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This book is dedicated to the
memory of Derek Waller

Preface

This book presents the proceedings of a one-day conference in Combinatorics and Graph Theory held at The Open University, England, on 12 May 1978. The first nine talks presented here were given at the conference, and cover a wide variety of topics ranging from topological graph theory and block designs to latin rectangles and polymer chemistry. The tenth author, Christopher Wright, had been invited to present his talk on traffic-flow problems, but was unable to do so due to other commitments; he has kindly allowed us to publish the talk he would have given. In all cases, the authors were chosen for their ability to combine interesting expository material in the areas concerned with an account of recent research and new results in these areas.

One of the special features of the conference was a Poster Session, and one of the presentations at this session appears here. I should like to thank my colleagues John Mason and Roger Duke for organizing the Poster Session, and to thank them and Roy Nelson for helping with the academic side of the programme. On the administrative side, I should like to express my thanks to everyone involved, and in particular to Marion Aldred, Jennifer Goldrei, Joan Street and Mike Bandle. Most of all, I should like to thank Frances Thomas for her excellent typing of the entire manuscript.

Finally, this volume is dedicated to the memory of Derek Waller, who was prevented by illness from attending the conference. Three weeks later he died of leukaemia, leaving a wife and three small children. His untimely death is a sad loss for British Combinatorics.

Robin J. Wilson

The Open University
January 1979

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L D Andersen and A J W Hilton

Generalized latin rectangles

1. INTRODUCTION

A latin square of size n × n based on the symbols 1, ..., n is an n × n matrix in which each cell is filled by exactly one symbol in such a way that each symbol occurs exactly once in each row and exactly once in each column. Instead of the n positive integers 1, ..., n any set of n symbols can be used.

A latin square can be thought of as a finite quasigroup: a quasigroup (Q, *) is a set Q with a binary operation * such that the equations a*x = b and y*a = b are uniquely solvable for each pair (a,b) of elements of Q. Thus the multiplication table of a quasigroup of order n is precisely a latin square of size n × n (with a headline and a sideline). Figure 1 gives an example.

Q, *	1	2	3	4
1	2	4	1	3
2	4	3	2	1
3	3	1	4	2
4	1	2	3	4

example
1*4 = 3
4*1 = 1

Latin square

Figure 1

Latin squares and quasigroups are extensively treated in [3], which contains several references also to works concerned with the application of latin squares in the design of experiments (the classic in this field is [7]).

One intriguing combinatorial problem concerning latin squares is that of *embedding*. There are no end of variations on this theme. We mention here two results.

H. J. Ryser [10] proved that if A is an r × s matrix where each cell contains one of the symbols 1, ..., n such that no symbol occurs more than once in any row or column, then A can be embedded in (found as a submatrix of) some latin square of size n × n on symbols 1, ..., n *if and only if* each symbol occurs at least r + s − n times in A.

A. Cruse [2] proved that if A is an r × r matrix where each cell contains one of the symbols 1, ..., n such that no symbol occurs more than once in any row or column and such that if cell (i,j) is occupied by the symbol k then so is cell(j,i), then A can be embedded in a symmetric latin square of size n × n on symbols 1, ..., n *if and only if* each symbol occurs at least 2r − n times in A and at least r different symbols occur a number of times congruent to n modulo 2. Other embedding theorems for quasigroups can be found in [9].

This paper is concerned with the following generalization of latin squares:

A *(p, q, x)-latin rectangle* of size r × s on symbols 1, ..., n is an r × s matrix in which each cell is filled by exactly x symbols in such a way that each symbol occurs at most p times in each row and at most q times in each column (p, q, x, r, s and n are positive integers).

1 2	2 3	4 4
2 4	1 3	1 2

A (2,2,2)-latin rectangle of size 2x3 on symbols 1, 2, 3, 4.

1 2 5	4 1 2
2 4 5	1 2 4

A (2,2,3)-latin square of size 2x2 on symbols 1, 2, 3, 4, 5.

Figure 2

A (p, q, x)-latin rectangle in which each symbol occurs exactly p times in each row and exactly q times in each column is called *exact*.

The x symbols in a cell of a (p, q, x)-latin rectangle need not be distinct; if they are distinct for each cell the rectangle is said to be *without repetition*. Note also that a (p, p, x)-latin square may be *symmetric*.

1 2	3 4	1 2	3 4
3 4	1 2	3 4	1 2

An exact (2,1,2)-latin rectangle A on symbols 1, 2, 3, 4 without repetition

2 3	1 2	1 3
1 2	3 3	1 2
1 3	1 2	2 3

An exact symmetric (2,2,2)-latin square B on symbols 1, 2, 3, 4.

Figure 3

A (p, p, 1)-square is a *frequency square* (F-square) with frequency vector (p, p, ..., p). F-squares are treated in [8], to which we refer the reader for further references. Here we investigate (p, q, x)-latin rectangles from other points of view.

In Section 5 we generalize the theorems of Ryser and Cruse by giving necessary and sufficient conditions for the embedding of (p, q, x)-latin rectangles, with or without repetition, symmetric or possibly unsymmetric.

Sections 3 and 4 contain construction and decomposition theorems, and we apply the results and methods to questions on quasigroups and on equitable edge-colourings of graphs, defined in the next section.

Many of the results stated here have quite long proofs. Most of the theorems given (but not all) can be found with proofs in [1] and will be published elsewhere.

We wish to thank Professor Trevor Evans for a helpful discussion.

2. EQUITABLE AND BALANCED EDGE-COLOURINGS OF GRAPHS

Equitable edge-colourings of graphs relate to the work of this paper in two ways: a result about them is used in several of our proofs, and on the other hand our results can be stated as results about such colourings.

An edge-colouring of a graph G with colours 1, ..., k is a partition of the edges and loops of G into k mutually disjoint subsets $c_1, \ldots, c_k$ (note that any partition will do, so that an edge-colouring need not be proper - that is, having no colour present more than once at any vertex). An edge or loop has colour i if it belongs to c_i.

Given an edge-colouring, we let $c_i(v)$ be the set of edges and loops on vertex v of colour i, and $c_i(u,v)$ be the set of edges joining vertices u and v (the set of loops on u, if u = v) of colour i.

An edge-colouring is equitable if, for all vertices v,

$$\text{(a)} \quad \max_{i,j} \Big| \, |c_i(v)| - |c_j(v)| \, \Big| \le 1.$$

It is balanced if it is equitable and for each pair of vertices u, v:

$$\text{(b)} \quad \max_{i,j} \Big| \, |c_i(u,v)| - |c_j(u,v)| \, \Big| \le 1.$$

Thus an edge-colouring is equitable if the colours occur as uniformly as possible at each vertex, and it is balanced if in addition the colours are shared as evenly as possible on each multiple edge.

An exact (p, q, x)-latin rectangle A corresponds to an equitable edge-colouring of a bipartite graph with a vertex for each row and a vertex for each column, where each row vertex ρ_i (corresponding to the i-th row) is joined to each column vertex γ_j (corresponding to the j-th column) by x edges being coloured with the symbols in the (i,j)-th cell of A, where the number of edges of any given colour joining ρ_i to γ_j is equal to the number of occurrences of the corresponding symbol in the cell (i,j). For example, from the rectangle A of Figure 3 we get the graph of Figure 4 with edge-colouring indicated. Since A is without repetition, the colouring is balanced.

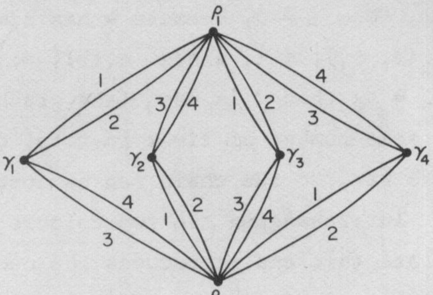

Figure 4

In the same way an exact symmetric (p, p, x)-latin square corresponds to an equitable edge-colouring of a complete graph with loops and multiple edges. We shall not go into details, but just remark that several of the following results can be seen in this light, although we only state one such result (Theorem 9 in Section 4).

The notion of equitable and balanced edge-colourings was introduced by D. de Werra, and he also proved the following important result ([4] - [6]). We give a short proof of our own.

Theorem 1. For each $k \geq 1$, any bipartite graph has a balanced edge-colouring with k colours.

Proof. Colour the edges of the graph in any way such that (b) is satisfied. The condition only affects each multiple edge by itself, so this is clearly possible. We then modify the colouring to make (a) be fulfilled without violating (b). Suppose that at some vertex v,

$$\max_{i,j} \Big| |c_i(v)| - |c_j(v)| \Big| > 1,$$

and suppose that the maximum is attained for colours 1 and 2. We can assume that $|c_1(v)| > |c_2(v)| + 1$. Let P be a maximal chain $v = v_0, v_1, v_2, \ldots, v_h$ such that the edges are coloured alternately 1 and 2 (the edge joining v_0 and v_1 having colour 1), and such that $|c_1(v_i, v_{i+1})| = |c_2(v_i, v_{i+1})| + 1$ when i is even, $|c_2(v_i, v_{i+1})| = |c_1(v_i, v_{i+1})| + 1$ when i is odd, and P uses only one edge from each multiple edge. Now $h \neq 0$, because v has some neighbour v_1 for which $|c_1(v, v_1)| = |c_2(v, v_1)| + 1$, since $|c_1(v)| > |c_2(v)| + 1$. Also $v_h \neq v_0$, because if $v_j = v_0$ then j is even (the graph is bipartite), i.e. both colours occur the same number of times in total on the multiple edges incident with v_0 used so far, so the chain can be continued because $|c_1(v)| > |c_2(v)| + 1$. Interchanging the two colours 1 and 2 on the chain P clearly does not violate (b), and it reduces the number of pairs of colours for which $\left||c_i(v)| - |c_j(v)|\right|$ was maximal (greater than 1) by at least 1. We only have to check that

$$\max_{i,j} \left||c_i(v_h)| - |c_j(v_h)|\right|$$

is not increased. Repeated application of the argument then proves the theorem. But if h is even the maximality of P implies that the colour 2 occurs at least once more than the colour 1 at v_h which implies the required result, and a similar argument holds if h is odd. This proves Theorem 1. □

It is not true in general that any graph has an equitable edge-colouring with k colours for all k.

3. EXISTENCE AND CONSTRUCTION OF EXACT (p, q, x)-LATIN RECTANGLES

An exact (p, q, x)-latin rectangle on xt symbols has pxt occurrences in each row, so a row must contain pt cells; similarly a column contains qt cells, so the rectangle has size $qt \times pt$.

Theorem 2. Let p, q, x be positive integers and let t be a rational number such that pt, qt, xt are positive integers. Then there exists an exact (p, q, x)-latin rectangle on xt symbols and an exact symmetric (p, p, x)-latin square on xt symbols. If $t \geq 1$, both can be taken to be without repetition. □

A simple method to obtain exact (p, q, x)-latin rectangles which are symmetric if p = q and without repetition if t ≥ 1 is the cyclic way of placing the symbols exemplified in Figure 5. The rectangle A of Figure 3 is also of this type.

1 2 3	4 5 6	7 1 2	3 4 5	6 7 1	2 3 4	5 6 7
4 5 6	7 1 2	3 4 5	6 7 1	2 3 4	5 6 7	1 2 3
7 1						

$p=3$
$x=3$
$t=\frac{7}{3}$

Figure 5

There are several ways of getting generalized latin rectangles from others. Some are rather trivial, like placing several rectangles on top of or next to each other. The following is more interesting, yet simple. Figure 6 provides an example.

1 2 3	4 5 6	1 3 6	2 4 5
4 5 6	1 2 3	2 4 5	1 3 6

An exact (2,1,3)-latin rectangle A on 1, 2, 3, 4, 5, 6.

↓ Identify 1 and 4, 2 and 5, 3 and 6.

1 2 3	1 2 3	1 3 3	2 1 2
1 2 3	1 2 3	2 1 2	1 3 3

An exact (4,2,3)-latin rectangle B on 1, 2, 3.

Figure 6

In brief, let m, n be positive integers and A an exact (p, q, x)-latin rectangle on 1, ..., mn. Identify symbols i and j whenever $i \equiv j \pmod{m}$. Then an exact (np, nq, x)-latin rectangle B on 1, ..., m is obtained. B is called the modulo m reduction of A. So in Figure 6, B is the modulo 3 reduction of A.

This process has a converse as stated in the next theorem.

Theorem 3. Let B be an exact (np, nq, x)-latin rectangle on 1, ..., m. Then
(i) B is the modulo m reduction of some exact (p, q, x)-latin rectangle A on 1, ..., mn;
(ii) if no symbol occurs more than n times in any cell of B then A can be taken to be without repetition. □

Since we go from B to A by splitting each symbol into n symbols, (ii) of Theorem 3 is best possible.

The theorem does not hold for symmetric squares - that is, we cannot be sure that A can be chosen to be a symmetric square even if B is (the square B of Figure 3 is a counterexample); we have only the following:

Theorem 4. Let B be an exact symmetric (2np, 2np, x)-latin square on symbols 1, ..., m. Then B is the modulo m reduction of some exact symmetric (2p, 2p, x)-latin square A on 1, ..., mn. □

In the next section we consider another way of obtaining (p, q, x)-latin rectangles from others.

4. MERGING OF ADJACENT CELLS

1	2	1	3	1	2	2	3	3
2	2	3	1	2	1	3	3	1
2	3	3	1	1	2	3	2	1
1	3	2	2	2	3	1	1	3
3	1	1	3	3	1	2	2	2
3	1	2	2	3	3	1	1	2

An exact (3,2,1)-latin rectangle A of size 6 x 9

erase lines in blocks

1 2 1 2 2 3	3 1 2 1 2 1	2 3 3 3 3 1
2 3 3 1 3 2	1 1 2 2 2 3	3 2 1 1 1 3
3 1 1 3 1 2	3 3 1 2 3 3	2 2 2 1 1 2

An exact (6,6,6)-latin square B of size 3 x 3

Figure 7

Figure 7 gives an example of the following process: Let A be an exact (p, q, x)-latin rectangle of size mr × ns for positive integers m, n, r, s. Merge (identify) cells (i_1, j_1) and (i_2, j_2) whenever $\lceil i_1/m \rceil = \lceil i_2/m \rceil$ and $\lceil j_1/n \rceil = \lceil j_2/n \rceil$ (where $\lceil z \rceil$ denotes the least integer not smaller than z). Then an exact (mp, nq, mnx)-latin rectangle B of size r × s is obtained. B is called the <u>(m, n)-merger</u> of A. For example, in Figure 7, B is the (2, 3)-merger of A.

In the following, we shall use the term "merging of cells" about a slightly more general situation as well.

We have a converse theorem:

Theorem 5. Let B be an exact (mp, nq, mnx)-latin rectangle. Then
(i) B is the (m, n)-merger of some exact (p, q, x)-latin rectangle A;
(ii) if no symbol occurs more than mn times in any cell of B then A can be taken to be without repetition. □

To go from B to A we subdivide each cell of B into mn cells, so (ii) is best possible.

We illustrate the proof of Theorem 5 by proving a related theorem about quasigroups. Let (Q,*) be a quasigroup. Suppose we have two partitions of the set Q, $Q = A_1 \cup \dots \cup A_r = B_1 \cup \dots \cup B_s$, the A_i's being mutually disjoint, likewise the B_j's. Form an r × s matrix M, where cell (i, j) contains each a*b with $a \in A_i$, $b \in B_j$, counting repetitions. Then cell (i, j) contains $|A_i| \cdot |B_j|$ elements. We call M the set-multiplication table for (Q,*) corresponding to the two partitions. It is easy to see that row i of M contains each element of Q $|A_i|$ times, and column j each element $|B_j|$ times.

The next theorem shows that if we write down any table with these properties, then it is the set-multiplication table for some pair of partitions of some quasigroup. In other words, we can postulate any set-multiplication table and be sure that there is a quasigroup satisfying it.

We define an SM-matrix on σ elements to be any matrix (say r × s) where there are positive integers ρ(i) associated with each row and γ(j) with each column such that

$$\sigma = \sum_{i=1}^{r} \rho(i) = \sum_{j=1}^{s} \gamma(j),$$

each cell (i, j) contains ρ(i)γ(j) elements, and each element occurs ρ(i) times in row i and γ(j) times in column j.

Theorem 6. Any SM-matrix on σ elements is the set-multiplication table for some quasigroup on these elements.

Proof. Let A be an SM-matrix of size r × s on elements $\alpha_1, \dots, \alpha_\sigma$. We show that if $r \neq \sigma$ then A can be obtained from an (r + 1) × s SM-matrix by merging the cells of two rows (such that any pair of cells in the same column are identified). Repeated application of this argument first on the rows, and then on the columns, shows that A can be obtained by "generalized merging" from a $\sigma \times \sigma$ SM-matrix B. But B is just a latin square, a quasigroup, and A is easily seen to be a set-multiplication table for it.

If $r \neq \sigma$ then some $\rho(i)$ is at least 2. Assume without loss of generality that $\rho(1) \geq 2$. We want to split the first row of A into new rows. Construct a bipartite graph G with vertex classes $\{v_1, \ldots, v_s\}$ and $\{\alpha_1, \ldots, \alpha_\sigma\}$, where v_i is joined to α_j by k edges if and only if the symbol α_j occurs k times in cell (1,i). Then v_i has degree $\rho(1)\gamma(i)$ and α_j has degree $\rho(1)$. Give G an equitable edge-colouring with $\rho(1)$ colours. Let c_1 be some colour class. Then each v_i has exactly $\gamma(i)$ edges of colour 1 on it, and each α_i is on exactly one such edge. Split row 1 of A into two rows 1' and 1" where a symbol α_j goes in row 1' in cell (1', i) if and only if there is an edge of colour 1 joining v_i and α_j, and symbol α_j goes in row 1" in cell (1", i) as many times as there are edges joining v_i and α_j of colour different from 1. Let $\rho(1') = 1$, $\rho(1'') = \rho(1) - 1$. It is easy to see that we have obtained an SM-matrix of size $(r + 1) \times s$. This proves Theorem 6. □

Corresponding to (i) of Theorem 5 we have the following result about symmetric squares.

Theorem 7. Let B be an exact symmetric (mp, mp, m^2x)-latin square. Then B is the (m, m)-merger of some exact symmetric (p, p, x)-latin square A if and only if at most mx distinct symbols occur an odd number of times in any given diagonal cell of B. □

The condition is due to the fact that a symbol occurring an odd number of times in a diagonal cell of B must occur in one of the corresponding diagonal cells of A. If we want to know when A can be taken to be without repetition, we get a rather complicated necessary and sufficient condition; we omit the result here.

We have the following related result about quasigroups.

Theorem 8. Any symmetric SM-matrix on σ elements with at most $\rho(i)$ distinct elements occurring an odd number of times in the i-th diagonal cell is the set-multiplication table for some commutative quasigroup on these elements. □