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J. Lindenstrauss V. D. Milman (Eds.)

Geometric Aspects of Functional Analysis

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FOREWORD

This is the third published volume of the proceedings of the Israel Seminar on Geometric Aspects of Functional Analysis. The first volume (1983-84) was published privately by Tel Aviv University and the second volume (1985-86) is volume 1267 of the Springer Lecture Notes in Mathematics. This volume covers the 1986-87 session of the seminar. As in previous years the seminar was partially supported by the Israel Mathematical Union.

The large majority of the papers in this volume are original research papers. As should be clear from the contents of this volume and the titles of the actual lectures delivered at the seminar, there was last year a strong emphasis on classical finite-dimensional convexity theory and its connection with Banach space theory. In recent years, it has become evident that the notions and results of the local theory of Banach spaces are useful in solving classical questions in convexity theory. We hope that the present volume will help in clarifying this point.

The papers in this volume are arranged in accordance with the order of their presentation at the seminar. The last four papers are based on talks delivered at conferences in the summer of 1987.

In preparing this volume we received invaluable help in editing and typing from Mrs. M. Hercberg. We are very grateful to her.

Joram Lindenstrauss, Vitali Milman

GAFA

1986–1987

GAFA 1986-1987

List of Seminar Talks

October 31, **G. Schechtman**(Weizmann Institute), "Balancing vectors in the max-norm" (after J. Spencer).

November 16, **L. Tzafriri** (Hebrew University), "On invertibility of large matrices in Euclidean space" (joint work with J. Bourgain).

November 23, **L. Tzafriri** (Hebrew University), "Invertibility of large submatrices" (joint work with J. Bourgain).

December 5, 1. **N. Alon** (Tel Aviv University), "Splitting measures".
2. **C.J. Read** (Cambridge University, England), "Solution to the invariant subspace problem on a class of Banach spaces".

December 19, **Y. Yomdin** (Beersheva University), "Approximational versus topological complexity of functions on infinite dimensional spaces".

January 2, **B. Bollobás** (Cambridge University, England), "Random graphs".

January 9, 1. **N. Alon** (Tel Aviv University), "The number of d-polytopes".
2. **J. Bourgain** (IHES, Paris), "Non-linear convolutions".

January 18, 1. **J. Bourgain** (IHES), "Approximation of zonoids by zonotopes" (joint work with J. Lindenstrauss and V. Milman)
2. **S. Reisner** (Haifa University), "On two theorems of Lozanovskii concerning intermediate Banach lattices".

February 1, 1. **J. Bourgain** (IHES), "Ruzsa's problem on recurrent sets".
2. **Y. Peres** (Tel Aviv University), "Applications of Banach limits to recurrence and equidistribution".
3. **Y. Gordon** (Technion, Haifa), "A simple new proof of Milman's inequality".

March 27, 1. **V. Milman** (Tel Aviv University), "A new approach to isomorphic symmetrization".
2. **R. Schneider** (University of Freiburg, Germany), "On the Alexandrov-Fenchel inequality involving zonoids".

April 10, **E. Bachmat** (Hebrew University), "Dense packing of $\mathbb{R}^n$ with balls" (after Kabatyanskii and Levenstein).

May 1, 1. **G. Schechtman** (Weizmann Institute), "Almost isometrical natural embeddings in L_p" (after A. Arias).
2. **M. Meyer** (Université Paris VI, France), "Sections of the unit ball of ℓ_p^n", (joint work with A. Pajor).

May 5, 1. **M. Zippin** (Hebrew University), "Banach spaces with a separable dual".
2. **A. Pajor** (Université Paris VII, France), "Centroid bodies".

The following talks were presented at the GAFA-Seminar in the framework of the D.P. Milman Memorial Conference on Geometric Aspects of Functional Analysis, organized by Tel Aviv University, May 28 - May 31.

May 28, 1. **D. Amir** (Tel Aviv University), Opening address.
2. **V. Milman** (Tel Aviv University), "A very old joint result with D.P. Milman".

3. **M. Gromov** (IHES, France), "Topological spectra of normed spaces".
4. **J. Lindenstrauss** (Hebrew University), "Some useful facts about Banach spaces".
5. **W.B. Johnson** (Texas A & M University), "Homogeneous Banach spaces".
6. **L. Tzafriri** (Hebrew University), "Invertibility of operators on ℓ_p^n and other finite dimensional spaces".

May 29, 1. **G. Pisier** (Université Paris VI, Texas A & M University) "On the p-th variations of martingales and related Banach spaces".
2. **B. Maurey** (Université Paris VII), "ℓ^p structures in B-spaces and related unsolved questions".
3. **M. Marcus** (City University of NY), "On the distribution of the maximum of Gaussian processes".
4. **J. Bourgain** (IHES, University of Illinois), "Some new pointwise ergodic theorems".
5. **M. Gromov** (IHES), "Atiyah's results on convex bodies in R^n".

May 31, 1. **K. Ball** (Cambridge University, England), "Norms generated by logarithmically concave functions".
2. **I. Gohberg** (Tel Aviv University), "Band bases and applications".
3.**P. Milman** (University of Toronto), Transforming an analytic function to normal crossings by blowing up (a variant of resolution of singularities)".
4. **N. Tomczak-Jaegermann** (University of Alberta, Edmonton), "Entropies of convex bodies and operators".
5. **M. Talagrand** (C.N.R.S., Ohio State University), "Mixed volumes of convex bodies and Gaussian processes".

June 11, **J. Bourgain** (IHES), "Finite-dimensional homogeneous Banach spaces".

Table of Contents

THE INVARIANT SUBSPACE PROBLEM ON A CLASS OF NONREFLEXIVE BANACH SPACES, 1

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Abstract

In [1] we gave a fairly short proof that there is an operator on the space ℓ_1, without nontrivial invariant subspaces, and we conjectured that the same might be true of any space $\ell_1 \oplus W$ where W is a separable Banach space. This conjecture turns out to be true, and by proving it here we give the first example of a reasonably large class of Banach spaces for which the solution to the invariant subspace problem is known. This continues the sequence of counter-examples which began on an unknown Banach space (Enflo [2], Read [4], Beauzamy [3], (simplification of [2])), proceeded to the space ℓ_1 (Read [5,1]) and here continues with the case of any separable Banach space containing ℓ_1 as a complemented subspace. No counter-example is known to the author for a Banach space which does not contain ℓ_1.

1. Introduction

Here and elsewhere in this paper, we follow as closely as possible the style of the proof in [1]. The proof here is a modified version of [1], changed as little as we could in order to incorporate the Banach $\ell_1 \oplus W$, rather than ℓ_1. Throughout this paper, the underlying field may be either $\mathbb{R}$ or $\mathbb{C}$.

1.1. Let $(X, \|\cdot\|)$ denote the Banach space $\ell_1 \oplus W$, where W is a given separable Banach space. We quote the following result of R. Ovsepian and A. Pelcynski ([6]).

Let us choose, once and for all, such a biorthogonal system $(x_i), (x_i^*)$ for the given Banach space W. We may assume that $\|x_i\| = 1$ for all i (1.1.1), and if we take the liberty of renorming our space W with the equivalent norm

$$\|x\|' = \inf\left\{20\|y\| + \sum_1^\infty |x_i^*(z)| : y + z = x\right\},$$

we may assume that $\|x_i^*\| = 1$ also (1.1.2). Let $(g_i)_{i=0}^\infty$ be the unit vector basis of ℓ_1, and let F be the dense subspace of X spanned by $\{x_i\} \cup \{g_i\}$. The norm of any linear map $S : W \cap F \to Z$ is then bounded by

$$\|S\| \leq \sum_{i=1}^\infty \|Sx_i\|, \tag{1.1.3}$$

and indeed the right hand side is an upper bound for the nuclear norm of S.

We may also assume that the norm on $X = \ell_1 \oplus W$ is taken in the sense of ℓ_1, that is

$$\|x + w\|_X = \|x\|_{\ell_1} + \|w\|_W \tag{1.1.4}$$

for every $x \in \ell_1$, $w \in W$. Then the norm of an operator $S : F \to Z$ is bounded by

$$\|S\| \leq \max\left(\sup_{i\geq 0} \|Sg_i\|, \sum_{i=1}^\infty \|Sx_i\|\right). \tag{1.1.5}$$

Definition 1.2. Let $\underset{\sim}{d} = (d_i)_{i=1}^\infty$ denote a strictly increasing sequence of positive integers. This sequence will be involved in most of the rest of our definitions, and it will be required to increase "sufficiently rapidly" in the following sense.

If $P(\underset{\sim}{d})$ is a proposition depending on the sequence $\underset{\sim}{d}$, we say (just as we did in [4, §1]) that $P(\underset{\sim}{d})$ is true "provided $\underset{\sim}{d}$ increases sufficiently rapidly", if there is a constant $c > 0$ and functions $f_i : \mathbb{N}^i \to \mathbb{N}$ $(i = 1, 2, 3, \ldots)$ such that the following holds.

If $d_1 > c_1$ and for all $i > 1$, we have

$$d_i > f_{i-1}(d_1, d_2, \ldots, d_{i-1})$$

then $P(\underset{\sim}{d})$ is true.

So if $P(\underset{\sim}{d})$ is true "provided $\underset{\sim}{d}$ increases sufficiently rapidly", there is a sequence $\underset{\sim}{d}$ such that $P(\underset{\sim}{d})$ is true; and if $P_1, P_2, \ldots, P_k$ are a finite collection of propositions, each of which is true provided $\underset{\sim}{d}$ increases sufficiently rapidly, then the proposition $\bigwedge_{i=1}^k P_i(\underset{\sim}{d})$ is true "provided $\underset{\sim}{d}$ increases sufficiently rapidly".

Given such a sequence $\underset{\sim}{d}$, let us write $a_i = d_{2i-1}$ and $b_i = d_{2i}$ for each $i \in \mathbb{N}$.

1.3. For the sake of convenience, we shall use the notation p.$\underset{\sim}{d}$. to mean 'provided $\underset{\sim}{d}$ increases sufficiently rapidly'.

1.4. We define $a_0 = 1$, $v_0 = 0$, $v_n = n(a_n + b_n)$ $(n \in \mathbb{N})$.

1.5. Let $|p|$ denote the sum of the absolute values of the coefficients of the polynomial p.

1.6. Given the space X with norm as in (1.1.4), with unit vector basis $(g_i)_{i=0}^{\infty}$ for ℓ_1, and a biorthogonal system $(x_i), (x_i^*)$ for W satisfying (1.1.0-2), we define a sequence $(f_i)_{i=0}^{\infty} \subset X$ such that $\{f_i\} = \{g_i\} \cup \{x_i\}$, in terms of the sequence $\underset{\sim}{d}$. We define

$$f_i = \begin{cases} x_n & \text{if } i = a_n - 1 \text{ for some } n \in \mathbb{N} \ , \\ g_i & \text{if } 0 \le i < a_1 - 1, \\ g_{i-n} & \text{if } a_n \le i < a_{n+1} - 1 \text{ for some } n \in \mathbb{N} \ . \end{cases}$$

1.7. If $S \subset \{g_i : i \geq 0\}$ we write Π_S for the norm 1 projection $X \to \overline{\text{lin}}(S)$ such that $\Pi_S(w) = 0$ if $w \in W$, and

$$\Pi_S(g_i) = \begin{cases} g_i \ , & \text{if } \quad g_i \in S \\ 0 \ , & \text{if } \quad g_i \notin S \end{cases}$$

2. Defining some operators on X, given the sequence $\underset{\sim}{d}$

Let the sequence $\underset{\sim}{d}$ be given. Certainly we have

$$0 = v_0 < a_1 + b_1 = v_1 < 2(a_2 + b_2) = v_2 < \dots \ ,$$

so the sets $\{0\}$, $(v_{n-1}, v_n]$ $(n = 1, 2, 3, \dots)$ form a partition of $\mathbb{Z}^+$. P.$\underset{\sim}{d}$, we have $v_{n-1} < a_n$ for each $n \in \mathbb{N}$, so each interval $(v_{n-1}, v_n]$ can be partitioned into the disjoint union of the sets

$$(v_{n-1}, na_n]$$

and

$$(na_n, v_n]$$

Moreover, $(v_{n-1}, na_n]$ is the disjoint union of the sets

$$(v_{n-1}, a_n) \quad , \quad [ra_n, ra_n + v_{n-r}] \qquad (r = 1, \dots, n) \ ,$$

and

$$(ra_n + v_{n-r}, (r+1)a_n) \qquad (r = 1, \dots, n-1) \ .$$

Furthermore, $(na_n, v_n]$ is the disjoint union of the sets

$$\big(na_n + rb_n, (r+1)(a_n+b_n)\big) \qquad (r = 0, \ldots, n-1)$$

and

$$\big[r(a_n+b_n), na_n + rb_n\big] \qquad (r = 1, \ldots, n) \ .$$

So $\mathbb{Z}^+$ is (p.$\mathcal{d}$) the disjoint union of all the intervals $\{0\}$, (v_{n-1}, a_n) $(n \in \mathbb{N})$, $[ra_n, ra_n + v_{n-r}]$ $(n \in \mathbb{N},\ r = 1, \ldots, n)$, $\big(ra_n + v_{n-r}, (r+1)a_n\big)$ $(n \geq 2,\ r = 1, \ldots, n-1)$, $\big(na_n + rb_n, (1+r)(a_n+b_n)\big)$ $(n \in \mathbb{N},\ r = 0 \ldots n-1)$, and $\big[r(a_n+b_n), na_n + rb_n\big]$ $(n \in \mathbb{N},\ r = 1 \ldots n)$.

Bearing this in mind, we make the following definition.

Definition 2.1. Let the sequence $\mathcal{d} \subset \mathbb{N}$ be given. We shall show that, p.$\mathcal{d}$., there is a unique sequence $(e_i)_{i=0}^{\infty} \subset F$ with the following properties:

(2.1.0)
$$f_0 = e_0 \ .$$

(2.1.1) If integers r, n, i satisfy $0 < r \leq n$, $i \in [o, v_{n-r}] + ra_n$, we have

$$f_i = a_{n-r} \cdot (e_i - e_{i-ra_n}) \ .$$

(2.1.2) if integers r, n, i satisfy $1 \leq r < n$, $i \in \big(ra_n + v_{n-r}, (r+1)a_n\big)$, (respectively, $1 \leq n$, $i \in (v_{n-1}, a_n)$) then

$$f_i = 2^{(h-i)/\sqrt{a_n}} \cdot e_i$$

where $h = \big(r + \frac{1}{2}\big)a_n$ (respectively, $h = \frac{1}{2}a_n$).
(2.1.3) If integers r, n, i satisfy $1 \leq r \leq n$, $i \in \big[r(a_n+b_n), na_n + rb_n\big]$, then

$$f_i = e_i - b_n \cdot e_{i-b_n} \ .$$

(2.1.4) If integers r, n, i satisfy $0 \leq r < n$, $i \in \big(na_n + rb_n, (r+1)(a_n+b_n)\big)$ then

$$f_i = 2^{(h-i)/\sqrt{b_n}} \cdot e_i$$

where $h = \big(r + \frac{1}{2}\big)b_n$.

Note 2.2. P.$\mathcal{d}$., Definition 2.1 gives $f_i = \sum_{j=0}^{i} \lambda_{ij} e_j$ uniquely for each $i \geq 0$, and since λ_{ii} is never zero, this linear relationship is invertible. So the e_i exist and are unique, and indeed

$$\operatorname{lin}\{e_i : 0 \leq i \leq n\} = \operatorname{lin}\{f_i : 0 \leq i \leq n\} = F_n \qquad \text{say} \ ,$$

for all $n \geq 0$. If $x = \sum_{i=0}^{N} \lambda_i e_i \in F$, we write $|x| = \sum_{i=0}^{N} |\lambda_i|$.

Definition 2.3. For each $n \geq 0$, let $I_n : (F_n, \|\cdot\|) \to (F_n, |\cdot|)$ be the identity map. We note that if $n = ma_m$ (respectively, $n = v_m$) for some $m \in \mathbb{N}$, then the norm $\|\cdot\|$ on F_n depends only on the underlying space W, the value of m, and the elements $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^{m-1}$ (respectively, $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$) of the sequence $\underset{\sim}{d}$. So for a given space X, we may choose functions $M_1 : \mathbb{N}^2 \to \mathbb{N}$ and $M_2 : \mathbb{N}^2 \to \mathbb{N}$ such that for all $m \in \mathbb{N}$, and for all $\underset{\sim}{d}$ such that definition 2.1 is meaningful, we have

$$\|I_{ma_m}\| \vee \|I_{ma_m}^{-1}\| \leq M_1(m, a_m) \tag{2.3.1}$$

and

$$\|I_{v_m}\| \vee \|I_{v_m}^{-1}\| \leq M_2(m, b_m) \ . \tag{2.3.2}$$

Definition 2.4. Let Q_m $(m \geq 1)$ denote the projection $F \to F_{ma_m}$ such that

$$Q_m(f_j) = \begin{cases} f_j \ , & 0 \leq j \leq ma_m \\ f_{j - ra_n + (r-n+m)a_m} \ , & j \in [o, v_{n-r}] + ra_n \ , \quad 0 < n - m < r \leq n \ , \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

Definition 2.5. Let Q_m^0 $(m \geq 1)$ denote the projection $F \to F_{ma_m}$ such that

$$Q_m^0(f_j) = \begin{cases} f_j \ , & 0 \leq j \leq ma_m \\ -a_{n-r} \cdot e_{j-ra_n} \ , & j \in [o, v_{n-r}] + ra_n \ , \quad 0 < n - m < r \leq n \ . \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

Definition 2.6. Let $P_{n,m}$ $(m > n \geq 1)$ be the operator $\tau_{n,m} \circ Q_m$, where $\tau_{n,m} : F_{ma_m} \to F_{ma_m}$,

$$\tau_{n,m}(e_j) = \begin{cases} e_j \ , & 0 \leq j < (m-n)a_m \\ 0 \ , & (m-n)a_m \leq j \leq ma_m \ . \end{cases}$$

Definition 2.7. Let $T : F \to F$ be the linear map such that $Te_i = e_{i+1}$ $(i \geq 0)$. Also, let I denote the identity map.

3. Continuity of Q_m and other projections

Lemma 3.1. *$\|Q_m\| \leq m$ for all m.*

Proof: By (1.1.5),

$$\|Q_m\| \leq \max\left(\sup_{i \geq 0} \|Q_m g_i\| \ , \ \sum_{i=1}^{\infty} \|Q_m x_i\|\right) \tag{3.1.1}$$

Now Definition 2.4 gives $Q_m f_i = f_j$ $(j \le i)$ or zero for every i; so $Q_m f_i$ is a vector of norm at most one. Since $\{f_i\} \supset \{g_i\}$, we conclude that

$$\|Q_m\| \le \max\left(1\ ,\ \sum_{i=1}^{\infty} \|Q_m x_i\|\right)\ . \tag{3.1.2}$$

But $Q_m x_i = Q_m f_{a_i - 1} = f_{a_i - 1}$ $(1 \le i \le m)$ or zero $(i > m)$(3.1.3) (by Definition 2.4). Hence,

$$\|Q_m^0\| \le m\ .$$

Lemma 3.2. *$\|Q_m^0\| \le a_m$ for all m, p.ḏ..*

Proof: Again, (1.1.5) gives

$$\|Q_m^0\| \le \max\left(\sup_{i \ge 0} \|Q_m^0 g_i\|\ ,\ \sum_{i=1}^{\infty} \|Q_m^0 x_i\|\right)\ .$$

Definition 2.5, like Definition 2.4 gives $Q_m^0(x_i) = x_i$ $(i \le m)$ or zero $(i > m)$ (3.2.1), so

$$\|Q_m^0\| \le \max\left(\sup_{i \ge 0} \|Q_m^0 g_i\|\ ,\ m\right)\ . \tag{3.2.2}$$

Now $\{g_i\} \subset \{f_i\}$, and $Q_m^0(f_i)$ is f_i or zero unless $i \in [o, v_{n-r}] + r a_n$, $0 < n - m < r \le n$, when

$$\begin{aligned} \|Q_m^0 f_i\| &= a_{n-r} \cdot \|e_{i - r a_n}\| \\ &\le a_{m-1} \cdot \|e_{i - r a_n}\| \\ &\le a_{m-1} \cdot \|I_{v_{m-1}}^{-1}\| \cdot |e_{i - r a_n}| \end{aligned}$$

since $i - r a_n \le v_{m-1}$;

$$\le a_{m-1} \cdot M_2(m-1, b_{m-1}) \quad \text{by} \quad (2.3.2)\ .$$

Hence,

$$\begin{aligned} \|Q_m^0\| &\le \max\left(a_{m-1} \cdot M_2(m-1, b_{m-1}), m\right) \\ &\le a_m \qquad \text{p.ḏ}\ . \end{aligned}$$

Lemma 3.3. *P.ḏ, we have $\|P_{n,m}\|, \|\tau_{n,m}\| \le \max(a_{n+1}, m)$, and $\|Q_n P_{n.m}\| \le a_{n+1}$ for all $n < m$.*

Proof: Using (1.1.5), we observe that (2.6) gives

$$\begin{aligned} P_{n,m}(x_i) &= \tau_{n,m} Q_m(x_i) \\ &= \begin{cases} \tau_{n,m}(x_i) & , \quad i \le m \\ 0 & , \quad i > m \end{cases} \\ &= \begin{cases} x_i & , \quad i \le m \\ 0 & , \quad i > m \end{cases} \end{aligned} \tag{3.3.1}$$

hence,

$$\|P_{n,m}\| \leq \max(\sup \|P_{n,m}g_i\|, m)$$
$$\leq \max(\sup \|P_{n,m}f_i\|, m) \ .$$

Now $P_{n,m}f_i = \tau_{n,m}Q_m f_i$ and by (2.4), $Q_m f_i$ is either zero of f_j for some $0 \leq j \leq ma_m$. Thus

$$\sup \|P_{n,m}f_i\| \leq \max_{0 \leq j \leq ma_m} \|\tau_{n,m}f_j\| \ . \tag{3.3.2}$$

But (2.6) and (2.1) give for all $0 \leq j \leq ma_m$,

$$\tau_{n,m}f_j = \begin{cases} -a_{m-r} \cdot e_{j-ra_m} \ , \ j \in [0, v_{m-r}] + ra_m \ , \ m-n \leq r \leq m \ , \\ f_j \quad \text{or zero} \ , \ \text{otherwise} \ . \end{cases}$$

Hence

$$\max_j \|\tau_{n,m}f_j\| \leq \max(1, \sup_{\substack{j \in [0,v_n]+ra_m \\ r \in [m-n,m]}} a_{m-r} \cdot \|e_{j-ra_m}\|)$$
$$\leq \max(1, a_n \cdot \sup_{i \in [0,v_n]} \|e_i\|)$$
$$\leq a_n \cdot M_2(n, b_n) \quad \text{by (2.3.2)} \ ; \tag{3.3.3}$$

then

$$\|P_{n,m}\| \leq \max\left(a_n \cdot M_2(n,b_n), m\right)$$
$$\leq \max(a_{n+1}, m) \qquad \text{p.}\underline{\text{d}}. \ .$$

We note also that since Q_m acts as the identity on the domain of $\tau_{n,m}$, we have $\|\tau_{n,m}\| \leq \|\tau_{n,m}Q_m\| = \|P_{n,m}\|$.

Using (1.1.5) to consider the operator $Q_n P_{n,m}$ we consider first the sum $\sum_{i=1}^{\infty} \|Q_n P_{n,m} x_i\|$. By (3.3.1),

$$Q_n P_{n,m} x_i = \begin{cases} Q_n x_i & , \ i \leq m \\ 0 & , \ i > m \ . \end{cases}$$
$$= \begin{cases} x_i & , \ i \leq n \\ 0 & , \ i > n \ , \ \text{by (3.1.3)} \ . \end{cases}$$

Thus $\sum_i \|Q_n P_{n,m} x_i\| = n$. Also, by (3.3.2), (3.3.3), (3.1), we have $\sup_i \|Q_n P_{n,m} f_i\| \leq \|Q_n\| \cdot \max_i \|P_{n,m} f_i\| \leq n a_n M_2(n, b_n)$. Therefore, (1.1.5) gives

$$\|Q_n P_{n,m}\| \leq \max\left(\sup_i \|Q_n P_{n,m} g_i\| \ , \ \sum_i \|Q_n P_{n,m} x_i\|\right)$$
$$\leq \max\left(n a_n M_2(n, b_n), n\right)$$
$$\leq a_{n+1} \qquad \text{p.}\underline{\text{d}}. \ .$$

4. Continuity of T

Lemma 4. *Let $\eta > 0$ be given. The following are true p.d.;*

(1) $\|T\| \le 1+\eta$, *indeed* $\|T|_{\ell_1}\| \le 1+\eta$ *while* $T|_W$ *has nuclear norm at most* η.

(2) $\|T^{a_n+b_n}(I-Q_n^0)\| < 1+\eta$ *for all* $n \in \mathbb{N}$.

4.1 Proof of Lemma 4(1).

By (1.1.5) it is sufficient to show that $\|T|_{\ell_1}\| < 1+\eta$, while $\sum_{i=1}^{\infty} \|Tx_i\| < \eta$ (4.1.1). (4.1.1) is easy to prove; by (2.1), (1.6.1) we have

$$x_i = f_{a_i-1} = 2^{(1-\frac{1}{2}a_i)/\sqrt{a_i}} \cdot e_{a_i-1} \; ;$$

$$Tx_i = 2^{(1-\frac{1}{2})/\sqrt{a_i}} \cdot e_{a_i} = 2^{(1-\frac{1}{2}a_i)/\sqrt{a_i}} \cdot (f_0 + a_{i-1}^{-1} \cdot f_{a_i})$$

(by (2.1));

$$\|Tx_i\| \le 4 \cdot 2^{-\frac{1}{2}\sqrt{a_i}}$$

since $\|f_j\| = 1$ for all j; then $\sum_{i=1}^{\infty} \|Tx_i\| \le \sum_{i=1}^{\infty} 2^{-\frac{1}{2}\sqrt{a_i}} \le \eta$ p.d. (4.1.2)

We now examine the behavior of T on ℓ_1. Let $R, S \subset \mathbb{Z}^+$ be as follows. $R = \bigcup_{i=0}^{4} R_i$ and $S = \bigcup_{i=1}^{4} S_i$ where

$$R_0 = \{o\} \; ,$$

$$R_1 = \{ra_n + v_{n-r} : 0 < r \le n\} \; ;$$

$$R_2 = \{ra_n - 1 : 2 \le r \le n\} \; ;$$

$$R_3 = \{na_n + rb_n : 1 \le r \le n\};$$

$$R_4 = \{r(a_n + b_n) - 1 : 1 \le r \le n\} \; ;$$

$$S_1 = \bigcup_{\substack{0<r\le n \\ n\ge 1}} \left([0, v_{n-r}) + ra_n\right) \; ;$$

$$S_2 = \bigcup_{n=1}^{\infty} S_{2,n} \quad \text{where} \quad S_{2,n} = \Big(\bigcup_{r=1}^{n-1} (ra_n + v_{n-r}, (r+1)a_n - 1)\Big) \cup (v_{n-1}, a_n - 1)$$

$$S_3 = \bigcup_{\substack{1\le r\le n \\ n\ge 1}} \left[r(a_n + b_n), na_n + rb_n\right] \; ;$$

$$S_4 = \bigcup_{n=1}^{\infty} S_{4,n} \quad \text{where} \quad S_{4,n} = \bigcup_{r=0}^{n-1} (na_n + rb_n, (r+1)(a_n + b_n) - 1) \; .$$

Note that, p.$\mathscr{d}$., $R \cup S = \mathbb{Z}^+ \setminus \{a_n - 1 : n \geq 1\}$ so for all i, $g_i = f_j$ for some $j \in R \cup S$. Thus

$$\sup_i \|Tg_i\| = \sup_{j \in R \cup S} \|Tf_j\| \ . \tag{4.1.3}$$

We now investigate the right hand side of (4.1.3), considering first those $j \in S$.

If $i \in S$ then by Definitions 2.1, 2.7, we have

$$Tf_i = \begin{cases} f_{i+1} & , \ i \in S_1 \cup S_3 \\ 2^{1/\sqrt{a_n}} \cdot f_{i+1} & , \ i \in S_{2,n} \\ 2^{1/\sqrt{b_n}} \cdot f_{i+1} & , \ i \in S_{r,n} \ . \end{cases}$$

So $\sup_S \|Tf_i\| = 2^{1/\sqrt{a_1}} < 1 + \eta$ p.$\mathscr{d}$..

Definitions 2.7 and 2.1 give the following slightly more elaborate action for T on those f_i with $i \in R$ (the 'bad cases'). If $i \in R$, then

$$Tf_i = \begin{cases} 2^{(1-\frac{1}{2}a_1)/\sqrt{a_1}} \cdot f_i \ , \ i = 0 \in R_0 \\ a_{n-r} \cdot (\varepsilon_1 f_{i+1} - \varepsilon_2 f_{v_{n-r}+1}) \ , \ i = ra_n + v_{n-r} \in R_1 \ , \text{ where} \\ \qquad \varepsilon_2 = 2^{(1+v_{n-r}-\frac{1}{2}a_{n-r+1})/\sqrt{a_{n-r+1}}} \ , \text{ and} \\ \qquad \varepsilon_1 = \begin{cases} 2^{(1+v_{n-r}-\frac{1}{2})/\sqrt{a_n}} & , r < n \ , \\ 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & , r = n \ . \end{cases} \\ 2^{(1-\frac{1}{2}a_n)/\sqrt{a_n}} \cdot \left(f_0 + \dfrac{1}{a_{n-r}} f_{i+1} \right) \ , \ i = ra_n - 1 \in R_2 \ , \ 2 \leq r \leq n \\ \varepsilon_1 f_{i+1} - b_n \varepsilon_2 f_{i+1-b_n} \ , \ i = na_n + rb_n \in R_3 \ , \text{ where} \\ \qquad \varepsilon_2 = 2^{(na_n+1-\frac{1}{2}b_n)/\sqrt{b_n}} \ , \\ \qquad \varepsilon_1 = \begin{cases} \varepsilon_2 \ , \ r < n \ , \\ 2^{(v_n+1-\frac{1}{2}a_{n+1})/\sqrt{a_{n+1}}} \ , \ r = n \ . \end{cases} \\ 2^{(1-(r+1)a_n-\frac{1}{2}b_n)/\sqrt{b_n}} \cdot \left(\sum_{j=0}^{r-1} b_n^j f_{i-jb_n} + b_n^r \cdot (f_0 + \dfrac{1}{a_{n-r}} f_{ra_n}) \right) \ , \\ \qquad i = r(a_n + b_n) - 1 \in R_4 \ . \end{cases} \tag{4.1.4}$$

It is not hard to see that, p.$\mathscr{d}$., these values have in common that they are all of norm less than η, for any fixed $\eta > 0$. So $\sup_R \|Tf_i\| < \eta$ p.$\mathscr{d}$.. But then (4.1.3) gives $\sup_i \|Tg_i\| \leq 1 + \eta$ (4.1.5), and then (1.1.5) gives

$$\|T\| \leq \max \left(\sup_i \|Tg_i\| \ , \ \sum_i \|Tx_i\| \right)$$
$$\leq \max(1 + \eta, \eta) = 1 + \eta \ .$$