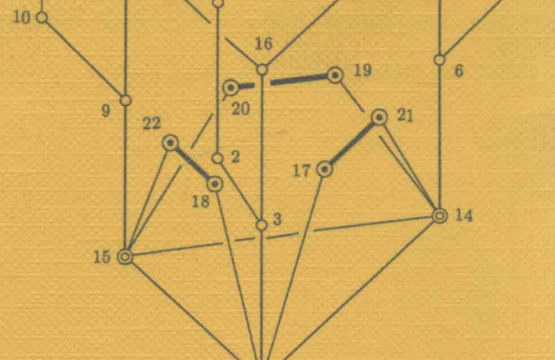
Lecture Notes in Mathematics
1548

Tohsuke Urabe

Dynkin Graphs and Quadrilateral Singularities



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Dynkin Graphs and Quadrilateral Singularities

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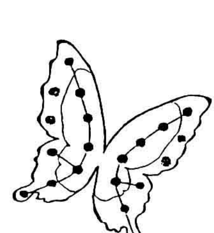
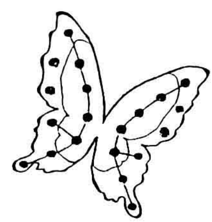
Contents



Chapter 4



Chapter 0

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Introduction

In this book we will study hypersurface quadrilateral singularities. Because the study of them can be reduced to the study of elliptic K3 surfaces with a singular fiber of type I_0^*, we will study such K3 surfaces, too. The combinations of rational double points that can occur on fibers in the semi-universal deformations of quadrilateral singularities are considered. We will show that the possible combinations can be described by a certain law from the viewpoint of Dynkin graphs. This is equivalent to saying that the possible combinations of singular fibers in elliptic K3 surfaces with a singular fiber of type I_0^* can be described by a certain law using classical Dynkin graphs appearing in the theory of semi-simple Lie groups.

In the appendix we explain that a similar description can be given for plane sextic curves. The theory developed in this book provides a long list of singular points that can occur on plane sextic curves. (Because of the complexity of the world of all plane sextic curves, the list is, however, probably not complete.)

In this book we always assume that the ground field is the complex field $\mathbf{C}$.

There are 6 types of hypersurface quadrilateral singularities (Arnold [1], [2]); each of them has the following normal form of the defining function and the Milnor number μ; all have modules number 2:

$J_{3,0}$: $x^3 + ax^2y^3 + y^9 + bxy^7 + z^2$, $(4a^3 + 27 \neq 0)$, $\mu = 16$.

$Z_{1,0}$: $x^3y + ax^2y^3 + bxy^6 + y^7 + z^2$, $(4a^3 + 27 \neq 0)$, $\mu = 15$.

$Q_{2,0}$: $x^3 + yz^2 + ax^2y^2 + bx^2y^3 + xy^4$, $(a^2 \neq 4)$, $\mu = 14$.

$W_{1,0}$: $x^4 + ax^2y^3 + bx^2y^4 + y^6 + z^2$, $(a^2 \neq 4)$, $\mu = 15$.

$S_{1,0}$: $x^2z + yz^2 + y^5 + ay^3z + by^4z$, $(a^2 \neq 4)$, $\mu = 14$.

$U_{1,0}$: $x^3 + xz^2 + xy^3 + ay^3z + by^4z$, $(a(a^2 + 1) \neq 0)$, $\mu = 14$.

Note that any connected Dynkin graph of type A, D or E corresponds to a singularity on a surface called a *rational double point* (Durfee [6]).

Let X be a class of quadrilateral singularities. Let $PC(X)$ denote the set of Dynkin graphs G with components of type A, D or E only such that there exists a fiber Y in the semi-universal deformation family of a singularity belonging to X satisfying the following two conditions depending on G:

(1) Fiber Y only has rational double points as singularities.

(2) The combination of rational double points on Y corresponds to graph G.

We will study set $PC(X)$ for $X = J_{3,0}$, $Z_{1,0}$, $Q_{2,0}$, $W_{1,0}$, $S_{1,0}$ and $U_{1,0}$.

For quadrilateral singularities we have important results due to Looijenga (Looijenga [9]). They enable us to reduce the study of the above $PC(X)$ to the study of lattice embeddings. A Dynkin graph G belongs to $PC(X)$ if, and only if, the root lattice $Q(G)$ has an embedding into Λ_3 satisfying certain conditions depending on X, where Λ_3 denotes the even unimodular lattice with signature (19, 3). (See Section 2.) On the other hand, for lattice embeddings we have excellent arithmetic results due to Nikulin (Nikulin [11]). With Nikulin's results we can always determine whether or not G belongs to $PC(X)$. This is the viewpoint of Mr. F.-J. Bilitewski, who made the list of $PC(X)$ for $X = J_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$ in his dissertation. In this book we show that his vast list has simple description from the view-point of the graph theory of Dynkin graphs. Moreover, we generalize this result to the remaining 3 quadrilateral singularities $W_{1,0}$, $S_{1,0}$ and $U_{1,0}$. Also, some results on plane sextic curves can be obtained.

The theory developed in this book was derived from the theory of elementary and tie transformations. These transformations are certain operations for Dynkin graphs. Fundamental theories of these transformations have been developed in Urabe [16] and in Urabe [18]. We will not repeat the descriptions in [16] and [18]. However, we believe readers will be able to understand the theory put forth in this book.

Dynkin graphs of type A, B, C, D, E, F or G related to simple Lie groups are used in our theory. Dynkin graphs associated with non-reduced root systems of type BC are used as aids. Bourbaki [3] contains explanations of Dynkin graphs of type A, B, C, D, E, F and G and root systems of type BC.

Let G be a connected Dynkin graph with r vertices. Recall that vertices in G have one-to-one correspondence with members of the corresponding root basis. This root basis is a basis of a certain Euclid space E of dimension r and consists of special vectors called roots. The collection of all roots in E is called the root system R. We can choose the normal inner product on E such that the longest root in R has length $\sqrt{2}$. Then, any root in R has length $\sqrt{2}$, 1, $\sqrt{2/3}$ or $1/\sqrt{2}$.

Now, associated with a finite subset S of a Euclid space consisting of vectors with length $\sqrt{2}$, 1, $\sqrt{2/3}$ or $1/\sqrt{2}$, we can draw a graph Γ by the following rules:

(1) Vertices in Γ have one-to-one correspondence with vectors in S.

(2) Any vertex in Γ has one of 4 different expressions depending on the length of the corresponding vector in S.

length	$\sqrt{2}$	1	$\sqrt{2/3}$	$1/\sqrt{2}$
expression	∘	•	◎	⊗

(3) If two vectors $\alpha, \beta \in S$ are orthogonal, then we do not connect the corresponding two vertices in Γ. α ∗ ∗ β. (∗ denotes one of ∘, •, ◎, and ⊗.)

(4) If α and $\beta \in S$ are not orthogonal, then the corresponding two vertices in Γ are connected by a single edge. α ∗——∗ β.

(5) If α and $\beta \in S$ are proportional, i.e., $\beta = t\alpha$ for some real number t, then the edge connecting the corresponding two vertices is replaced by a bold edge. α ∗**━━**∗ β.

If S is a root basis of a finite root system, then the graph Γ is the corresponding Dynkin graph in our theory. A Dynkin graph is a finite union of connected Dynkin graphs.

If S is an extended root basis, i.e., the union of a finite root basis Δ and (-1) times maximal roots associated with irreducible components of Δ, then the graph Γ is the corresponding extended Dynkin graph.

Note here that our Dynkin graphs are slightly different from the standard Dynkin graphs in Bourbaki [3]. Our graphs carry more information on the length of vectors than the standard ones. Let G be a connected Dynkin graph in our sense. We can produce the corresponding standard Dynkin graph by the following procedure:

(1) If G has a part like ○——● or ●——⊗, replace it by ○⇒○.
(2) If G has a part like ○——◎, then replace it by ○⇛○.
(3) Finally, replace all vertices by a small white circle ○.

Also, applying the same procedure to an extended Dynkin graph in our theory, we get the standard extended Dynkin graph.

A root having a length of $\sqrt{2}$ is called a *long root*, and a root having a length shorter than $\sqrt{2}$ is called a *short root*.

For any connected Dynkin graph with only vertices ○ corresponding to long roots, our graph in our definition coincides with the standard graph in Bourbaki [3].

See Section 5 for the exact explanation of Dynkin graphs.

To state theorems we need two definitions (Urabe [15], [16], [18]).

Definition 0.1. Elementary transformation: The following procedure is called an *elementary transformation* of a Dynkin graph:

(1) Replace each connected component by the corresponding extended Dynkin graph.
(2) Choose in an arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuing from them.

Definition 0.2. Tie transformation: Assume that by applying the following procedure to a Dynkin graph G we have obtained the Dynkin graph $\overline{G}$. Then, we call the following procedure a *tie transformation* of a Dynkin graph:

(1) Attach an integer to each vertex of G by the following rule:
Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the root basis associated with a connected component G' of G. Let $\sum_{i=1}^{k} n_i\alpha_i$ be the associated maximal root. Then, the attached integer to the vertex corresponding to α_i is n_i.
(2) Add one vertex and a few edges to each component of G and make it into the extended Dynkin graph of the corresponding type. Attach the integer 1 to each new vertex.
(3) Choose, in an arbitrary manner, subsets A, B of the set of vertices of the extended graph $\widetilde{G}$ satisfying the following conditions:
 ⟨a⟩ $A \cap B = \emptyset$.
 ⟨b⟩ Choose arbitrarily a component $\widetilde{G}''$ of the extended graph $\widetilde{G}$ and let V be the set of vertices in $\widetilde{G}''$. Let l be the number of elements in $A \cap V$. Let $n_1, n_2, \ldots, n_l$ be the numbers attached to $A \cap V$. Also, let N be the sum of the numbers attached to elements in $B \cap V$. (If $B \cap V = \emptyset$, $N = 0$.) Then, the greatest common divisor of the $l+1$ numbers $N, n_1, n_2, \ldots, n_l$ is 1.
(4) Erase all attached integers.
(5) Remove vertices belonging to A together with the edges issuing from them.

(6) Draw another new vertex called θ corresponding to a long root. Connect θ and each vertex in B by a single edge.

Remark. After following the above procedure (1)–(6) the resulting graph $\overline{G}$ is often *not* a Dynkin graph. We consider only the cases where the resulting graph $\overline{G}$ is a Dynkin graph and then we call the above procedure a *tie transformation*.

The number $\#(B)$ of elements in the set B satisfies $0 \leq \#(B) \leq 3$. $l = \#(A \cap V) \geq 1$.

When the Dynkin graph G contains a_k of connected components of type A_k, b_l of components of type D_l, c_m of components of type E_m, d_n of components of type B_n, ..., we identify the formal sum $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m + \sum d_n B_n \cdots$ with graph G.

The following is the first part of our main results.

Theorem 0.3. *Consider one of $J_{3,0}$, $Z_{1,0}$, and $Q_{2,0}$ as the class X of hypersurface quadrilateral singularities. A Dynkin graph G belongs to $PC(X)$ if, and only if, either (1) or (2) of the following holds:*

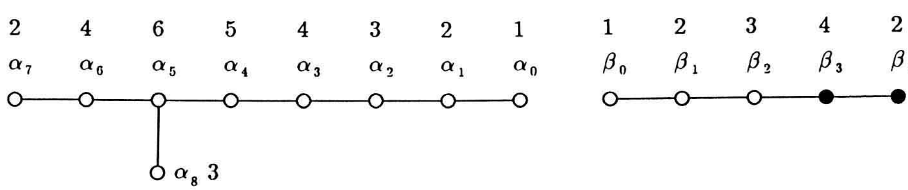
(1) G is one of the following exceptions.

(2) G can be made from one of the following essential basic Dynkin graphs by elementary or tie transformations applied 2 times (Four kinds of combinations — i.e., "elementary" twice, "tie" twice, "elementary" after "tie", and "tie" after "elementary" — are all permitted.) and G contains no vertex corresponding to a short root.

	The essential basic Dynkin graphs:	*The exceptions:*
The case $X = J_{3,0}$:	$E_8 + F_4$	$3A_3 + 2A_2$
The case $X = Z_{1,0}$:	$E_7 + F_4, E_8 + BC_3$	*None*
The case $X = Q_{2,0}$:	$E_6 + F_4, E_8 + F_2$	$3A_3 + A_2$

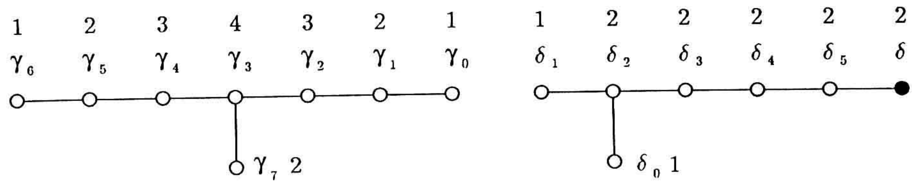
Example. Let us show that $2E_7$ and $A_7 + D_6$ are members of $PC(J_{3,0})$.

Consider the Dynkin graph $E_8 + F_4$ first. This is the essential basic Dynkin graph for $J_{3,0}$. We apply a tie transformation to this graph. At the second step we have the following graph:



Set $A = \{\alpha_1, \beta_4\}$ and $B = \{\alpha_0, \beta_0\}$. We can check to determine if the condition on G.C.D. is satisfied for each component. Under this choice we get the graph $E_7 + B_6$ as the result of the tie transformation.

Now, we can apply a transformation to $E_7 + B_6$ again. At the start we have the following graph:



If an elementary transformation is applied and if we erase the vertices γ_7 and δ_6, we get graph $A_7 + D_6$. If a tie transformation is applied and if we choose $A = \{\gamma_0, \delta_6\}$ and $B = \{\delta_1\}$, we get graph $2E_7$.

By the above theorem one knows $A_7 + D_6$, $2E_7 \in PC(J_{3,0})$.

Remark. In the above theorem type BC_3 and type F_2 Dynkin graphs appear. We explan them briefly here:

Following is an F_2 type graph: •——•. Since it is a subgraph of F_4, we call it an F_2 type Dynkin graph. The extended F_2 type Dynkin graph is nothing more than a triangle with three black circles • at three angles. The 3 coefficients of the maximal root are 1, 1, and 1.

Let $L = \sum_{i=1}^{k} \mathbf{Z}v_i$ be a positive definite unimodular lattice equipped with the bilinear form $(\ ,\ )$ satisfying $v_i^2 = (v_i, v_i) = 1$ for $1 \le i \le k$ and $(v_i, v_j) = 0$ for $i \neq j$. Set $R_m = \{\alpha \in L \mid \alpha^2 = (\alpha, \alpha) = m\}$. The union $R = R_1 \cup R_2$ is a root system of type B_k, and $\Delta = \{v_1\} \cup \{-v_i + v_{i+1} \mid 1 \le i < k\}$ is the root basis of type B_k. The maximal root is $v_{k-1} + v_k$.

Set $R' = R \cup \{2\alpha \mid \alpha \in R_1\}$. This R' is the root system of type BC_k. It is *non-reduced*, i.e., it has an element $\alpha \in R'$ with $2\alpha \in R'$. The normalized bilinear form $(\ ,\ )'$ for R' is given by $(\ ,\ )' = \frac{1}{2}(\ ,\ )$. The root basis for R' is the same Δ as above, but here, v_1 has length $1/\sqrt{2}$ and $-v_i + v_{i+1}$ has length 1. The maximal root is $2v_k$, which has length $\sqrt{2}$.

Following is the type BC_3 Dynkin graph: ⊗——•——•, while the following is the extended type BC_3 Dynkin graph: ⊗——•——•——o. The coefficients of the maximal root are 2, 2, 2, and 1 from the left.

Remark. Note the phenomenon called "exceptional deformations" pointed out in Wall [25]. Let $J_{3,0}(a)$ be the subclass of the class of quadrilateral singularity $J_{3,0}$ with a fixed value a of the parameter a in the above normal form of $J_{3,0}$. By Wall, there exist finite exceptional special values a_0, such that the set $PC(J_{3,0}(a))$ for any general value a is a proper subset of $PC(J_{3,0}(a_0))$. Wall's exception seems to have no relation to our exceptions in Theorem 0.3. Note that by our definition $PC(J_{3,0}) = \bigcup_a PC(J_{3,0}(a))$.

For those who are interested in elliptic surfaces, we would like to explain the relation between the above theorem and elliptic K3 surfaces (Kodaira [8]). Let $\Phi : Z \to C (\cong \mathbf{P}^1)$ be an elliptic K3 surface. It has no multiple fibers. By Kodaira's result we have an elliptic K3 surface $\Phi' : Z' \to C'$ with a section $s : C' \to Z'$, $\Phi s =$ the identity on C'

whose combination of singular fibers is same as that of Φ. Therefore, we can assume from the beginning that Φ itself has a section. Then, we can associate each singular fiber with a connected Dynkin graph of type A, D or E naturally.

$$\begin{array}{llll} I_b \longrightarrow A_{b-1}, & & I_b^* \longrightarrow D_{b+4} \\ II \longrightarrow \emptyset, & & II^* \longrightarrow E_8 \\ III \longrightarrow A_1, & & III^* \longrightarrow E_7 \\ IV \longrightarrow A_2, & & IV^* \longrightarrow E_6 \end{array}$$

Above symbols I_b, ..., IV^* are Kodaira's symbols corresponding to singular fibers of elliptic surfaces.

Let $\hat{G}$ denote the formal sum of all connected Dynkin graphs associated with the singular fibers of Φ. Let PC be the set of all Dynkin graphs $\hat{G}$ obtained from elliptic K3 surfaces $\Phi : Z \to C$. Note that $\hat{G}$ has a component of type D_4 if, and only if, Φ has a fiber of type I_0^*. Now, by Looijenga [9] it is known that $G + D_4$ belongs to PC if, and only if, G belongs to $PC(J_{3,0})$. (See Section 2.) Therefore, one knows by the above theorem that possible combinations of singular fibers in elliptic K3 surfaces with a singular fiber of type I_0^* are subject to the law described above. For one X of the other 5 types of quadrilateral singularities, the set $PC(X)$ describes possible combinations of singular fibers in elliptic K3 surfaces with a I_0^*-fiber satisfying certain additional conditions. (See Section 2.)

The list of all maximal graphs in $PC(J_{3,0})$ with respect to the inclusion relation was first given by F.-J. Bilitewski. Here, I wish to express my sincere thanks to Mr. Bilitewski for showing me his list.

Bilitewski has given the following description for $PC(Z_{1,0})$ and $PC(Q_{2,0})$: First, we consider $PC(Z_{1,0})$. Set

$$\mathcal{M}_1 = \{E_6,\ E_7,\ E_8,\ A_1\} \cup \{D_l \mid l = 4, 5, \ldots\}$$
$$\mathcal{G}_1 = \{(G,\ G_0) \mid G \in PC(J_{3,0}),\ G_0 \in \mathcal{M}_1,\ G_0 \text{ is a component of } G.\}.$$

Consider an element $(G,\ G_0) \in \mathcal{G}_1$. We can write $G = G'' + G_0$. We associate G_0' with G_0 in the following manner, depending on the type of G_0. Then, we set $G' = G'' + G_0'$:

$$\begin{array}{lll} G_0 \longrightarrow G_0' & & \\ \hline E_8 \longrightarrow E_7, & D_4 \longrightarrow 3A_1, & \\ E_7 \longrightarrow D_6, & D_5 \longrightarrow A_3 + A_1, & \\ E_6 \longrightarrow A_5, & D_l \longrightarrow D_{l-2} + A_1 & (l \geq 6), \\ A_1 \longrightarrow \emptyset. & & \end{array}$$

Let $\mathcal{G}_1'$ be the set of all G' obtained from elements $(G,\ G_0) \in \mathcal{G}_1$. Then, $\mathcal{G}_1' = PC(Z_{1,0})$.

For $PC(Q_{2,0})$ his description is like the following: Set

$$\mathcal{M}_2 = \{E_6,\ E_7,\ E_8,\ A_2\}$$
$$\mathcal{G}_2 = \{(G,\ G_0) \mid G \in PC(J_{3,0}), G_0 \in \mathcal{M}_2, G_0 \text{ is a component of } G.\}.$$

For $(G, G_0) \in \mathcal{G}_2$, we can write $G = G'' + G_0$. Associating G_0' with G_0 in the following manner, we set $G' = G'' + G_0'$:

$G_0 \longrightarrow G_0'$	
$E_8 \longrightarrow E_6$,	$E_6 \longrightarrow 2A_2$,
$E_7 \longrightarrow A_5$,	$A_2 \longrightarrow \emptyset$.

Let $\mathcal{G}_2'$ be the set of all G' obtained from elements in $\mathcal{G}_2$. Then, $\mathcal{G}_2' = PC(Q_{2,0})$.

Bilitewski's replacement depends on the theory of singular fibers in elliptic surfaces. It is plain and easy to understand if the set $PC(J_{3,0})$ is known.

To state the theorems for $W_{1,0}$, $S_{1,0}$ and $U_{1,0}$, we must introduce another new concept called "obstruction components". Some of components of type A_k with $k \geq 4$ of a Dynkin graph are distinguished from the others as *obstruction components* and they follow special rules. (See Definition 5.9 (2) and Theorem 5.11.)

Definition 0.4. When a component G_0 of type A_k with $k \geq 4$ of the Dynkin graph G is an *obstruction component*, G_0 follows the below rules:

[The rule under an elementary transformation]

Assume that making the corresponding extended Dynkin graph $\widetilde{G}$ from G, and erasing several vertices and edges issuing from them, we have obtained the Dynkin graph G'.

(1) Let $\widetilde{G}_0$ be the component of $\widetilde{G}$ corresponding to G_0. If the vertex erased from $\widetilde{G}_0$ is unique, then the component G_0' of G' derived from $\widetilde{G}_0$ is of type A_k. We can make this G_0' an obstruction component. Also, we can make G_0' a non-obstruction component, if we so desire.

(2) When two or more vertices are erased from $\widetilde{G}_0$, any component of G' derived from $\widetilde{G}_0$ is *not* an obstruction component.

(3) Obstruction components of G' are only those obtained from the obstruction components of G following the above rules (1) and (2).

[The rule under a tie transformation]

Assume that making the extended Dynkin graph $\widetilde{G}$ from G and choosing subsets A and B of the set of vertices in $\widetilde{G}$ satisfying the condition, we have made the new Dynkin graph G' depending on A and B. Let V_0 be the set of all vertices in the connected component $\widetilde{G}_0$ of the extended Dynkin graph $\widetilde{G}$ corresponding to G_0.

(1) Assume that the sets A and B satisfy the following condition #:

$$\# \quad V_0 \cap B = \emptyset \text{ and } V_0 \cap A \text{ contains only a unique element.}$$

Then, $V_0 - A$ is the set of vertices in a component G_0' of G'. (G_0' is also of type A_k.) This G_0' is *necessarily* an obstruction component of G'.

(2) When the sets A and B do not satisfy the condition #, any component of G' containing a vertex belonging to $V_0 - A$ is *not* an obstruction component.

(3) Obstruction components of G' are only those obtained from obstruction components of G by following rules (1) and (2) above.

We can state the theorem for $W_{1,0}$.

Theorem 0.5. *A Dynkin graph G belongs to $PC(W_{1,0})$ if, and only if, G can be made from one of the following basic Dynkin graphs with distinguished obstruction components by elementary or tie transformations applied 2 times (Four kinds of combinations — i.e., "elementary" twice, "tie" twice, "elementary" after "tie", and "tie" after "elementary" — are all permitted.) and G contains no vertex corresponding to a short root and no obstruction component:*

The basic Dynkin graphs:
A_{11}, $B_9 + G_2$, $E_8 + G_2 + B_1$, $E_7 + B_3 + G_1$.
(A_{11} is the only obstruction component.)

In the above theorem the graph • with only one black vertex is called a type B_1 Dynkin graph, while ◎ with one double vertex is called a type G_1. Their extended Dynkin graphs are •——• and ◎——◎ respectively. The coefficients of the maximal root are 1 and 1 for both graphs.

Our main result for $S_{1,0}$ is as follows.

Theorem 0.6. *A Dynkin graph G belongs to $PC(S_{1,0})$ if, and only if, G contains no vertex corresponding to a short root and no obstruction component, and either following (1) or (2) holds:*

(1) G can be made from one of the following sub-basic Dynkin graphs by one elementary transformation or one tie transformation:

The sub-basic Dynkin graphs:
$B_{10} + A_1$, $B_9 + A_2$, $E_7 + B_4$,
$A_6 + B_5$, $B_6 + A_3 + A_2$.
(No obstruction component.)

(2) G can be made from one of the following basic Dynkin graphs with distinguished obstruction components by elementary or tie transformations applied 2 times (Four kinds of combinations — i.e., "elementary" twice, "tie" twice, "elementary" after "tie", and "tie" after "elementary" — are all permitted.):

The basic Dynkin graphs:
$A_9 + BC_1$, $B_8 + A_1$, $E_8 + BC_1$, $E_7 + BC_2$, $E_6 + B_3$,
(A_9 is the only obstruction component.)

The last remaining case is $U_{1,0}$. We must introduce an additional new concept.

Recall that an extended Dynkin graph is associated with the extended root basis, i.e., the root basis plus (-1) times maximal roots associated with an irreducible component of the basis. Note here that if an irreducible root system contains roots with 2 kinds of different length, the maximal short root ζ is uniquely defined among shorter roots depending on the root basis Δ. When the irreducible root system is the BC_k type with $k \geq 2$, the system contains roots with 3 different lengths and by ζ we denote the maximal root with length 1. ζ is defined among roots with the middle length 1. We call the union $\Delta^\bullet = \Delta \cup \{-\zeta\}$ the *dual extended root basis*, and call the associated graph the *dual extended Dynkin graph*.

For example, the dual extended Dynkin graph of type G_2 is similar to the following: ◎——◎——o.

For the irreducible root system whose roots are of the same length we define that *the dual extended Dynkin graph is equal to the extended Dynkin graph.*

Definition 0.7. Dual elementary transformation: The following procedure is called a *dual elementary transformation* of a Dynkin graph:

(1) Replace each connected component by the corresponding *dual* extended Dynkin graph.

(2) Choose, in an arbitrary, at least one vertex from each component (of the dual extended Dynkin graph) and then remove these vertices together with the edges issuing from them.

Definition 0.4. (Addition) Under a dual elementary transformation an obstruction component follows the same rule as for an elementary transformation.

Theorem 0.8. *A Dynkin graph G belongs to $PC(U_{1,0})$ if, and only if, G contains no vertex corresponding to a short root and no obstruction component, and either (1), (2) or (3) of the following holds:*

(1) (Exceptions) $G = E_6 + D_4 + A_2$, $E_6 + A_2 + 3A_1$, $D_4 + 3A_2 + A_1$ or $3A_2 + 4A_1$.

(2) G can be made from one of the following basic Dynkin graphs with distinguished obstruction components by elementary or tie transformations applied 2 times (Four kinds of combinations — i.e., "elementary" twice, "tie" twice, "elementary" after "tie", and "tie" after "elementary" — are all permitted):

The basic Dynkin graphs:
$E_8 + A_2(3)$, $E_7 + G_2$, $E_6 + A_2 + A_2(3)$, $A_8 + G_2$.
(A_8 is the only obstruction component.)

(3) G can be made from the following dual basic Dynkin graph by two transformations one of which is a dual elementary transformation and the other is an elementary or a tie transformation (Thus, only four kinds of combinations of transformations — i.e., "dual elementary" after "elementary," "elementary" after "dual elementary," "dual elementary" after "tie," and "tie" after "dual elementary" — are permitted.):

The dual basic Dynkin graphs:
$E_8 + G_2$. *(No obstruction component.)*

Remark. The $A_2(3)$ Dynkin graph in the above theorem is explained herewith.

We can consider a root system $R = \{\alpha, \beta, \alpha+\beta, -\alpha, -\beta, -\alpha-\beta\}$ with 6 roots satisfying the conditions $\alpha^2 = \beta^2 = 2/3$ and $(\alpha, \beta) = -1/3$. In this book, R is denoted to be of type $A_2(3)$ (See Section 5.) since it becomes a root system of type A_2 if the bilinear form $3(\, , \,)$ is given.

$\Delta = \{\alpha, \beta\}$ is its root basis, and the Dynkin graph is as follows: ◎——◎. The extended type $A_2(3)$ Dynkin graph is a triangle with three double circles ◎ at three angles. The 3 coefficients of the maximal root are all 1.

As for above $PC(W_{1,0})$, $PC(S_{1,0})$ and $PC(U_{1,0})$, Mr. Bilitewski informed me that he had a complete listing of them.

Now, apart from the above theorems, we can consider the set of all general elliptic K3 surfaces. It is not difficult to formulate the corresponding theorem about combinations of singular fibers in them. The basic Dynkin graphs in this case are $2E_8$ and D_{16}. No sub-basic Dynkin graphs appear. Perhaps there are several exceptions such as for $J_{3,0}$ and $U_{1,0}$. Yet, not having given a great deal of consideration to this case we are not certain.

I would like to give a theorem dealing with all elliptic K3 surfaces in a forthcoming article.

Also, there exist similar theorems for 14 exceptional hypersurface singularities with modules number 1 (Arnold [1], [2]). We would like to study them in a forthcoming article.

Mr. Bilitewski informed that he had a complete listing of Dynkin graphs of $PC(X)$ for any one X of 14 exceptional hypersurface singularities.

Our theory seems to have two advantages over Nikulin's arithmetic theory. First, it gives a consistent view-point. Second, we can avoid long tiresome calculations in determining all overlattices of lattices appearing in complicated cases in our problem.

The main ideas of this book are outlined here. (See also Section 1, Summary.)

The starting point is Looijenga's result, which follows from the surjectivity of the period mapping for K3 surfaces. The lattice P of rank $22-\mu$ is defined for each one X of 6 quadrilateral singularities. By his result our problem is reduced to show the existence of an embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ of lattices satisfying certain conditions (L1) and (L2), where G is a Dynkin graph. $Q(G)$ is the root lattice of type G, and Λ_N stands for the even unimodular lattice with signature $(16+N, N)$. If $N \geq 1$, then Λ_N is unique up to isomorphism, and is isomorphic to $Q(E_8) \oplus Q(E_8) \oplus H^{\oplus N}$. Here, $Q(E_8)$ denotes the root lattice of type E_8 and H denotes the hyperbolic plane, i.e., the even unimodular lattice of rank 2 with signature $(1,1)$.

Next, we translate Looijenga's conditions (L1) and (L2) to a simpler equivalent condition. They are satisfied if, and only if, the induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is *full* and satisfies a certain condition related to obstruction components. Here, we say that the embedding is full, if the root system in $Q(G)$ including short roots and that in the primitive hull of $Q(G)$ coincide. Here is the essential reason we have to introduce the concept of short roots. The condition "no short roots and no obstruction components" implies conditions (L1) and (L2) are satisfied. Short roots play an interesting but mysterious role in our theory.

We can apply the theory of elementary transformations and tie transformations here. Let G' be a Dynkin graph made from a given Dynkin graph G by one elementary transformation or one tie transformation. Assume that a full embedding $Q(G) \hookrightarrow \Lambda_N/P$ is given. Then, we can define another full embedding $Q(G') \hookrightarrow \Lambda_{N+1}/P$. Note that the suffix of Λ has increased by one. A direct summand H is added under the process of one transformation.

Conversely, assume that we have a primitive isotropic vector u in Λ_{N+1}/P in a *nice position* with respect to any given full embedding $Q(G') \hookrightarrow \Lambda_{N+1}/P$. Here, we say that u is in a nice position either if u is orthogonal to $Q(G')$ or for some root basis $\Delta \subset Q(G')$ and for some long root $\theta \in \Delta$ $u \cdot \alpha = 0$ for any $\alpha \in \Delta$ with $\alpha \neq \theta$ and $u \cdot \theta = 1$. (In the case of $U_{1,0}$ we have to assume moreover that if u is not orthogonal to $Q(G')$, then $u \cdot x \in \mathbf{Z}$ for every $x \in \Lambda_N/P$.) Under this assumption we have a Dynkin