SIGNAL PROCESSING II: THEORIES AND APPLICATIONS

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Proceedings of EUSIPCO-83
Second European Signal Processing Conference

Erlangen, W.-Germany
September 12-16, 1983

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Erlangen, W.-Germany

1983

NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

ISBN: 0 444 86743 0

Published by:

ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributors for the U.S.A. and Canada:

ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 Vanderbilt Avenue
New York, N.Y. 10017
U.S.A.

Library of Congress Cataloging in Publication Data

EUSIPCO (2nd : 1983 : University of Erlangen-Nuernberg)
Signal processing II.

Includes index.
1. Signal processing--Congresses. I. Schüssler, H. W. (Hans Wilhelm), 1928- . II. Title.
TK5102.5.E93 1983 621.38'043 83-14056
ISBN 0-444-86743-0

PRINTED IN THE NETHERLANDS

FOREWORD

The European Signal Processing Conference 1983 is the second in a sequence of international conferences promoted and organized by EURASIP, the European Association for Signal Processing. It was held at the new campus of the Engineering Department (Technische Fakultaet) of the University of Erlangen-Nuernberg on September 13 to 15, 1983. The conference provided a forum for all aspects of signal processing, this exciting and still fast growing field of engineering science. It gave a chance to get aquainted with scientists and activities especially out of Europe.

It was a special feature of the conference thaŧ it included a large number of tutorial papers as introductions to the different sessions. They were presented either by the chairman or by another distinguished expert in that particular field. All efforts have been made to arrange the program such that there was no overlapping of these tutorial papers. So it was possible to attend not only the particular session of direct interest, but to listen to other overview type papers in related fields.

These Proceedings include these tutorial papers as well as the accepted regular contributions, reporting about recent progress in signal processing. The order follows the organization of the conference, which was composed of 27 sessions, grouped into the following 7 fields:

A. Signal Processing 1 D
B. Signal Processing 2 D
C. Speech and Sound Processing
D. Detection and Estimation
E. Applications
F. Software
G. Hardware

Unfortunately some papers, considered in the program, did not arrive in time, to be included in the Proceedings.

The large number of papers submitted for EUSIPCO 83 from all over the world reflects the importance of our field. We are grateful to all scientists, who submitted papers. Their presentation formed the essential part of the conference, their contributions form the main part of these Proceedings as well. Thanks are especially due to the members of the Scientific Committee, who supervised the reviewing process of all papers, took care of the tutorial papers, either by providing them themselves or by inviting another specialist and who prepared their sessions. Their contribution to the success of the conference and to these Proceedings can not be overestimated.

We are grateful to the University of Erlangen-Nuernberg for its support and for the possibility to use their facilities. Our thanks are also due to the cosponsoring organizations and institutions. The preparation of the whole conference was the result of an outstanding cooperation within the Steering Committee. Only one member, Mrs. U. Arnold, the Conference Secretary, is named here especially for the enormous work she carried out.

Finally, thanks are due to North Holland Publishing Company for an excellent collaboration.

Erlangen,
W-Germany

H.W. Schüssler

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Elsevier Science Publishers B.V. (North-Holland)

75 YEARS ALIASING ERROR IN THE SAMPLING THEOREM

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The reconstruction of a signal function by Shannon's sampling series is not exact whenever the signal is not precisely bandlimited. The arising so-called aliasing error was first treated by de La Vallée Poussin in a paper on interpolation in 1908. Our aim is to summarize theoretical results dealing with bounds on the aliasing error. This will include those in historical papers, work enclosing the bound of Weiss of 1963, orders of convergence of the error to zero, several generalizations of the sampling series, and combined errors. Some papers containing related results are mentioned only but included in the list of references which is by no means complete.

1. THE SAMPLING THEOREM

The sampling theorem, as it is attributed to Shannon [24], Kotel'nikov [21], Whittaker [45], and others, states that each signal function f with highest frequency Ω, i.e.,

$$(1)\quad f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{i\omega t}\,d\omega \qquad (t \in \mathbb{R}),$$

can be reconstructed from its samples taken at instants kT, $T = \pi/\Omega$, $k \in \mathbb{Z}$, by

$$(2)\quad f(t) = s_\Omega(f;t) \stackrel{\Delta}{=} \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{si}\{\Omega(t-kT)\} \quad (t \in \mathbb{R}).$$

Due to the Paley-Wiener theorem the assumption (1) with $F \in L^2[-\Omega,\Omega]$ is equivalent to that of $f \in L^2(\mathbb{R})$ being the restriction to $\mathbb{R}$ of an entire function of exponential type Ω. Thus f is arbitrarily often continuously differentiable and can neither be time-limited nor causal unless $f \equiv 0$. This is one fundamental reason for studying the behaviour of the series (2) for functions which are not bandlimited.

2. THE ALIASING ERROR

Starting from a given set of sampling points kT and neglecting condition (1), there is of course a whole class of functions g which have the same sampled values $g(kT) = f(kT)$ for all $k \in \mathbb{Z}$. They are all interpolated by the series in (2) but in general are not represented by it; they are called aliases of f or elements of a cotabular set of functions according to Whittaker. The reconstruction error

$$e_\Omega(f;t) \stackrel{\Delta}{=} f(t) - s_\Omega(f;t)$$

is then called the aliasing error, certainly depending on the arbitrarily chosen parameter Ω. It is also called foldover error [38] having in mind that the sampling series result from the convolution (folding) of the series of sampled values and the impulse response of the ideal lowpass. It is also described as the error arising from misapplication of the sampling theorem [44], or the error incurred by reconstructing f(t) as if it were properly bandlimited [2].

3. HISTORICAL RESULTS

The main problems concerning the aliasing error are to find useful bounds, to examine classes of functions for which the error becomes small with Ω getting large, and to investigate the speed (order) of convergence with which the error tends to zero for $\Omega \to \infty$. The first results in these directions are already 75 years old and were obtained by de La Vallée Poussin [42] who, however, knew nothing about the role which the series in (2) would play in signal theory, nor did he prove anything about exact representations[1]. Instead he examined functions f which vanish outside a finite interval [a,b], are integrable there, continuous in some $t_0 \in [a,b]$, and of bounded variation (BV) in a neighbourhood of t_0. In that case he showed that the aliasing error tends to zero in t_0 and uniformly in any subinterval of [a,b] where f is continuous and BV. He also dealt with the speed of convergence in the latter case: if f is continuous in (a,b) and its first derivative f' belongs to BV with V being the total variation of f' on (a,b), then he showed that

$$|\varepsilon_\Omega(f;t)| < \frac{V+\mu}{2\Omega}\,|\sin\Omega t| \leqslant \frac{V+\mu}{2\Omega}$$

where $\mu = f'(a+) + f'(b-)$. In other words, for the error he derived the order $\mathcal{O}(1/\Omega), \Omega \to \infty$,

[1] There is, however, a publication [15] by J. Hadamard of 1901 in which he already noticed that the series in (2) represents an entire function of exponential type Ω.

which tends to zero uniformly in $t \in \mathbb{R}$ [2] in case the signal function is timelimted. Additionally he gives a negative result, namely that the sampling series do not converge to f in t_0 if f has a jump in t_0, indeed the aliasing error will take each value between zero and $|f(t_0+) - f(t_0-)|$.

A few years later two papers appeared which were based on [42], one by J.F. Steffenson (1914) dealing with applications in number theory and another by M. Theis [39] giving a new proof of de La Vallée Poussin's results. There she also generalized the sampling series in the sense that she replaced the si-function by it's square or, in other words, used Fejer's kernel which lead to the sampling series

$$(3)\quad s_{\Omega}^{F}(f;t) \triangleq \frac{1}{2} \sum_{k=-\infty}^{\infty} f(kT)(si\{\tfrac{\Omega}{2}(t-kT)\})^2 .$$

Although these series will converge faster than those in (2) for non-differentiable functions as will be seen later [28], she could show that the error fails to vanish in case of jumps of the function.

4. RESULTS FOLLOWING SHANNON'S PAPER

Several years after Shannon had made the sampling theorem familiar to communication theorists, a number of papers appeared which contained essentially the same bound on the aliasing error. There it was usually assumed that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \qquad (t \in \mathbb{R})$$

has an absolutely integrable Fourier transform F in order to derive the estimate

$$(4)\quad |e_{\Omega}(f;t)| \leq \frac{1}{\pi} \int_{|\omega|>\Omega} |F(\omega)| \, d\omega ,$$

the left hand side tending to zero for $\Omega \to \infty$. This bound was first announced by P. Weiss [44] without proof and also investigated in [46] with a wrong proof, both in 1963. A correct proof was given in 1967 by J.L. Brown, Jr. [2], who also considered the bandpass case and proved [3] that the bound is best possible. Also D.C. Stickler [38] and later J.B. Kioustelidis [20] and R.P. Boas, Jr. [1] derived the same error bound. Earlier comparable results were given by J.B. Thomas and B. Liu [40] in 1964, reviewing bounds for several types of errors and by A. Papoulis [23], who showed in 1966 that

$$|e_{\Omega}(f;t)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega) - S_{\Omega}(f;\omega)| \, d\omega \, |\sin \Omega t| .$$

Also the Russians A.I. Shmukler and T.A. Shul'man [26] were very near to the bound (4); they started with the inequality

$$(5)\quad |e_{\Omega}(f;t)| \leq |f(t) - f_{\Omega}(t)| + \left| f_{\Omega}(t) - \sum_{k=-\infty}^{\infty} f(kT) si\{\Omega(t-kT)\} \right|$$

where

$$(6)\quad f_{\Omega}(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) e^{i\omega t} \, d\omega \qquad (t \in \mathbb{R}),$$

but then used a poor estimate for the first difference on the right hand side of (5). On the other hand, they derived the order of convergence $\mathcal{O}(\Omega^{-\varepsilon})$ for each $\varepsilon < 1/2$, in case f and f' vanish faster than $|t|^{-1}$ at infinity. They also investigated BV-functions as de La Vallée Poussin did, and this was considered again in 1977 [12]. Another Russian author, I.T. Turbovich, also derived lower bounds for e_{Ω} [41].

5. RECENT RESULTS

In the last six years the main investigations on the aliasing error were concerned with improvements of the orders of convergence, extensions and generalizations of the sampling theorem, and combined errors.
For time-limited signal functions the bound (4) was used to investigate an order of convergence for Lipschitz-continuous functions [6]. This order has been improved e.g. in [27],[29], even for functions of arbitrary duration, by circumventing (4) and using a different approach. This approach is comparable to that of [26] based on (5) where, however, the truncated Fourier inversion integral (6) is replaced by certain convolution integrals with bandlimited kernels which do approximate f better than the f_{Ω} do. This leads to the order $\mathcal{O}(\Omega^{-r-\alpha}\log\Omega)$ in case $f = f^{(o)}$ or the r-th derivative $f^{(r)}$ belongs to $\mathrm{Lip}\alpha$, $\alpha \in (0,1]$; compare also [33] and [36] in the time-limited case.
If the si-function in (2) is replaced by Fejér's kernel as in (3) the order is improved to $\mathcal{O}(\Omega^{-\alpha})$ for $f \in \mathrm{Lip}\alpha$, $\alpha \in (0,1)$, and to $\mathcal{O}(\Omega^{-r-\alpha})$ for $f^{(r)} \in \mathrm{Lip}\alpha$, if one uses de La Vallée Poussin's kernel, the kernel with trapezoidal Fourier transform, which leads to the series

$$s_{\Omega}^{V}(f;t) \triangleq \frac{3}{4} \sum_{k=-\infty}^{\infty} f(kT) si\{\tfrac{\Omega}{4}(t-kT)\} si\{\tfrac{3\Omega}{4}(t-kT)\}.$$

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[2] The $\mathcal{O}$-notation $g(\Omega) = \mathcal{O}(\varphi(\Omega))$, $\Omega \to \infty$, means that there exists a constant c such that $|g(\Omega)/\varphi(\Omega)| \leq c$ for Ω sufficiently large.

The connection between generalized sampling series of the latter type and convolution integrals together with orders of convergence is explained in [27],[28],[35], where also further examples can be found.
Concerning generalizations, let us also mention the aliasing error in the sampling approximation of signal functions with more than one variable [31], of polynomially bounded functions [22], of weak sense stationary processes [4],[30], of harmonizable random processes [13],[19], and sampling approximations containing higher derivatives [7],[43].
In several papers bounds have been calculated for combined errors, e.g. if also time jitter and quantization is taken into account [5],[9], or in case the sampling series of non-bandlimited signals is truncated [10],[16]. In this respect note that C.J. Standish [34] has shown that the partial sums of (2) need not to be convergent, even if the energy portion of F outside $[-\Omega,\Omega]$ is small.

6. RELATED RESULTS

In this last section let us mention some related topics where bounds for a respective aliasing error play a role, but which we cannot treat in detail here. The first is the Walsh sampling theorem stating that sequency-bandlimited functions can be represented from their sampled values. Since the condition of limitation in the sequency-band is even more restrictive than in the frequency-band, the error arising if this condition is not fulfilled was estimated in [8], [14], and for random signals in [11].
Secondly the formal differentation or Hilbert transformation of the series (2) lead to series representations of the derivative or the Hilbert transform of bandlimited functions in terms of samples of the function itself. The associated aliasing error was treated e.g. in [7],[37].
There are also investigations to reconstitute a bandlimited signal from samples from the past exclusively. For the estimation of the error resulting from the non-bandlimitedness there we refer e.g. to [32].
Last not least several authors have dealt with so-called alias-free sampling representations, possibly the first being H.S. Shapiro and R.A. Silverman [25]. This approach led to unequally spaced instants of time and random sampling schemes.
Finally note that we did not intend to review on the more practical investigations concerning aliasing, as the construction of pre- and post-filters, nor did we include the applied aspects of the given error bounds. In this respect let us only mention a most recent paper [18] where an application of the bound (4) is given for the problem of spacing the rail cross-ties in a railroad track.
For further papers on the sampling theorem we refer the reader to the review article [17].

7. REFERENCES

[1] Boas, R.P., Jr., Summation formulas and band-limited signals, Tôhoku Math. J. 24 (1972), 121 - 125.

[2] Brown, J.L., Jr., On the error in reconstructing a non-bandlimited function by means of the bandpass sampling theorem, J. Math. Anal. Appl. 18 (1967), 75 - 84. Erratum. Ibid. 21 (1968), 699.

[3] Brown, J.L., Jr., A least upper bound for aliasing error, IEEE Trans. Automatic Control. AC-13 (1968), 754 - 755.

[4] Brown, J.L., Jr., On mean-square aliasing error in the cardinal series expansion of random processes, IEEE Trans. Inform. Theory IT-24 (1978), 254 - 256.

[5] Butzer, P.L., A survey of the Whittaker-Shannon sampling theorem and some of its extensions, J. Math. Res. Exposition, 3 (1983), 185 - 212.

[6] Butzer, P.L., and Splettstößer, W., A sampling theorem fur duration-limited functions with error estimates, Inform. and Control 34 (1977), 55 - 65.

[7] Butzer, P.L., and Splettstößer, W., Approximation und Interpolation durch verallgemeinerte Abtastsummen, Forsch. Ber. NRW Nr. 2708. Westdeutscher Verlag Opladen, 1977.

[8] Butzer, P.L. and Splettstößer, W., Sampling principle for duration limited signals and dyadic Walsh analysis, Inform. Sci. 14 (1978), 93 - 106.

[9] Butzer, P.L. and Splettstößer, W., On quantization, truncation and jitter errors in the sampling theorem and its generalizations. Sign. Proc. 2 (1980), 101 - 112.

[10] Cambanis, S., and Habib, M.K., Finite sampling approximation for non-bandlimited signals, IEEE Trans. Inform. Theory IT-28 (1982), 67 - 73.

[11] Engels, W. and Splettstößer, W., On Walsh differentiable dyadically stationary random processes, IEEE Trans. Inform. Theory IT-28 (1982), 612 - 619.

[12] Giardina, C.R., On the sampling rate for cardinal series representations, Amer. Math. Soc. Notices 24 (1977), A - 310.

[13] Habib, M.K. and Cambanis, S., Sampling approximations for non-bandlimited harmonizable random signals, Inform. Sci. 23 (1981), 143 - 152.

[14] Habib, M.K. and Cambanis, S., Dyadic sampling approximations for non-sequency-limited signals, Inform. and Control 49 (1981), 199 - 211.

[15] Hadamard, J., La série de Taylor, Scientia, Phys.-Math. (1901).

[16] Honda, I., Approximation for a class of signal functions by sampling series rep-