/MPQS	Quadratic Sieve/Multiple Polynomial Quadratic Sieve
ECPP	Elliptic Curve Primality Proving
DHM	Diffie-Hellman-Merkle
RSA	Rivest-Shamir-Adleman
DSA/DSS	Digital Signature Algorithm/Digital Signature Standard
$a \mid b$	a divides b
$a \nmid b$	a does not divide b
$p^\alpha \mid\mid n$	$p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$
$\gcd(a, b)$	greatest common divisor of (a, b)
$\text{lcm}(a, b)$	least common multiple of (a, b)
$\lfloor x \rfloor$	floor: also denoted by $[x]$; the greatest integer less than or equal to x
$\lceil x \rceil$	ceiling: the least integer greater than or equal to x
x mod n	remainder: $x - n \left\lfloor \frac{x}{n} \right\rfloor$
$x = y \bmod n$	x is equal to y reduced to modulo n
$x \equiv y \pmod{\text{n}}$	x is congruent to y modulo n
$x \not\equiv y \pmod{\text{n}}$	x is not congruent to y modulo n
$f(x) \equiv g(x) \pmod{\text{h(x)}, \text{n}}$	$f(x)$ is congruent to $g(x)$ modulo $h(x)$, with coefficients modulo n
$[a]_n$	residue class of a modulo n
$+_n$	addition modulo n
$-_n$	subtraction modulo n
$\cdot_n$	multiplication modulo n
$\sqrt{x} \pmod{n}$	square root of x modulo n
$\sqrt[k]{x} \pmod{n}$	kth root of x modullo n
$x^k \bmod n$	x to the power k modulo n
$\log_x y \bmod n$	discrete logarithm of y to the base x modulo n
x^k	x to the power k
kP	$kP = \underbrace{P \oplus P \oplus \cdots \oplus P}_{k \text{ summands}}$, where P is a point (x, y) on elliptic curve $E: \; y^2 = x^3 + ax + b$

$kP \bmod n$	kP modulo n, where P is a point on E
$\log_P Q \bmod n$	elliptic curve discrete logarithm of Q to the base P modulo n, where P and Q are points on elliptic curve E
$\mathrm{ord}_n(a)$	order of an integer a modulo n; also denoted by $\mathrm{ord}(a, n)$
$\mathrm{ind}_{g,n} a$	index of a to the base g modulo n; also denoted by $\mathrm{ind}_g a$ whenever n is fixed
$\sim$	asymptotic equality
$\approx$	approximate equality
∞	infinity
$\Longrightarrow$	implication
$\Longleftrightarrow$	equivalence
$\Box$	blank symbol; end of proof
$\sqcup$	space
Prob	probability measure
$\vert S \vert$	cardinality of set S
$\in$	member of
$\subset$	proper subset
$\subseteq$	subset
$\star, *$	binary operations
$\oplus$	binary operation (addition)
$\odot$	binary operation (multiplication)
$f(x) \sim g(x)$	$f(x)$ and $g(x)$ are asymptotically equal
$\perp$	undefined
$f(x)$	function of x
f^{-1}	inverse of f
$\binom{n}{i}$	binomial coefficient: $\binom{n}{i} = \frac{n!}{i!(n-i)!}$
$\int$	integration
$\mathrm{Li}(x)$	logarithmic integral: $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\ln t}$
$\sum_{i=1}^{n} x_i$	sum: $x_1 + x_2 + \cdots + x_n$

$\prod_{i=1}^{n} x_i$	product: $x_1 x_2 \cdots x_n$
$n!$	factorial: $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$
$\log_b x$	logarithm of x to the base b ($b \neq 1$): $x = b^{\log_b x}$
$\log x$	binary logarithm: $\log_2 x$
$\ln x$	natural logarithm: $\log_e x$, $e = \sum_{n \geq 0} \frac{1}{n!} \approx 2.7182818$
$\exp(x)$	exponential of x: $e^x = \sum_{n \geq 0} \frac{x^n}{n!}$
$\pi(x)$	number of primes less than or equal to x: $\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1$
$\tau(n)$	number of positive divisors of n: $\tau(n) = \sum_{d \mid n} 1$
$\sigma(n)$	sum of positive divisors of n: $\sigma(n) = \sum_{d \mid n} d$
$\phi(n)$	Euler's totient function: $\phi(n) = \sum_{\substack{0 \leq k < n \\ \gcd(k,n)=1}} 1$
$\lambda(n)$	Carmichael's function: $\lambda(n) = \operatorname{lcm}(\lambda(p_1^{\alpha_1})\lambda(p_2^{\alpha_2})\cdots\lambda(p_k^{\alpha_k}))$ if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$
$\mu(n)$	Möbius function
$\zeta(s)$	Riemann zeta-function: $\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{n^s}$, where s is a complex variable
$\left(\frac{a}{p}\right)$	Legendre symbol, where p is prime
$\left(\frac{a}{n}\right)$	Jacobi symbol, where n is composite
Q_n	set of all quadratic residues of n
$\overline{Q}_n$	set of all quadratic non-residues of n
J_n	$J_n = \left\{ a \in (\mathbb{Z}/n\mathbb{Z})^* : \left(\frac{a}{n}\right) = 1 \right\}$
$\tilde{Q}_n$	set of all pseudo-squares of n: $\tilde{Q}_n = J_n - Q_n$
$K(k)_n$	set of all kth power residues of n, where $k \geq 2$
$\overline{K(k)}_n$	set of all kth power non-residues of n, where $k \geq 2$
$[q_0, q_1, q_2, \cdots, q_n]$	finite simple continued fraction
$C_k = \frac{P_k}{Q_k}$	kth convergent of a continued fraction

$[q_0, q_1, q_2, \cdots]$	infinite simple continued fraction
$[q_0, q_1, \cdots, q_k, \overline{q_{k+1}, q_{k+2}, \cdots, q_{k+m}}]$	periodic simple continued fraction
e_k	encryption key
d_k	decryption key
$E_{e_k}(M)$	encryption process $C = E_{e_k}(M)$, where M is the plain-text
$D_{d_k}(C)$	decryption process $M = D_{d_k}(C)$, where C is the cipher-text
$\mathcal{O}(\cdot)$	upper bound: $f(n) = \mathcal{O}(g(n))$ if there exists *some* constant $c > 0$ such that $f(n) \leq c \cdot g(n)$
$\mathcal{O}(N^k)$	polynomial-time complexity measured in terms of arithmetic operations, where $k > 0$ is a constant
$\mathcal{O}\left((\log N)^k\right)$	polynomial-time complexity measured in terms of bit operations, where $k > 0$ is a constant
$\mathcal{O}\left((\log N)^{c \log N}\right)$	superpolynomial complexity, where $c > 0$ is a constant
$\mathcal{O}\left(\exp\left(c\sqrt{\log N \log\log N}\ \right)\right)$	subexponential complexity, $\mathcal{O}\left(\exp\left(c\sqrt{\log N \log\log N}\ \right)\right) = \mathcal{O}\left(N^{c\sqrt{\log\log N/\log N}}\right)$
$\mathcal{O}(\exp(x))$	exponential complexity, sometimes denoted by $\mathcal{O}(e^x)$
$\mathcal{O}(N^\epsilon)$	exponential complexity measured in terms of bit operations; $\mathcal{O}(N^\epsilon) = \mathcal{O}\left(2^{\epsilon \log N}\right)$, where $\epsilon > 0$ is a constant

Table of Contents

1. Number-Theoretic Preliminaries

Mathematics is the Queen of the sciences, and number theory is the Queen of mathematics.

C. F. GAUSS (1777–1855)

1.1 Introduction

The theory of numbers is primarily the theory of the *properties* of integers (whole numbers), such as parity, divisibility, primality, additivity, multiplicativity, and unique factorization, etc. One of the important features of number theory is that problems in number theory are generally easy to state but often very difficult to solve. The following are just some examples:

1) Fermat Last Theorem: The Fermat Last Theorem asserts that if $n > 2$, the equation

$$x^n + y^n = z^n \tag{1.1}$$

cannot be solved in integers x, y, z, with $xyz \neq 0$. Fermat (1601–1665) claimed in a margin of his copy of Diophantus's book that he had found a beautiful proof of this theorem, but the margin was too small to contain his proof. Later on mathematicians everywhere in the world struggled to find a proof for this theorem but without success. The theorem remained open for more than 300 years and was finally settled in June 1995 by Andrew Wiles (partly in joint work with Richard Taylor).

2) Goldbach Conjecture: In a letter to Euler (1707–1783), dated 7 June 1742, Christian Goldbach (1690–1764) conjectured that *every even integer greater than 4 is the sum of two prime numbers*. Despite much effort has been made, this conjecture remains unsolved to this day, and the best result is still that *Every sufficiently large even integer can be written as the sum of a prime and a product of at most two primes*, proved by