MATRIX ALGEBRA

DAVID J. WINTER

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University of Michigan, Ann Arbor

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To the memory of I. N. Herstein

PREFACE

Matrix algebra is vitally important as a tool in such subjects as chemistry, economics, engineering, mathematics, physics, and scientific computation. Important problems in these fields can be reduced to problems in matrix algebra, which can be solved accurately on high-speed computers. For this reason, students often get a first course in linear algebra fairly early in their curriculum. A casualty of this is that geometric aspects of linear algebra often get little attention. This is unfortunate, for much of linear algebra owes its existence to the geometric intuitions of its creators, and many of its methods can best be understood in connection with their geometric interpretations.

This book, *Matrix Algebra*, presents computational linear algebra within a geometric context. The general theme is that of matrix manipulation, which keeps things at a concrete and computational level. Geometric phenomena are illustrated by diagrams that complement and simplify the discussion and help the reader gain a deeper, more conceptual grasp of the subject. In this way, the book can be comprehensive as well as accessible.

Although this book can be used as the text for a two-semester course covering all sections, it is intended primarily for use in a one-semester course, wherein Chapters 1 and 2 are covered rapidly and many or all sections marked with asterisks (*) are omitted. This gives the instructor flexibility to add topics appropriate for the particular course, and it makes the book more useful to students for reference purposes later on. Some sample courses are outlined in the Instructor's Manual.

Since a true understanding of linear algebra requires using it from the outset to solve problems, many problems are included to make up a kind of "laboratory," or "proving ground," for developing problem-solving skills. The Instructor's Manual provides solutions to all problems. The Instructor's Manual also provides a detailed Student's Manual, which the instructor can make available to students through any copy center.

The book is divided into three parts:

Part 1 (Chapter 1) is a prologue, or preview in the microcosm of the plane, of what is to come. As such, it should be covered rapidly.

Part 2 (Chapters 2–4) is concerned with vector spaces and linear transformations. To set the stage, Chapter 2 introduces Cartesian n-space. This can be done quickly. Chapter 3 is about linear transformations of Cartesian n-space and should be covered more slowly. The remainder of Part 2, Chapter 4, is concerned with general n-dimensional vector spaces and their linear transformations. In this general setting, bases are discussed and the effect of a change of basis on the matrix of a linear transformation is described. Chapter 4 ends by showing how problems in this general context can be reduced to concrete problems about Cartesian n-space. Specifically, it explains the fundamental principle that problems stated in terms of linear transformations of a vector space of dimension n can be translated by an isomorphism to problems stated in terms of matrices viewed as linear transformations of F^n.

Part 3 (Chapters 5–7) then goes on to develop the linear algebra and geometry of Cartesian n-space, including the theory of quadratic forms.

Appendix A ("Sets, Elements, and Functions"), Appendix B ("Real Numbers"), and Appendix C ("Complex Numbers") are included for students to consult or review as needed.

Each chapter and appendix ends with references to other works that provide background or illustrate some of the richness and diversity of linear algebra.

We take this opportunity to thank the many people who have read the manuscript and made useful comments and contributions. We thank Bill Blair, Dan Britten, and Gene Klotz for their many and varied suggestions, which led to very substantial improvements. We also thank Bob Pirtle, Phyllis Niklas, and Linda Thompson for their excellent editorial help in bringing this book into being. Finally, we thank Molly Thornton and Reese Thornton of Folium for their intricate and creative preparation of the many supporting illustrations.

CONTENTS

* Optional sections; see the Preface.

PART 2: VECTOR SPACES AND LINEAR TRANSFORMATIONS

* Optional sections; see the Preface.

PART 3: THE LINEAR ALGEBRA AND GEOMETRY OF n-SPACE

* Optional sections; see the Preface.

LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers	2, 504
x_1-axis, x_2-axis	Axes of the Cartesian plane	2
a, b, etc.	Scalars	3
$\mathbf{v}$, $\mathbf{w}$, etc.	Vectors	3
$\mathbf{0}$	Zero vector	3
$\mathbf{v} = \mathbf{w}$	Equality of vectors	4
$\mathbb{R}^2$	Set of vectors $\begin{bmatrix} a \\ b \end{bmatrix}$	4
(a, b)	Point, ordered pair, tip of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$	4
$\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$, $-\mathbf{v}$	Sum, difference, negative of vectors	5, 6, 7
$t\mathbf{v}$	Scalar times vector	5
$-\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}$	Negative of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$	6
$\mathbf{v} = \overrightarrow{\mathbf{0}\,\mathbf{v}}$	$\mathbf{v}$ as directed line segment from $\mathbf{0}$ to $\mathbf{v}$	7
$\overrightarrow{\mathbf{v}\,\mathbf{w}}$	Directed line segment from $\mathbf{v}$ to $\mathbf{w}$	7
$\overline{\mathbf{v}\,\mathbf{w}}$	Line segment from $\mathbf{v}$ to $\mathbf{w}$	7
$\lvert A \rvert$	Determinant of a matrix A	15
R, S	Linear transformations	15, 16
$\mathbf{e}_1$, $\mathbf{e}_2$	Standard unit vectors of the Cartesian plane	25
0	Zero linear transformation	28

I	Identity linear transformation	28
$R + S, R - S, -S$	Sum, difference, negative of linear transformations of $\mathbb{R}^2$	32, 35
tR	Scalar times linear transformation	32
RS	Product of linear transformations	32
R^{-1}	Inverse of the linear transformation R	36
A, B	Matrices	40
$A\mathbf{x}$	Matrix times a vector	41
$0, I$	Zero and identity matrices	42
$m(R) = A$	Matrix A of a linear transformation R	42
$A + B, A - B, -B$	Sum, difference, negative of matrices	43, 44
rA	Scalar r times matrix A	43, 44
AB	Product of matrices	43, 44
A^T	Transpose of the matrix A	46
A^{-1}	Inverse of the matrix A	46
$\mathbf{v} \cdot \mathbf{w}$	Dot product of vectors $\mathbf{v}$, $\mathbf{w}$	49
$\mathbf{v} \cdot \mathbf{v}$	Squared length of the vector $\mathbf{v}$	50
$\lVert\mathbf{v}\rVert$	Length of $\mathbf{v}$	50
$d(\mathbf{v}, \mathbf{w}) = \lVert\mathbf{w} - \mathbf{v}\rVert$	Distance from $\mathbf{v}$ to $\mathbf{w}$	50
$\dfrac{\mathbf{v}}{\lVert\mathbf{v}\rVert}$	Direction of the vector $\mathbf{v}$	52
$\mathbf{v_w}, \mathbf{v_w^\perp}$	Components of the vector $\mathbf{v}$ parallel and perpendicular to the vector $\mathbf{w}$	55
α	Angle between vectors $\mathbf{v}$ and $\mathbf{w}$	58
$R_\alpha(\mathbf{x})$	Rotation through α	58
$\sin\alpha, \cos\alpha$	Sine and cosine of angle between $\mathbf{v}$ and $\mathbf{w}$	59
$\lvert\mathbf{v}, \mathbf{w}\rvert$	Determinant of $[\mathbf{v} \quad \mathbf{w}]$ (signed area)	59
$\mathbf{1}, \mathbf{v}, \mathbf{vw}, \mathbf{v}\#\mathbf{w}, \mathbf{v}^*$	Circle group elements and operations	65–66
$\mathbf{uv}$	Complex multiplication	66
$S_\alpha\mathbf{x} = R_\alpha\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}$	Reflection corresponding to α	73
$r_\mathbf{w}$	Reflection across the normal to $\mathbf{w}$	74
$f(x_1, x_2)$	Quadratic function	80
c	Characteristic root for A	86
$\mathbb{C}$	Field of complex numbers	94, 514
F	Field of scalars (usually F is $\mathbb{R}$ or $\mathbb{C}$)	94
F^n	Cartesian n-space, space of column vectors	94
$\mathbf{a}$	Point, vector, n-tuple of scalars	94
$\overrightarrow{\mathbf{v}\,\mathbf{w}}$	Directed line segment from $\mathbf{v}$ to $\mathbf{w}$	96, 107
$\mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w}, -\mathbf{v}$	Sum, difference, negative of vectors	97, 102
$t\mathbf{v}$	Scalar times vector	97
$\mathbf{a} \cdot \mathbf{b}$	Dot product of vectors $\mathbf{a}$, $\mathbf{b}$	99
$\bar{\mathbf{z}}$	Complex conjugate of the vector $\mathbf{z}$	100
$\langle\mathbf{a}, \mathbf{b}\rangle$	Inner product of vectors $\mathbf{a}$, $\mathbf{b}$	100

PART 1

PROLOGUE

CHAPTER

1

THE GEOMETRY AND LINEAR ALGEBRA OF THE CARTESIAN PLANE

The **geometry** of the plane is concerned with points, distance, and angle, as well as with **rigid motions** (congruences), which move lines, circles, etc. without disturbing their shape. Such motions are, simply, *those mappings from the plane to itself that preserve distance and angle.*

When coordinate axes are introduced in the plane, the result is the *Cartesian plane*, where points are denoted by pairs of numbers. Using these numbers, the geometry of the Cartesian plane can be developed quantitatively. The purpose of this chapter is to develop this geometry with the help of vectors. Along the way we preview, in the microcosm of the plane, what is to follow in the remaining chapters.

1.1 THE CARTESIAN PLANE

The **Cartesian plane** is a plane equipped with two **coordinate axes**, the horizontal x_1-axis and the vertical x_2-axis. The x_1-axis is a copy of the real line $\mathbb{R}$ (Appendix B), and the x_2-axis is a copy of the x_1-axis obtained by rotating