GLOBAL RIEMANNIAN GEOMETRY

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ELLIS HORWOOD LIMITED
Publishers · Chichester

Halsted Press: a division of
JOHN WILEY & SONS
New York · Chichester · Brisbane · Toronto

First published in 1984 by

ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street, Chichester, West Sussex, PO19 1EB, England

The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:

Australia, New Zealand, South-east Asia:
Jacaranda-Wiley Ltd., Jacaranda Press,
JOHN WILEY & SONS INC.,
G.P.O. Box 859, Brisbane, Queensland 40001, Australia

Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada.

Europe, Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England.

North and South America and the rest of the world:
Halsted Press: a division of
JOHN WILEY & SONS
605 Third Avenue, New York, N.Y. 10016, U.S.A.

British Library Cataloguing in Publication Data
Willmore, T.J.
Global Riemannian geometry. –
(Ellis Horwood series in mathematics and its applications)
1. Riemannian manifolds
I. Title II. Hitchin, N.J.
516.3'73 QA649

Library of Congress Card No. 83-26675

ISBN 0-85312-699-2 (Ellis Horwood Limited)
ISBN 0-470-20017-0 (Halsted Press)

Printed in Great Britain by Unwin Brothers of Woking.

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Introduction

The subject of differential geometry has shifted its emphasis in recent years, largely in response to the demands of mathematical physics. The situation resembles in some ways the stimulus which general relativity gave to Riemannian geometry in the first quarter of the century. The main difference is that the modern approach is essentially global. The integer invariants of topology, transmuted into 'charges' and 'anomalies' are familiar to a large number of physicists, who now look to the language and machinery of pure mathematics for the answers to some of their basic questions.

This is the background to this book, which records the proceedings of the Symposium on Global Riemannian Geometry, held at the University of Durham in July 1982. It was designed to reflect the change in the character of the subject since the first Durham symposium in 1974, and to concentrate on the important new results obtained in the past few years.

The major themes dealt with are those of the Yang–Mills equations, Einstein metrics, harmonic maps and the relationship between curvature and topology. Recent advances in these subjects have been due on the one hand to powerful existence theorems in partial differential equations, notably the work of S. T. Yau, and on the other to the possibility of explicit constructions based on algebraic geometry, relying on the ideas of R. Penrose. There has also been a better appreciation of the way in which curvature and topology are related, especially in the work of M. Gromov and R. Hamilton.

The interplay of ideas which the symposium provoked is well represented in this book. They range from black holes to algebraic geometry, but at heart have a common global differential geometric viewpoint.

The symposium was supported by the London Mathematical Society and the organizers wish to thank the LMS, all participants and especially the contributors to this volume for contributing to such a successful meeting. We hope the reader acquires from these proceedings a glimpse of the network of ideas which make up global Riemannian geometry today.

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Chapter 1

Yang–Mills Equations

The Yang–Mills equations and the structure of 4-manifolds

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1. INTRODUCTION

I shall explain here the very recent and striking results of Donaldson [4], in which the Yang–Mills equations are used to solve a long-standing problem on the structure of smooth 4-manifolds. Before embarking on this I should like to make some general remarks about the application of partial differential equations (PDEs) to problems in geometry.

In topology one can distinguish broadly between *covariant* functors such as homology or homotopy and *contravariant* functors such as cohomology, *K*-theory or more general bundle theory. In applying PDEs to either of these situations one can see four stages: (1) physical background and motivation, leading to a particular variational problem and an associated Euler–Lagrange equation; (2) construction of many explicit solutions, often involving algebra or algebraic geometry; (3) general existence and regularity theorems; (4) application of the PDEs to problems in geometry and topology.

On the covariant side the theory of minimal surfaces is the most familiar example and it is one that has been extensively reported on at this symposium. The physical background is classical and comes from soap films, while algebraic curves in Kähler manifolds give natural families of examples. Existence and regularity have been studied for a long time and the results now available have enabled Yau and others (cf. Yau's contribution in Chapter 3 of this volume) to apply minimal surface theory as a geometric tool.

On the contravariant side the standard example is the Hodge theory of harmonic forms, motivated in part by Maxwell's equations and having important connections with algebraic geometry through the use of Kähler metrics. However, the Hodge theory is *linear*, unlike the theory of minimal surfaces which is *non-linear*. In recent years the Yang–Mills equations have arisen in elementary particle physics and they provide a non-linear theory of Hodge type. Roughly speaking, whereas Hodge theory relates to cohomology, Yang–Mills theory relates to G-bundles. The two theories overlap when $G = U(1)$ so that the curvature is just a 2-form. Explicit solutions of the Yang–Mills equations have been extensively studied on R^4 where the general solution (the multi-instantons) has been constructed by algebraic geometry [1]. Also, time-independent solutions on R^3, described as 'magnetic monopoles', have been constructed by similar methods [6]. On more general 4-manifolds a general existence theorem has recently been proved by Taubes [8] and important regularity results have been established by Uhlenbeck [9, 10].

All this activity was devoted to studying the Yang–Mills equations for their own sake, and most of the motivation came from physics. Donaldson's results, which I shall describe shortly, constitute the first case of the Yang–Mills equations as a geometric tool. In view of my general comments, and the analogy with minimal surfaces, this was perhaps to be expected sooner or later. What is surprising is the particular way in which the Yang–Mills equations are used by Donaldson and the beautiful result that eventually emerges. In comparing Donaldson's use of the Yang–Mills equations with the use of minimal surfaces by Yau and others one crucial difference should be emphasized. In the minimal surface theory one uses a *single minimal surface* (i.e. one solution of the PDEs) *as a geometric object*. By contrast Donaldson uses the *parameter space of all solutions* of the Yang–Mills equations *as his geometric object*.

2. Donaldson's theorem

Theorem *Let X be a compact simply-connected differentiable 4-manifold and assume the quadratic form on $H^2(X, Z)$ is (positive) definite. Then this is standard, i.e. equivalent over Z to $\Sigma_i x_i^2$.*

Note The quadratic form depends on a choice of orientation, so there is no loss of generality in making the definite form positive. Also Poincaré duality implies that the form is unimodular, i.e. represented by a matrix of determinant ± 1.

Arithmetically there are a finite number of unimodular positive definite quadratic forms over Z of given rank [7]. Donaldson's theorem asserts that only one of these can actually occur as the quadratic form of a smooth X.

An especially interesting case arises when X is a spin manifold. The quadratic form is then even and clearly by Donaldson's theorem such an X

cannot exist unless $H^2 = 0$. Arithmetically one knows that the rank must be a multiple of 8 and for the first few values, namely 8, 16, 24 the number of quadratic forms is 1, 2, 24, respectively (but for rank 32 it is greater than 8×10^6 [7]). For rank 8 the unique quadratic form is that associated to the exceptional Lie group E_8: its matrix is

$$\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix}$$

where all other entries are 0. An old result of Rohlin already tells us that E_8 cannot occur as the quadratic form of a smooth X. The reason is that the spinor index (i.e. index of the Dirac operator) must be even. However, this does not exclude $E_8 \oplus E_8$ occurring for a smooth X with H^2 of rank 16. In fact much effort over the years has gone into trying to construct precisely such a 4-manifold. The idea has been to start with a K3 surface (essentially a quartic algebraic surface in $P_3(C)$) which has a quadratic form

$$E_8 \oplus E_8 \oplus H \oplus H \oplus H,$$

where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the basic hyperbolic form.

Now it is clear that the quadratic form of a connected sum of two 4-manifolds is the direct sum of the two quadratic forms. Conversely, given X with quadratic form $Q_1 \oplus Q_2$ one may attempt to decompose X accordingly as a connected sum $X_1 \# X_2$. If this could be done, starting with a K3 surface we would end up with X_1 having $E_8 \oplus E_8$ as quadratic form. Needless to say there are geometric obstacles to this process and no way around them was found despite much ingenuity by Casson, Kirby and others. We now know, by Donaldson's theorem, that this process was inherently doomed to failure.

Although surgery techniques did not work for smooth 4-manifolds, they have recently been applied successfully in the topological context by Freedman [5], who has shown that *every unimodular quadratic form occurs as the quadratic form of some topological 4-manifold*. This is clearly in sharp contrast to Donaldson's result. Moreover, by combining both results it can apparently be shown that R^4 has a non-standard differentiable structure! (I owe this observation to M. Freedman and R. Kirby.)

3. Method of proof

We begin with an elementary algebraic observation. Let $\pm \alpha_i$ $(i = 1,\ldots, r)$ be the elements of $H^2(X, Z)$ with $\alpha^2 = 1$. Then $r \leqslant n$ (the rank of H^2) and equality holds if and only if the form is standard.

We come now to Donaldson's brilliant idea. Although the theorem makes no reference to the Yang–Mills equations, we introduce this as an auxiliary tool in the following way. *We let M be the moduli space of instantons on X*, i.e. M is the parameter space of self-dual Yang–Mills fields on X with group $SU(2)$ and $c_2 = -1$ (definitions will be recalled in Section 4). Then M has the following properties:

(1) $\dim M = 5$.
(2) M has X as natural boundary (more precisely outside a compact set M looks like $X \times R$).
(3) M is a differentiable manifold except for singularities described below.
(4) There are point singularities $A_1,\ldots, A_r$ (corresponding to $\alpha_1,\ldots, \alpha_r$) which are cones on $P_2(C)$.
(5) Other singularities are local complete intersections and can be deformed away.
(6) M is orientable.

Granted these properties, if we deform M to eliminate the singularities in (5), and cut off the cone singularities in (4), we obtain an oriented cobordism between X and r copies of $P_2(C)$. If a of these copies have the standard orientation and $b = r - a$ have the negative orientation, then the cobordism invariance of signature shows that

$$n = \text{sign}\,(X) = a - b \leqslant a + b = r \leqslant n.$$

Hence we must have $b = 0$ and $r = n$, proving that the quadratic form is standard as required.

If we reflect on the nature of Donaldson's proof we see that the general instanton (i.e. point of M) 'interpolates' between two quite different extremes. At one end we have the ideal instantons concentrated at points of X, while at the other we have the abelian solutions representing cohomology classes α with $\alpha^2 = 1$ (see Section 4). As Roger Penrose has pointed out to me; this is rather suggestive of a physical system which has two sorts of limiting situations: a classical one (at points of X) and a quantum one (given by the linear abelian fields).

4. The Yang–Mills equations

In this section I shall rapidly review the basic definitions of Yang–Mills theory and I shall discuss in outline how one establishes properties (1)–(6) of the moduli space.

We fix a principal $SU(2)$-bundle P over X with $c_2 = -k$, and consider a connection A with curvature F. The Yang–Mills functional is the global L^2-norm of F:

$$\|F\|^2 = \int_X |F|^2 \, \mathrm{dv},$$

where a Riemannian metric on X has been used to define the local norm $|F|^2$ and the volume element dv. One can then show that we have a lower bound,

$$\|F\|^2 \geqslant 8\pi^2 |k|,$$

with equality if and only if $^*F = \pm F$ (the sign depends on the sign of k). Thus for $k = 1$, $\|F\|^2$ is minimized if $^*F = F$ (the self-duality equations). Connections A with self-dual F are called *instantons* and we identify two connections which are 'gauge equivalent', i.e. which differ by an element of the group $\mathscr{G} = \mathrm{Aut}\, P$.

To get an idea of the nature of the moduli space M of instantons let us consider first the infinitesimal theory. We suppose A is a fixed instanton (giving a point of M) and we look for nearby instantons by linearizing the self-duality equations around A. This leads us to introduce the elliptic complex:

$$0 \to \Omega^0(\mathrm{ad}\, P) \xrightarrow{d_A} \Omega^1(\mathrm{ad}\, P) \xrightarrow{d_A^-} \Omega^2_-(\mathrm{ad}\, P) \to 0,$$

where $\Omega^q(\mathrm{ad}\, P)$ denotes differential q-forms with values in the adjoint bundle of P, d_A is the skewed covariant derivative (defined by A), Ω^2_- is the (-1)-eigenspace of $*$ and d_A^- is obtained from d_A by projecting onto Ω^2_-. Since $(d_A)^2 = F$, the condition $^*F = F$ asserts precisely that $d_A^- d_A = 0$, so that we do in fact have a complex.

The solutions of the linearized problem can then be identified with H^1 of this complex (the image of $d_A\Omega^0$ corresponds to infinitesimal gauge transformations). If $H^0 = H^2 = 0$ then every infinitesimal solution generates a genuine solution, so that H^1 can be identified with the tangent space to M at A. An index computation [2] then yields a formula for dim $H^1 = \dim M$ and this proves (1).

If $H^0 = 0$ but $H^2 \neq 0$ then M is locally given by the zeros of a quadratic function $H^1 \to H^2$. This explains the singularities of type (5). If $H^0 \neq 0$, the connection reduces to a $U(1)$ subgroup of $SU(2)$ and we have an abelian solution, i.e. one coming from a line-bundle L. The vector bundle is $L \oplus$

L^{-1} and so $c_2 = -c_1(L)^2$; since $c_2 = -1$ this means that $c(L) = \pm\alpha_i$ with $\alpha_i^2 = 1$. Thus the points $A_1, \ldots, A_r$ of (4) correspond to the abelian solutions and they are singular cones precisely because, on dividing by the gauge group $\mathcal{G}$, they have a non-trivial isotropy $U(1)$ group.

The orientability of M (property (6)) is proved by a more refined application of the index theorem. We need to take any closed loop in M and then apply the index theorem for real families of elliptic operators parametrized by a circle [3].

This accounts for all properties except the crucial (2) which connects M with X. To understand (2) we consider the basic example when X is the 4-sphere. M is then known explicitly and can be identified with the hyperbolic 5-space, i.e. the interior of the unit ball B^5 in R^5. Moreover, this is naturally acted on by the conformal group $SO(5, 1)$ because the Yang–Mills functional is actually a conformal invariant. From this we see that a sequence of points $A_i \in B^5$ which converge to a boundary point $x \in S^4$ represent instantons whose curvature converges to a delta function at x. The picture for a general manifold X is qualitatively similar. Thus the points of X needed to compactify M appear as 'delta function curvatures'.

The justification of these statements constitutes the hard technical part of the proof. It depends heavily on the earlier work of Taubes and Uhlenbeck. In particular an essential starting point is Taubes' existence theorem [8] which asserts that M is non-empty. This requires the hypothesis that the quadratic form on $H^2(X, Z)$ is positive definite.

5. Further comments

It seems likely that Donaldson's theorem is only the first of many geometric applications of the Yang–Mills equations. This suggests that a more systematic study of the moduli spaces M_k for higher values of $k = -c_2$ should be undertaken. In particular one will need an extension of Taubes' existence theorem to cover manifolds with indefinite quadratic form. It seems likely that M_k should be non-empty provided that $k > \dim H^2_-$, where H^2_- is the 'negative part' of H^2 (or equivalently the space of harmonic 2-forms ω with $^*\omega = -\omega$).

Another potentially important problem is to understand how the topology of M_k depends on the choice of metric (or conformal structure) on X. Whenever H^2 of the complex in Section 3 vanishes then M_k is non-singular except for the conical singularities A_i due to the abelian solutions. As we vary the metric on X we will occasionally find a non-zero H^2. One would expect this to occur on a codimension one set in the space of metrics and that, on crossing this set, the topology of M_k would be altered by the attachment of a suitable handle. At present nothing is