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E. Benoît (Ed.)

Dynamic Bifurcations

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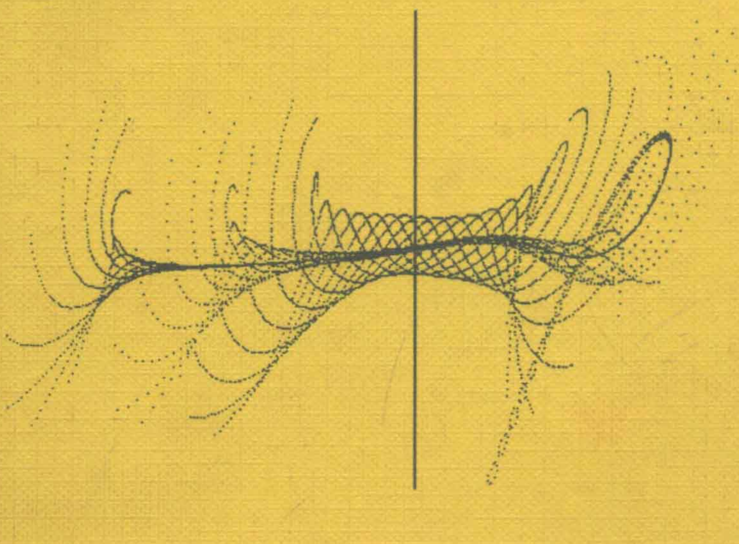

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Preface

Dynamic Bifurcations Theory is concerned by the phenomena that occur in one parameter families of dynamical systems (usually ordinary differential equations), when the parameter is a slowly varying function of time. It turns out that during the last decade these phenomena were observed and studied by various mathematicians, both pure and applied, from eastern and western countries, using classical and nonstandard analysis.

It is the purpose of this book to give an account of these developments. The first paper of C. Lobry is an introduction : the reader will find an explanation of the problems and some easy examples, he will understand also the right place of every paper in the book.

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Dynamic Bifurcations

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The organisation of this introductory paper is the following. First I shall explain what is the "Delay"phenomenon in a Poincaré, Andronov, Hopf Bifurcation (P.A.H. Bifurcation). After that I shall develop in details a linear example which contains all the main features of the theory giving references to papers in the present volume. I will conclude by few historical comments.

The present paper uses the terminology of Non Standard Analysis (N.S.A.) but due to its expository character I tried to make it understandable by just taking the words in their intuitive meaning.

1 The delay effect

Consider the differential system :

$$\begin{cases} x' &= -y + \mu x - x(x^2 + y^2) \\ y' &= x + \mu y - y(x^2 + y^2) \end{cases}$$

which exhibits a P.A.H. Bifurcation for $\mu = 0$. In most classical textbook on the subject one can find a description of the phenomenon. Let us take two from well known authors : The first one (fig 1) is taken from the book of Arnold [5] on differential equations, the second one is taken from the book of Marsden and McKracken [103]

For the reader who is not familiar with the precise statement of the P.A.H. theorem these unformal descriptions are somewhat misleading. Such a reader will probably understand that the parameter μ is a function of the time. Now it is interesting to actually consider a P.A.H. Bifurcation with the parameter slowly growing with time. Let us integrate the system :

$$\begin{cases} x' &= -y + \mu x - x(x^2 + y^2) \\ y' &= x + \mu y - y(x^2 + y^2) \\ \mu' &= \varepsilon \end{cases}$$

with ε small. We obtain the picture shown in fig 3.

Traitons d'abord le cas $c < 0$. Lorsque ε passe par 0, le foyer de l'origine des coordonnées perd sa stabilité. Pour $\varepsilon = 0$, l'origine des coordonnées est également un foyer stable mais non structurellement stable : les trajectoires ne se rapprochent pas exponentiellement de 0 (fig. 127).

Pour $\varepsilon > 0$ les trajectoires s'éloignent du foyer à une distance proportionnelle à $\sqrt{\varepsilon}$ et s'enroulent autour du cycle limite stable. Donc, lorsque ε passe par 0, $c < 0$, la perte de stabilité s'accompagne de la naissance d'un cycle limite stable dont le rayon croît comme $\sqrt{\varepsilon}$.

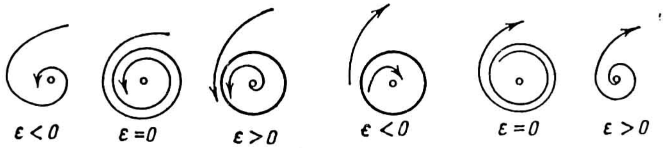

Fig. 127 Fig. 128

En d'autres termes, l'état stationnaire perd sa stabilité et il apparaît un régime périodique stable dont l'amplitude est proportionnelle à la racine carrée de l'écart du paramètre par rapport à la valeur critique. Les physiciens parlent dans ce cas d'une *excitation douce d'auto-oscillations.*

Figure 1: From V.I. Arnold [6]

The appearance of the stable closed orbits (= periodic solutions) is interpreted as a "shift of stability" from the original stationary solution to the periodic one, i.e., a point near the original fixed point now is attracted to and becomes indistinguishable from the closed orbit. (See Figures 1.4 and 1.5).

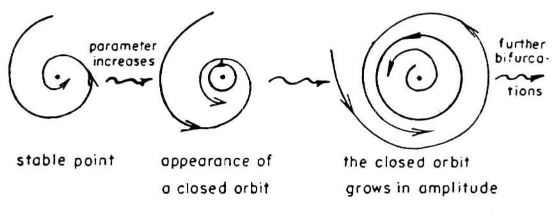

Figure 2: From J.E. Marsden M.F. Mc Cracken [103]

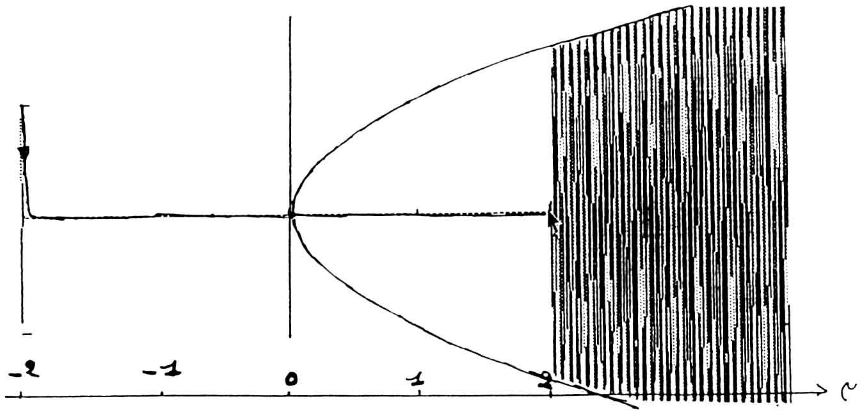

Figure 3: $\varepsilon = 0.05$

This has nothing to do with the canonical description ! The mathematical explanation is rather simple. Consider the equations in polar coordinates. We have :

$$\begin{cases} \theta' & = & 1 \\ \rho' & = & \rho(\mu - \rho^2) \\ \mu' & = & \varepsilon \end{cases}$$

Forget the angular coordinate. We see that $\rho = 0$ is a trivial solution. Every solution which starts from (ρ_0, μ_0) with $\mu_0 < 0$ first tends in finite time to a point which is very close to $(0, \mu_0)$ because 0 is a stable equilibrium point for the equation :

$$\rho' = \rho(\mu_0 - \rho^2)$$

and the motion is infinitesimally slow with respect to μ. It is clear that ρ will remain close to 0 until $\mu(t)$ will be positive. Now we perform the following change of coordinates :

$$y = \ln(\rho)$$

as soon as ρ takes the value 1. We have :

$$\begin{aligned} y(t)' &= \mu(t) - \rho^2(t) \\ y(0) &= 0 \end{aligned}$$

As long as $\mu_0 + \varepsilon t$ is negative the term $\rho^2(t)$ is negligible and one has :

$$y(t) \simeq \mu_0 t + \varepsilon \frac{t^2}{2}$$

and thus, for $t = -\frac{\mu_0}{\varepsilon}$ we get :

$$y\left(\frac{-\mu_0}{\varepsilon}\right) \simeq \frac{-\mu_0^2}{2\varepsilon}$$

Hence we see that ρ is then of the order of $\exp(-\frac{\lambda}{\varepsilon})$. From that we can deduce that $\rho(t)^2$ remains negligible in the equation until :

$$t = -\frac{\mu_0}{\varepsilon} \quad \text{and} \quad \mu = -\mu_0$$

This proves that from the value $\mu = 0$ to $\mu = -\mu_0$ the solution remains infinitesimal and then departs very suddently to an oscillation of large amplitude. This is certainly surprising if compared to the classical description, but, in view of this very particular example (the system factorizes through polar coordinates, it has 0 as an explicit solution etc...) one is allowed to suspect an exceptional phenomenon.

Thus we address the following problem : Consider a system of the form :

$$x' = f(x,\mu)$$

with an equilibrium $\varphi(\mu)$ which is stable for μ negative and unstable for μ positive. The static bifurcation theory is concerned by the evolution of the phase portrait of this system with respect to the parameter μ, or, in other words, by the phase portrait of the system :

$$\begin{cases} x' = f(x,\mu) \\ \mu' = 0 \end{cases}$$

By contrast we consider :

$$\begin{cases} x' = f(x,\mu) \\ \mu' = \varepsilon \end{cases}$$

and we ask the following questions which are the basic questions of Dynamic Bifurcation theory :

1.1 Existence of a delay in the bifurcation

- Is there a solution which is close to $\varphi(\mu)$ up to certain positive value of μ ?
- Is this true for every initial condition (x_0, μ_0) ?

1.2 Predictability

Is it possible to compute the delay from the differential equation ?

1.3 Structural stability

If (1.1) holds for some system, what about neighbourhing systems ?

1.4 Robustness

What about the delay in the presence of deterministic or stochastic noise ?

The theory of Dynamic Bifurcations tries to give an answer to all these questions.

2 The linear case

2.1 Motivation for the study of certain linear equation

We start with a system depending on a parameter :

$$\begin{cases} x' &= f(x,\mu,a) \\ \mu' &= \varepsilon \end{cases}$$

with $f(\varphi(\mu),\mu,0) = 0$, and $\varphi(\mu)$ stable for μ negative, unstable for μ positive. If we make the change of coordinates :

$$X = x - \varphi(\mu)$$

we obtain :

$$X' = f(X+\varphi(\mu),\mu,a) - \varepsilon\varphi'(\mu)$$

$$\begin{aligned} X' &= Xf'_x(X+\varphi(\mu),\mu,a) - \varepsilon\varphi'(\mu) + af'_a(X+\varphi(\mu),\mu,a) + \text{higher order terms} \\ \mu' &= \varepsilon \end{aligned}$$

In one dimension the hypothesis on stability says that $f'_x(X+\varphi(\mu),\mu,a)$ has the same sign as μ. All this suggests to study the following linear equation :

$$\begin{cases} X' &= \mu X + \varepsilon\psi(\mu) + a \\ \mu' &= \varepsilon \end{cases}$$

where $\psi(\mu)$ is refered as the "perturbation"and a as the "control". After division by ε (change of time) one gets :

$$\begin{cases} X' &= \dfrac{\mu X + \varepsilon\psi(\mu) + a}{\varepsilon} \\ \mu' &= 1 \end{cases}$$

or simply, because μ is just t, up to a constant :

$$X(t)' = \frac{t}{\varepsilon}X(t) + \psi(t) + \frac{a}{\varepsilon}$$

This is the equation we discuss in details now. Notice that the homogeneous equation is definitely trivial and has the solutions :

$$X(t) = X(t_0)\exp\left(\frac{t^2}{2\varepsilon} - \frac{t_0^2}{2\varepsilon}\right)$$

Thus the interest in this equation is due to the presence of the pertubating term $\psi(t)+\frac{a}{\varepsilon}$ which is not avoidable because it is related to the slow motion of the equilibrium point.

2.2 Integration

The equation :

$$X'(t) = \frac{t}{\varepsilon}X(t) + \psi(t) + \frac{a}{\varepsilon}$$

has the following two explicit solutions :

$$X_1(t) = \int_{-\infty}^{t} \exp\left(\frac{t^2}{2\varepsilon} - \frac{s^2}{2\varepsilon}\right)(\psi(s) + \frac{a}{\varepsilon})ds$$

$$X_2(t) = -\int_{t}^{+\infty} \exp\left(\frac{t^2}{2\varepsilon} - \frac{s^2}{2\varepsilon}\right)(\psi(s) + \frac{a}{\varepsilon})ds$$

and one sees easily that the first one is an infinitesimal of the order of ε for t negative and not equivalent to 0, and of the order of $\sqrt{\varepsilon}$ for $t = 0$. A priori we dont know the behaviour of X_1 for positive t. We have a symetric conclusion for X_2. Now let us say that :

$$X_1(t) = X_2(t) + (X_1(t) - X_2(t))$$

That is to say :

$$X_1(t) = X_2(t) + \exp\left(\frac{t^2}{2\varepsilon}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right)(\psi(s) + \frac{a}{\varepsilon})ds$$

We know that $X_1(t)$ is infinitesimal for negative values of t. From the above formula we obtains that it is also an infinitesimal for positive values of t if the integral is exponentially small.

We shall take advantage of its particular form :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right)(\psi(s) + \frac{a}{\varepsilon})ds$$

in order to compute an estimate for the control parameter a.

2.3 Adjustment of the parameter and the corresponding phase portrait

For the value :

$$a_0 = -\frac{\sqrt{\varepsilon}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right)\psi(s)ds$$

of the parameter a, the term :

$$\exp\left(\frac{t^2}{2\varepsilon}\right)\int_{-\infty}^{+\infty} \exp\left(\frac{s^2}{2\varepsilon}\right)\left(\psi(s) + \frac{a}{\varepsilon}\right)ds$$

is just 0 and we have $X_1(t) = X_2(t) \simeq 0$. In this case the control parameter just compensate the perturbation. Let us call a_0 the "value for maximal Canard" and "maximal Canard solution" the corresponding $X_0(t)$. (This terminology has an historical explanation [19] and seems to be recognized now. We adopt it but we emphasize that it has no particular significance in our problem). In the general non linear case there is no

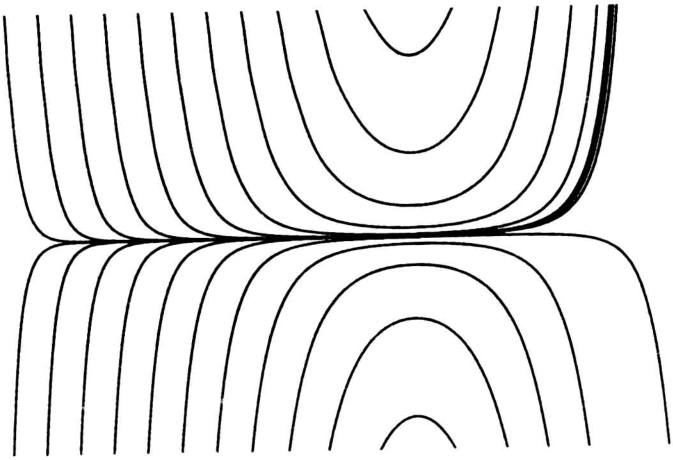

Figure 4: Phase portrait for a exponentially close to a_0

closed formula giving the value of such an a_0. Nevertheless a_0 exists; this is proved by G. Wallet (See in this volume : *Overstability in Arbitrary Dimension*).

Suppose now that a is not just equal to a_0 but :

$$a = a_0 + \alpha \exp\left(-\frac{T^2}{\varepsilon}\right)$$

where α is of the order of unity, then, $X_1(t)$ is infinitely close to 0 for negative t as we already knew, and also infinitely close to 0 for times t significantly smaller than T. Let us call such a value a "Canard" value and the corresponding solution a "Canard" solution. Now consider another solution $X(t)$ of the linear system; its difference with X_1, let us say Y, is solution of the homogeneous system :

$$Y'(t) = \frac{t}{\varepsilon} Y(t)$$

whose solutions are symmetric with respect to the vertical axis. A solution which starts from some limited point (t_0, Y_0) moves very fast to the t axis, follows it until it is close to the symmetric point and then goes very fast to infinity. The consequences for X are the following : very quickly $X(t)$ enters the infinitesimal neighbourhood of $X_1(t)$, follows $X_1(t)$ up to time $-t_0$. If $-t_0$ is smaller than T then it leaves very fast to infinity, otherwise, if $-t_0$ is significantly greater than T, $X(t)$ has to remain close to $X_1(t)$ which means that it has to leave to infinity. In other words, for every initial condition, except those for which t_0 is infinitely close to $-T$ the solution has to leave to infinity before T (see Fig 4). We see that there is an upper bound for the length of a "Canard" solution. This upper bound depends on the choice of the parameter a.

The fact that the existence of "Canard" solutions is valid for an open set of values of a can be interpreted as a kind of "structural stability", but the fact that the size of this set is of the order of $\exp(-\frac{k}{\varepsilon})$ tels us that this stability is not very robust. The surprising fact is that delay occurs in physical experiments, for instance in laser experiments (see T. Erneux et al. *Slow Passage Through Bifurcation and Limit Points. Asymptotic Theory and Applications* in this volume).

The fact that a solution which enters the infinitesimal neighbourhood of X_0 after $-T$ leaves it symetrically about 0 is due to the particular coefficient of X in the equation. Consider the more general (complex) case :

$$X'(t) = \frac{1}{\varepsilon}(p(t) + iq(t))X(t) + \psi(t) + \frac{a}{\varepsilon}$$

with $\text{sign}(p(t)) = \text{sign}(t)$. In this case the same method works, we just replace $\frac{t^2}{2}$ by a primitive of $p(t) + iq(t)$ and we see that a solution which enters in the infinitesimal neigbourhood of X_1 at a time t_0 will leave it at a time t^* defined by :

$$\int_{t_0}^{t^*} p(s)ds = 0$$

This relation is called the "entrance exit" relation and the function which defines t^* from t the "entrance exit" function [15]. Its intuitive significance is that the stability created during the time the system is stable is measured as the integral of the "real part of the eigenvalue" and this amount of stability as to be destroyed up to the same amount.

2.4 Estimation of the control parameter from the perturbation

The maximal Canard value is given by :

$$a_0 = -\frac{\sqrt{\varepsilon}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right) \psi(s)ds$$

When the function ψ is a standard analytic function, an asymptotic expansion of a_0 is easily computed using the "moment formula" :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{2}\right) u^{2n}du = \sqrt{2\pi}\,\frac{(2n)!}{2^n n!}$$

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{2}\right) u^{2n+1}du = 0$$

from which we see that the expansion is generally not convergent but Gevrey (growing like a factorial). The question of the computation of an asymptotic expansion for a "canard" value is considered by M. Canalis (see *Formal Expansion of van der Pol Equation Canard Solutions are Gevrey* in this volume).

2.5 The case of a fast oscillating perturbation

There is an other case which is of interest. The case where the function $\psi(s)$ is of the form :

$$\psi(s) = \exp\left(\frac{i\beta s}{\varepsilon}\right) \xi(s)$$

where β is a parameter of the order of the unity. Due to the presence of ε which is responsible for fast oscillations in the expression, the above estimations are no longer valid. Let us study this case.

We are looking for the "Canard" solutions of :

$$X'(t) = \frac{1}{\varepsilon} t X(t) + \exp\left(\frac{i\beta t}{\varepsilon}\right) \xi(t) + \frac{a}{\varepsilon}$$

We have seen that, for a non oscillating perturbation, the "Canard" solutions are obtained for values of a which are exponentially close to a_0 ; it is interesting to notice that in the case of a fast oscillating perturbation like above this set of values is much larger and contains the value 0. This means that the equation without parameter :

$$X'(t) = \frac{1}{\varepsilon} t X(t) + \exp\left(\frac{i\beta t}{\varepsilon}\right) \xi(t)$$

does contain "Canard" solutions. Let us show it. We assume for simplicity, but it is not essential, that $\xi(t) = 1$. In this case the integral :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right) (\psi(s) + \frac{a}{\varepsilon}) ds$$

is just :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right) \exp\left(\frac{i\beta s}{\varepsilon}\right) ds$$

which is easily computed as :

$$\sqrt{\varepsilon 2\pi} \exp\left(-\frac{\beta^2}{2\varepsilon}\right)$$

and we know by the same argument than in 2-3 that the X_1 solution is infinitely close to the t axe from $-\infty$ up to $t \simeq \beta$ where it leaves it for infinity.

Thus we observe the presence of an upper bound for the lengh of "canards" which is related to the imaginary part β (see fig. 5). Whether this property is true in general is not completely known. Some particular exemple is studied in the paper of F. and M. Diener *Maximal Delay* in this volume.

2.6 Return to the Hopf bifurcation

The case of a fast oscillating perturbation is interesting because if we look for solutions of the form :

$$X(t) = \exp\left(\frac{i\beta t}{\varepsilon}\right) Z(t)$$

it turns out that Z(t) satisfies :

$$Z'(t) = \frac{1}{\varepsilon}(t - i\beta) Z(t) + 1$$

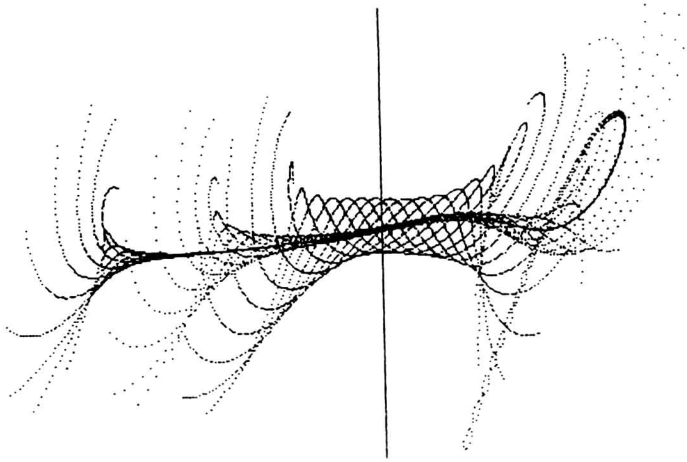

Figure 5: Upper bound for "Canard" solutions

which is just the linear part of the Hopf bifurcation situation. This means that if we consider the Dynamic Hopf Bifurcation :

$$\begin{cases} x' &= f(x,\mu) \\ \mu' &= \varepsilon \end{cases}$$

whithout a control parameter, we can expect a delay in the bifurcation. This is actually the case from a recent result of A.I. Neishtadt [109].

2.7 Robustness

What we mean by robustness is the persistence of the delay phenomenon under external disturbances either deterministic or stochastic. Despite the fact that the presence of a stochatic noise in the case of the one dimentional linear system is not too difficult to discuss, I will not do it in this introduction because it involves concepts like white noise and diffusion processes which need some space to be precised. See the papers by C. Baesens : *Noise Effects of Dynamic Bifurcations the Case of a Period-doubling Cascade* and E. Benoît : *Linear Dynamic Bifurcation with Noise* in this volume.

Let me just say a few words on the deterministic case. Consider the linear system of section 2 :

$$\begin{cases} X'(t) &= \mu X(t) + (a_0 + \varepsilon\psi(\mu)) + k\xi(t) \\ \mu'(t) &= \varepsilon \end{cases}$$

perturbed by the deterministic noise $k\xi(t)$ where we assume that k is an infinitesimal in order to preserve the fact that the equilibrium is close to 0. After the change of time the system turns out to be :

$$X'(t) = \frac{t}{\varepsilon}X(t) + \left(\frac{a_0}{\varepsilon} + \psi(t)\right) + \frac{k}{\varepsilon}\xi\left(\frac{t}{\varepsilon}\right)$$

and performing the same computations as in section 2 we know that the delay is up to time L if we have :

$$\frac{k}{\varepsilon}\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right)\xi\left(\frac{s}{\varepsilon}\right)ds = \exp\left(-\frac{L^2}{\varepsilon}\right)$$

for some limited number L. Due to the presence of ε in the argument of ξ one has to be careful in the conclusions. For instance if $\xi(t)$ is a periodic perturbation of the form :

$$\xi(t) = \exp(i\beta(t))$$

the quantity :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right) \exp\left(\frac{i\beta s}{\varepsilon}\right) ds$$

is equal to :

$$\sqrt{\varepsilon 2\pi} \exp\left(-\frac{\beta^2}{2\varepsilon}\right)$$

and k can be fairly large. Conversely, if $\xi(t)$ is a slow perturbation of the form :

$$\xi(t) = \eta(\varepsilon t)$$

then the amplitude of the perturbation has to be very (exponentially) small.

Consider now the case of the perturbed linearized Hopf bifurcation :

$$\begin{cases} X'(t) &= (\mu + i\beta)X(t) + (a_0 + \varepsilon\psi(\mu)) + k\xi(t) \\ \mu'(t) &= \varepsilon \end{cases}$$

Now the integral under consideration is :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon}\right) \exp\left(\frac{-i\beta s}{\varepsilon}\right) \xi\left(\frac{s}{\varepsilon}\right) ds$$

and we see that its magnitude depends very much whether $\xi(\frac{s}{\varepsilon})$ is "resonant" or not, that is to say if $\xi(t)$ is close to $\exp(i\beta t)$ or not. If the perturbation is not resonant then fairly large amplitudes are acceptable, if it is resonant then only exponentially small amplitudes are acceptable.

3 Discrete Systems

The theory of bifurcations for discrete dynamical systems has also its dynamical version described by the process :

$$\begin{cases} x_{n+1} &= f(x_n, \mu_n) \\ \mu_{n+1} &= \mu_n + \varepsilon \end{cases}$$

The delay effect is also present in this case. This theory is important for itself and because discrete systems modelize many natural situations. But there is an other reason for which those systems are important : If one wants to understand more deeply the effect of noise discrete systems are certainly more tractable. For the deterministic case this volume contains the paper by A. Fruchard : *Existence of Bifurcation Delay : The discrete case* and for the non deterministic one the already mentionned paper by C. Baesens.

4 Historical comments

These historical comments are by no mean an historical study of the subject but just a guide for the reader interested.

The scientific significance of the questions adressed in section 1 was clearly emphasized at the very beginning of eighties in a sery of papers by T. Erneux, P. Mandel [100,101,9] (See also in this volume). They showed by means of various problems from physics and chemistry that this question is not academic at all. Striking pictures of delays observed in nature can be found from these authors. From the mathematical viewpoint they compared static versus dynamic phase portrait mainly for the nonlinear one dimensional case that they solved almost completely with the methods of classical asymptotic analysis.

Actually the mathematics of the one dimensional dynamic bifurcation problem where already known at that time, but not related to the question of dynamic bifurcations. In a series of papers by E. Benoît, J.L. Callot, F. Diener and M. Diener [19] dealing with equations of the form :

$$\begin{cases} x'(t) &= \frac{1}{\varepsilon} f(x,y) \\ y'(t) &= g(x,y) \end{cases}$$

"Canard" solutions were defined, using N.S.A. techniques, as solutions which are first in an infinitesimal neighbourhood of the attracting part of the slow manifold (The slow manifold is the set of points for which $f(x, y)$ is equal to 0) , cross a critical point, and remain for a while in the infinitesimal neighbourhoud of the repelling part.

Essentially, all the properties shown in section 2-3 for the linear case where shown to be true for the nonlinear one-dimensional case in the above mentioned papers. By contrast, the computation of the asymptotic expansion of the control parameter and the proof of its divergent (Gevrey) nature was difficult. It has been recently proved by M. Canalis [34,35].

Four papers in this volume (See S.N. A. Delcroix, S.N. Samborski,F. Blais, I.P. van den Berg) are indirectly connected to "dynamic bifurcations" because they study "canard" problems.

In 1985 G. Wallet and the author discovered the pertinence of the results on "canard solutions" interpreted in terms of dynamic bifurcations [96]. They looked for the the existence of "canards solutions" in the general Hopf bifurcation case but did not succeded. The first proof of the existence of a delay for the general Hopf bifurcation without restriction on the dimension, was done by A.I. Neishtadt [109]. This was a decisive step. V.I. Arnold stressed recently the importance of this result and recognized by the way the importance of the problem of dynamic bifurcations in [4] (but apparently was not aware that the subject has been already studied by various authors). The connection between the Hopf case for which the existence of a delay is not governed by the accurate choice of an extra control parameter and the singular case (when an eigenvalue vanishes) which needs control parameter is quite clear in the linear one dimensional case through the introduction of a fast oscillating perturbation. For the general non linear case, in arbitrary dimension, the connection was made by G. Wallet [131].

Very recently (January 1991), J.L. Callot [33] found a very short and elegant proof for a global version of A.I. Neishtadt result (by global I mean that the proof includes the determination of the length of the upper bounds for "canard" solutions).