

STANISLAW R. MASSEL

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HYDRODYNAMICS OF COASTAL ZONES

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HYDRODYNAMICS OF COASTAL ZONES

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COASTAL OCEANOGRAPHY OF WASHINGTON AND OREGON

To my family

PREFACE

Coastal waters are receiving greatly increasing attention by the scientific and engineering community. The scientific interest is motivated mainly by the fact that in the coastal region the interaction between atmosphere, ocean and land can be the most clearly observed. Crests of wind waves or swell can be seen arriving at the beach at intervals of some seconds and appreciable variations in the water level with about 12 or 24 hours between occurrences of maximum height are often detected. The proximity of a coast line leads to a piling-up of water and the development of surface slopes. The surface slopes give rise to horizontal pressure gradients in the water and these in turn generate currents which are superimposed on the initial wind drift. On the other hand, the river water, which often gives coastal regions their relatively low salinity, has frequently been modified by its passage through estuaries.

Coastal waters also play a special role from economic and environmental points of view. The expansion of world trade requires construction of large numbers of harbours and terminals. A striking example of the increased use of coastal waters is the exploration for oil and gas resources and their exploitation. At the same time, the recreational use of coastal zone is a very important factor in many areas.

This book does not cover all the problems mentioned above. It is intended to discuss the selected theoretical topics of coastal hydrodynamics, including basic principles and applications in coastal oceanography and coastal engineering. However, the book is not a handbook; the emphasis is placed on presentation of a number of basic problems, rather than giving detailed instructions for their application.

The bulk of the material deals with surface waves. In the author's opinion there is still a strong need for a book on wave phenomena in the coastal waters, as general textbooks on sea surface dynamics focus most of their attention on the deep ocean. The book is intended to cover this need by concentrating on the phenomena typical for the coastal zone.

The way of handling in the book is mainly based on courses of lectures given in the Institute of Hydroengineering, Polish Academy of Sciences in Gdansk. The approach throughout is a combination of the theoretical and observational. A certain amount of mathematics must play its part as the

contributions by mathematicians have always been prominent in this field. The necessary mathematical background is a basic knowledge of ordinary and partial differential equations, as well as the statistical and spectral analysis of the time series. The reader should also be familiar with fundamental hydrodynamic concepts.

The book comprises nine chapters. The governing equations and conservation laws are treated in Chapter One, using the variational principles. The theory of regular surface waves is covered in Chapters Two to Four. The nonlinear effect of wave train modulation and their breaking on beaches is examined in Chapter Five. Chapters Six and Seven focus on the statistical and spectral treatment of waves induced by wind. The current generation and the circulation pattern are the subject of Chapter Eight and the sea level variations are examined in Chapter Nine. References for further reading, including both general references and some related to specific sections, can be found at the end of each chapter.

I owe a lot to many people over many years for forming the ideas expressed in the book. In particular, Professors Cz.Druet, and J.Onoszko are those who introduced the author to the field of marine hydromechanics. I wish to acknowledge the stimulating discussions provided by many colleagues and members of the staff at the Institute of Hydroengineering of the Polish Academy of Sciences in Gdansk. I am also very grateful to my wife, Barbara, for her accurate and patient typing of the manuscript and for keeping me up during the writing.

November 1988

S.R. Massel

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Chapter 1

INTRODUCTION

1.1 Basic ideas and assumptions

1.1.1 Distinctive features of coastal waters

The term *coastal waters* will be taken for the purpose of this book to define the areas enclosed, towards the ocean by the continental shelf, and towards land by the upper limit for the direct action of the sea, but not extending into estuaries (Fig. 1.1). Coastal waters have features which are sufficiently distinctive from the deep oceans. The presence of the bottom at a relatively shallow depth forms a constraint on water movement, tending to divert currents so that they flow nearly parallel to it. Wind-driven currents are also strongly affected by the presence of the coastline and the bottom. These mechanisms rise in some areas the storm surges or produce the upwelling and the coastal jets.

The shallow-water waves have many specific properties which distinguish them from deep-water waves. As waves travel into shallower water, their dynamics is progressively more nonlinear and dissipative. Energy is transferred away from the peak of the spectrum to higher and lower frequencies. This is mainly due to water depth changes. In the shallow water a variety of processes composes very complicated picture of the sea surface and the wave spectrum. Among them there are nonlinear interactions between spectral components, wave transfer due to shoaling water, wave breaking, etc. However, at present we have not the theory which is able to explain upon experimental results. This is the reason that due to complexity of the wave motion in the coastal zone, our present knowledge is based mainly on the experiments and the application of statistical and spectral analysis.

The influx of fresh water run-off from the land, often passing through estuaries, has the effect of reducing the salinity, and hence also the density of coastal water. As a result of these effects, coastal waters are usually areas of relatively large horizontal gradients of density often associated with changes

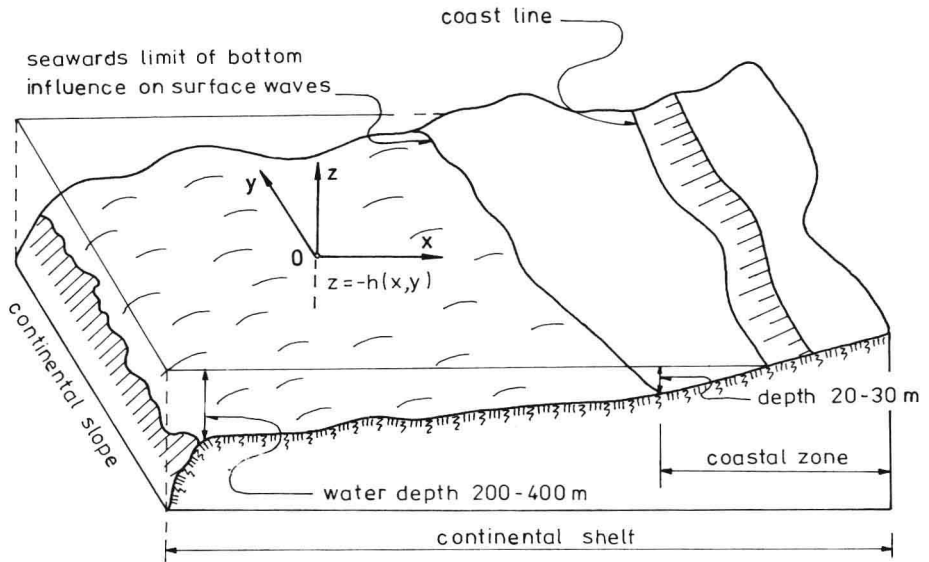


Figure 1.1: The coastal zone. Definition sketch

in currents. The coastal waters are, of course, interconnected and dependent on the adjacent ocean and land. Therefore, they cannot be considered in isolation.

1.1.2 Classification of coastal water oscillations

The periodic rise and fall of the sea surface are experienced by anyone who lives near at the coastline. As many oscillations known in physics text books, all surface variations can be classified. It is quite natural to use the period of time scale as a measure of the distinction. Example of such crude classification is given in Table 1.1, which is mainly due to Svendsen and Jonsson (1982). Of course, such classification has rather descriptive than analytical character but it reflects the variety of oscillation periods observed in the coastal zone.

Prior to introducing the basic principles of hydrodynamics, we will adopt the rectangular coordinates system $O(x, y, z)$ or $O(x_i, z)$; $i = 1, 2$. The origin of the system is at the mean sea surface (Fig. 1.1). The axes x and y are horizontal and the x axis is directed towards coastline while the y axis is parallel to the coastline. The z axis is directed opposite to the force of gravity. For some purposes, the coastal region, defined above, is quite large and diverse from the hydrodynamical point of view. Particularly, the most dynamic part of the coastal waters is the vicinity of the shore line. Thus, it will be useful to define, within the coastal waters, a region called the *coastal zone*. It is enclosed towards the ocean by the water depth contour $h = h_g$, and towards

Table 1.1: Waves, physical mechanisms, and periods

Wave type	Physical mechanism	Periods
Wind waves	Wind shear	<15s
Swell	Wind waves	<30s
Surf beat	Wave groups	1–5min
Seiche	Wind variation	2–40min
Harbour resonance	Surf beat	2–40min
Tsunami	Earthquake	10min–2h
Tides	Gravitational action of the moon and sun, earth rotation	12–24h
Storm surges	Wind stresses and atmospheric pressure variation	1–30days

the land by an upper limit for the direct action of the sea. In the coastal zone, which water depth $h \leq h_g$, the water motion (especially wave motion) is strongly influenced by the bottom configuration. From the linear wave theory (see Chapter 2) we find that the bottom influence on the surface waves is observed when:

$$h \leq h_g = \frac{L}{2}, \quad (1.1)$$

where: L - wave length.

Using the classical dispersion relation (eq. 2.50), eq. (1.1) gives:

$$h_g \approx \frac{\pi g}{\omega^2}, \quad (1.2)$$

in which: ω - angular frequency of wave motion.

For the wind-induced waves, the frequency ω should be identified with the peak frequency ω_p of wave energy spectrum (see Chapter 7). For example, in the small, semi-enclosed seas, the typical peak frequency ω_p is of order $\sim 1 \text{ rad/s}$. Therefore:

$$h_g \approx \pi g \approx 30.0 \text{ m}, \quad (1.3)$$

If the typical bottom slope in the coastal zone is estimated as $0.01 \div 0.015$, the average width (D) of the coastal zone is:

$$D \approx 2000 \div 3000m. \quad (1.4)$$

1.1.3 Continuous fluid and water particle concepts

One of the most important physical principles of hydrodynamics is that the fluid is *continuous*. By continuous fluid in continuous motion, we mean that the velocity \vec{u} is everywhere finite and continuous while its space derivatives of the first order are finite (but not necessarily continuous). Thus, any closed surface S which moves with the fluid, permanently and completely separates the fluid matter inside S from that outside. Sometimes the following definition of continuous fluid is used, i.e., the fluid can be treated as continuous when the flow past an obstacle of the dimension A which is much larger than the average free path of the molecule l_0 (for water $l_0 \approx 3 \cdot 10^{-10}m$). The rate (l_0/A) is known as Knudsen number (**Kn**). If:

$$\mathbf{Kn} = \frac{l_0}{A} < 0.01, \quad (1.5)$$

the fluid can be treated as continuous (Puzyrewski and Sawicki, 1987). In the continuous fluid, we can define a *fluid particle* as consisting of the fluid contained within an infinitesimal volume, that is to say, a volume whose size may be considered so small that for the particular purpose in hand its linear dimensions are negligible. We can then treat a fluid particle as a geometrical point.

In general, the equation of motion for the fluid particle depends on the physical properties of the fluid and motion itself. In order to render the subject amenable to exact mathematical treatment, we make simplifying assumptions on the fluid and motion, i.e.:

a) *water is an inviscid or perfect fluid*. An inviscid fluid is a continuous fluid which can exert no shearing stress. However, the real fluids do have viscosity which creates the stresses and additional dissipations within the fluid. Thus:

$$\vec{\tau} = \mu \frac{d\vec{u}}{d\vec{n}}, \quad (1.6)$$

where: $\vec{\tau}$ - tangential stress, $\vec{u} = (u, v, w)$ - fluid velocity vector, \vec{n} - vector, normal to vector \vec{u} , μ - coefficient of viscosity of the fluid.

Sometimes for convenience, we will represent the vector \vec{u} as $\vec{u} = (u_i, w; i = 1, 2)$; therefore $u_1 = u, u_2 = v$. Thus, for the ideal fluid should be:

$$\vec{\tau} = 0 \quad \text{or} \quad \mu = 0 \quad \text{when} \quad \vec{u} \neq \vec{0}, \quad (1.7)$$

in which: $\vec{0} = (0, 0, 0)$.

To justify the identity (1.7), we adopt the differential length scale L in which the velocity varies in magnitude by U . The ratio $\mathbf{Re} = \rho UL/\mu$ (Reynolds number) represents the relative magnitudes of the inertial and viscous terms; in many oceanic motions, the Reynolds number is very large. Thus, the viscous influence is often quite negligible over most of the field of motion. The viscous forces are important only in narrow regions of the flow, where the local inertial and viscous forces are comparable. In the ocean the interfacial layer between the air and the water, as well as the bottom boundary layer, illustrates such regions quite clearly. The thickness δ of the surface boundary layer is of order $\delta \approx (2\nu/\omega)^{1/2}$, where ν - kinematic coefficient of viscosity (for water $\nu \approx 1.2 \cdot 10^{-6} \text{ m}^2/\text{s}$). For the typical frequencies, the thickness $\delta \sim 0.001 \text{ m}$. For the boundary layer near the natural sea bottom, the eddy viscosity is much higher ($\sim 100\nu$); the thickness of the boundary layer is then about 0.1 m , which is still quite small. Therefore, the boundary layer regions are but a very small fraction of a fluid volume, and the influence of the viscosity on the wave motion can be neglected.

b) *water is an incompressible fluid*. The compressibility of water is rather small and the Young modulus is of order $E \approx 3.05 \cdot 10^8 \text{ N/m}^2$ (Dera, 1983). As the typical velocity of sea water is much smaller than the sound speed, the very small water compressibility has not influence on the water motion. Therefore, the equation of continuity for the homogeneous incompressible fluid becomes (Milne-Thomson, 1974):

$$\text{div } \vec{u} = \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \quad (1.8)$$

where:

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}. \quad (1.9)$$

In general, vector \vec{u} represents the sum of the current and wave velocities.

c) *motion is irrotational*. It means that the individual elementary particles of the fluid do not rotate. The mathematical expression of this is:

$$\text{rot } \vec{u} = \text{curl } \vec{u} = \nabla \times \vec{u} = \vec{0}, \quad (1.10)$$

When the vorticity is different from zero, the motion is defined as *rotational*. It was indicated above that in many oceanic motions, the influence of the viscous terms are quite negligible. In this event, the Lagrange theorem (Kochin

et al., 1963) indicates that if, at some initial instant, the vorticity vanishes everywhere in the field of flow, the motion is irrotational. In the absence of viscous effects, it remains so. The consequence of eq. (1.10) is that the velocity \vec{u} can be represented as the gradient of a scalar function, the *velocity potential* Φ :

$$\vec{u} = \nabla \Phi; \quad (1.11)$$

then in virtue of the continuity equation (1.8), the potential Φ obeys Laplace equation:

$$\text{div}(\nabla \Phi) \equiv \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}. \quad (1.12)$$

Additionally, we assume that the current velocity in the coastal zone is a slowly varying function of the horizontal coordinates (x_1, x_2) and time t . The characteristic distance (\hat{L}) and time (\hat{T}) scales of the current are much greater than those of the waves (Mei, 1983):

$$(\omega \hat{T})^{-1} \sim (k \hat{L})^{-1} \sim \frac{h}{\hat{L}} = O\left(\frac{\nabla_h h}{kh}\right) = O(\mu) \ll 1, \quad (1.13)$$

in which: ω - characteristic frequency of wave motion, k - corresponding wave number and:

$$\nabla_h = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j}. \quad (1.14)$$

We shall also assume, that the characteristic water depth h varies slowly in horizontal coordinates $x_i (i = 1, 2)$. The horizontal velocity components $U_i (i = 1, 2)$ usually are $O(\sqrt{gh})$. Because of (1.13), the vertical current velocity is small, $W = O(\mu)$. Then, the horizontal components of the vorticity vector take the form:

$$\left[\frac{\partial U_1}{\partial z} - \frac{\partial W}{\partial x_1} \right] \vec{i} \quad \text{and} \quad \left[\frac{\partial U_2}{\partial z} - \frac{\partial W}{\partial x_2} \right] \vec{j}, \quad (1.15)$$

where: $\vec{U} = (U_j, W), j = 1, 2$.

As the term $\partial W / \partial x_j$ is $O(\mu^2)$, the horizontal components are:

$$\frac{\partial U_j}{\partial z} = O(\mu^2). \quad (1.16)$$