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Saint-Venant's Problem

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INTRODUCTION

A major concern throughout the history of elasticity has been with problems dictated by the demands of engineering. Interest in the construction of a theory for the deformation of elastic cylinders dates back to Coulomb, Navier and Cauchy. However, only Saint-Venant has been able to give a solution of the problem.

The importance of Saint-Venant's celebrated memoirs [132,133], on what has long since become known as Saint-Venant's problem requires no emphasis. To review the vast literature to which the work contained in [132,133] has given impetus is not our intention. An account of the historical developments as well as references to various contributions, may be found in the books and in some of the papers cited.

We recall that Saint-Venant's problem consists in determining the equilibrium of a homogeneous and isotropic linearly elastic cylinder, loaded by surface forces distributed over its plane ends. Saint-Venant proposed an approximation to the solution of the three-dimensional problem, which only requires the solution of two-dimensional problems in the cross section of the cylinder. Saint-Venant's formulation leads to the four basic problems of extension, bending, torsion and flexure. His analysis is founded on physical intuition and elementary beam theory. Saint-Venant's extension, bending, torsion, and flexure solutions are well-known (see, for example, Love [98], Chapters 14,15 and Sokolnikoff [139], Chapter 4).

Saint-Venant's approach of the problem is based on a relaxed statement in which the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. Justification of the procedure is twofold. First, it is difficult in practice to determine the actual distribution of applied stresses on the ends, although the resultant force and moment can be measured accurately. Second, one invokes Saint-Venant's principle. This principle states, roughly speaking, that if two sets of loadings are statically equivalent at each end, then the difference in stress fields and strain fields are negligible, except possibly near the ends. The precise meaning of Saint-Venant's hypothesis and its justification have been the subject of many studies, almost from the time of the original Saint-Venant's papers. Reference to some of the early investigations of the question will be found in [98],[139],[140]. In recent years important steps toward clarifying Saint-Venant's principle have been made. The classic paper in linear elasticity is by Toupin [146] (see also, e.g. Roseman [128], Knowles [84] and Fichera [39,40] for further important developments). For the history of the problem and the detailed analysis of various results on Saint-Venant's principle we refer to the works of M.E.Gurtin [47], G.Fichera [38], C.O.Horgan and J.K.Knowles [53].

The relaxed Saint-Venant's problem continues to attract attention both from the mathematical and the technical point of view.

It is obvious that the relaxed statement of the problem fails to characterize the solution uniquely. This fact led various authors to establish characterizations of Saint-Venant's solution. Thus, Clebsch [24] proved that Saint-Venant's solution can be derived from the assumption that the stress vector on any plane normal to the cross-sections of the cylinder is parallel to its generators. In [155], Voigt rediscovered Saint-Venant's solution by using another assumption regarding the structure of the stress field. Thus, Saint-Venant's extension, bending and torsion solutions are derived from the hypothesis that the stress field is independent of the axial coordinate, and Saint-Venant's flexure solution is obtained if the stress field depends on the axial coordinate at most linearly.

E.Sternberg and J.K.Knowles [143] characterized Saint-Venant's solutions in terms of certain associated minimum strain-energy properties. Other intrinsic criteria that distinguish Saint-Venant's solutions among all the solutions of the relaxed problem were established in [79]. In [79], a rational scheme of deriving Saint-Venant's solutions is presented. The advantage of this method is that it does not involve artificial a priori assumptions. The method permits to construct a solution of the relaxed Saint-Venant's problem for other kinds of constitutive equations (anisotropic media, Cosserat continua, etc.) where the physical intuition or semi-inverse method cannot be used.

In [148]-[150], C.Truesdell proposed a problem which, roughly speaking, consists in the generalization of Saint-Venant's notion of twist so as to apply to any solution of the torsion problem. Recently an elegant solution of Truesdell's problem has been established by W.A.Day [25]. In [123], P.Podio-Guidugli studied Truesdell's problem rephrased for extension and pure bending. The case of flexure was considered in [79]. The results of [25,123] are related to the results of Sternberg and Knowles [143] concerning the minimum energy characterizations of corresponding Saint-Venant's solutions.

A generalization of the relaxed Saint-Venant's problem consists in determining the equilibrium of an elastic cylinder which - in the presence of body forces - is subjected to surface tractions arbitrarily prescribed over the lateral boundary and to appropriate stress resultants over its ends. The study of this problem was initiated by Almansi [1] and Michell [102] and was developed in various later papers (see,for example, Sokolnikoff [139], Djanelidze [29] and Hatiashvili [49]).

As pointed out before, Saint-Venant's results were established within the equilibrium theory of homogeneous and isotropic elastic bodies. A large number of papers are concerned with the relaxed Saint-Venant's problem for other kinds of elastic materials (see, for example, Lekhnitskii [96], Lomakin [97], Brulin and R.K.T.Hsieh [15] and Reddy and Venkatasubramanian [90]). References to recent results are cited in the text. No attempt is made to provide a complete list of works on Saint-Venant's problem. Neither the contents, nor the list of works cited are exhaustive. Nevertheless, it is hoped that the developments presented reflect the state of knowledge in the study of the problem.

The purpose of this work is to present some of the recent researches on Saint-Venant's problem. An effort is made to provide a systematic treatment of the subject.

Chapter 1 is concerned mainly with results where Saint-Venant's solutions are involved. We give a rational method of construction of these solutions and then we characterize them in terms of certain associated minimum strain-energy properties. A study of Truesdell's problem is presented. This chapter also includes a proof of Saint-Venant's principle.

In Chapter 2, an interesting scheme of deriving a solution of

Almansi-Michell problem is presented. Almansi's problem, where the body forces and the surface tractions on the lateral boundary are polynomials in the axial coordinate, is also studied. The results are used to study a statical problem in the linear thermoelasticity.

Chapter 3 is concerned with the relaxed Saint-Venant's problem for anisotropic elastic bodies. We first establish a solution of the foregoing problem. The method does not involve artificial a priori assumptions and permits a treatment of the problem even for nonhomogeneous bodies, where the elastic coefficients are independent of the axial coordinate. It is shown that the well-known boundary-value problem for the torsion function derives from a special problem of generalized plane strain. Then, minimum energy characterizations of the solutions are presented. Also included in this chapter is a study of Truesdell's problem.

Chapter 4 deals with the relaxed Saint-Venant's problem for heterogeneous elastic cylinders. We consider the case of a composed cylinder when the generic cross-section is occupied by different anisotropic solids. The problems of Almansi and Michell are also studied. Applications to the linear thermoelastostatics are given.

In Chapter 5 we study Saint-Venant's problem within the linearized theory of Cosserat elastic bodies. We first present a proof of Saint-Venant's principle in the theory of Cosserat elasticity. Then, a solution of the relaxed Saint-Venant's problem is derived. Truesdell's problem and a theory of loaded cylinders are also studied. Illustrative applications are presented.

A number of results included in this work have not appeared or been discussed previously in literature.

CONTENTS

1. THE RELAXED SAINT-VENANT PROBLEM

1.1. Preliminaries

We refer to a bounded regular region B of three-dimensional Euclidian space as the body (cf.M.E.Gurtin [47], Sect.5). We let $\bar{B}$ denote the closure of B, call ∂B the boundary of B, and designate by $\underline{n}$ the outward unit normal of ∂B. Letters marked by an underbar stand for tensors of an order $p \geq 1$, and if $\underline{v}$ has the order p, we write $v_{ij...k}$ (p subscripts) for the components of $\underline{v}$ in the underlying rectangular Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1,2), whereas Latin subscripts - unless otherwise specified - are confined to the range (1,2,3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

We assume that the body occupying B is a linearly elastic material. Let $\underline{u}$ be a displacement field over B. Then

$$\underline{E}(\underline{u}) = \operatorname{sym} \nabla \underline{u} ,$$

is the strain field associated with $\underline{u}$. Here $\nabla \underline{u}$ denotes the displacement gradient and $(\operatorname{sym} \nabla \underline{u})_{ij} = (u_{i,j} + u_{j,i})/2$. The stress-displacement relation may be written in the form

$$\underline{S}(\underline{u}) = \underline{C}[\nabla \underline{u}]. \qquad (1.1)$$

Here $\underline{S}(\underline{u})$ is the stress field associated with $\underline{u}$, while $\underline{C}$ stands for the elasticity field. We assume that $\underline{C}$ is positive-definite, symmetric, and smooth on $\bar{B}$. For the particular case of the isotropic elastic medium the tensor field $\underline{C}$ admits the representation

$$C_{ijk\ell} = \lambda \delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \qquad (1.2)$$

where λ and μ are the Lamé moduli and δ_{ij} is the Kronecker delta.

We call a vector field $\underline{u}$ an equilibrium displacement field for B if $\underline{u} \in C^1(\bar{B}) \cap C^2(B)$ and

$$\operatorname{div} \underline{S}(\underline{u}) = \underline{0} , \tag{1.3}$$

holds on B. Clearly, $(\operatorname{div} \underline{S}(\underline{u}))_{ij} = (S_{ij}(\underline{u}))_{,j}$.

Let $\underline{s}(\underline{u})$ be the surface traction at regular points of ∂B corresponding to the stress field $\underline{S}(\underline{u})$ defined on $\bar{B}$, i.e.

$$\underline{s}(\underline{u}) = \underline{S}(\underline{u})\underline{n} . \tag{1.4}$$

The strain energy $U(\underline{u})$ corresponding to a smooth displacement field $\underline{u}$ on $\bar{B}$ is (cf.[47], Sect.32)

$$U(\underline{u}) = \frac{1}{2} \int_B \nabla \underline{u} \cdot \underline{C}[\nabla \underline{u}] dv . \tag{1.5}$$

In the following, two displacement fields differing by an (infinitesimal) rigid displacement will be regarded identical.

The functional $U(\cdot)$ generates the bilinear functional

$$U(\underline{u},\underline{v}) = \frac{1}{2} \int_B \nabla \underline{u} \cdot \underline{C}[\nabla \underline{v}] dv .$$

The set of smooth fields over $\bar{B}$ can be made into a real vector space with the inner product

$$\langle \underline{u},\underline{v} \rangle = 2\, U(\underline{u},\underline{v}) . \tag{1.6}$$

This inner product generates the energy norm

$$\| \underline{u} \|_e^2 = \langle \underline{u},\underline{u} \rangle . \tag{1.7}$$

For any equilibrium displacement fields $\underline{u}$ and $\underline{v}$, one has (cf. [47], Sect.30)

$$\langle \underline{u},\underline{v} \rangle = \int_{\partial B} \underline{s}(\underline{v}) \cdot \underline{u} \, da , \tag{1.8}$$

which implies the well known relation

$$\int_{\partial B} \underline{s}(\underline{u}) \cdot \underline{v} \, da = \int_{\partial B} \underline{s}(\underline{v}) \cdot \underline{u} \, da . \tag{1.9}$$

1.2. Properties of the Solutions to the Relaxed Saint-Venant Problem

We proceed now to Saint-Venant's problem and for this purpose stipulate that the region B from here on refers to the interior of a right cylinder of length h with open cross-section Σ and the lateral boundary Π. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the x_3-axis is parallel to the generators of B and the x_1Ox_2 plane contains one of the terminal cross-sections. We denote by Σ_1 and Σ_2, respectively, the cross-section located at $x_3 = 0$ and $x_3 = h$. We assume that the generic cross-section Σ is a simply connected regular region. We denote by Γ the boundary of Σ.

Saint-Venant's problem consists in the determination of an equilibrium displacement field $\underline{u}$ on B, subject to the requirements

$$\underline{s}(\underline{u}) = \underline{0} \quad \text{on } \Pi, \qquad \underline{s}(\underline{u}) = \underline{s}^{(\alpha)} \quad \text{on } \Sigma_\alpha (\alpha = 1,2), \tag{1.10}$$

where $\underline{s}^{(\alpha)}$ is a vector-valued function preassigned on Σ_α. Necessary conditions for the existence of a solution to this problem are given by

$$\int_{\Sigma_1} \underline{s}^{(1)} da + \int_{\Sigma_2} \underline{s}^{(2)} da = \underline{0}, \qquad \int_{\Sigma_1} \underline{x} \times \underline{s}^{(1)} da + \int_{\Sigma_2} \underline{x} \times \underline{s}^{(2)} da = \underline{0},$$

where $\underline{x}$ is the position vector of a point with respect to O.

Under suitable smoothness hypotheses on Γ and on the given forces, a solution of Saint-Venant's problem exists (cf. Fichera [37]).

In the relaxed formulation of Saint-Venant's problem the conditions (1.10) are replaced by

$$\underline{s}(\underline{u}) = \underline{0} \quad \text{on } \Pi, \quad \underline{R}(\underline{u}) = \underline{F}, \quad \underline{H}(\underline{u}) = \underline{M}, \tag{1.11}$$

where $\underline{F}$ and $\underline{M}$ are prescribed vectors representing the resultant force and the resultant moment about O of the tractions acting on Σ_1. Accordingly, $\underline{R}(\cdot)$ and $\underline{H}(\cdot)$ are the vector-valued linear functionals defined by

$$\underline{R}(\underline{u}) = \int_{\Sigma_1} \underline{s}(\underline{u}) da, \quad \underline{H}(\underline{u}) = \int_{\Sigma_1} \underline{x} \times \underline{s}(\underline{u}) da. \tag{1.12}$$

If $\varepsilon_{\alpha\beta}$ is the two-dimensional alternator, (1.12) appears as

$$R_i(\underline{u}) = -\int_{\Sigma_1} S_{3i}(\underline{u})da,$$

$$H_\alpha(\underline{u}) = -\int_{\Sigma_1}\varepsilon_{\alpha\beta}x_\beta S_{33}(\underline{u})da, \quad H_3(\underline{u}) = -\int_{\Sigma_1}\varepsilon_{\alpha\beta}x_\alpha S_{3\beta}(\underline{u})da \ . \qquad (1.13)$$

The necessary conditions for the existence of a solution to Saint-Venant problem lead to the following relations, which are needed subsequently

$$\int_{\Sigma_2} S_{3i}(\underline{u})da = -R_i(\underline{u}) \ , \int_{\Sigma_2}\varepsilon_{\alpha\beta}x_\alpha S_{3\beta}(\underline{u})da = -H_3(\underline{u}),$$

$$\int_{\Sigma_2} x_\alpha S_{33}(\underline{u})da = -hR_\alpha(\underline{u}) + \varepsilon_{\alpha\beta}H_\beta(\underline{u}) \ . \qquad (1.14)$$

It is obvious that the relaxed statement of the problem fails to characterize the solution uniquely.

By a solution of the relaxed Saint-Venant's problem we mean any equilibrium displacement field that satisfies (1.11).

We denote by (P) the relaxed Saint-Venant's problem corresponding to the resultants $\underline{F}$ and $\underline{M}$. Let $K(\underline{F},\underline{M})$ denote the class of solutions to the problem (P).

The classification of the relaxed problem rests on various assumptions concerning the resultants $\underline{F}$ and $\underline{M}$. Throughout this work it is convenient to use the decomposition of the relaxed problem into problems (P_1) and (P_2) characterized by

(P_1) (extension-bending-torsion) : $F_\alpha = 0$,

(P_2) (flexure) : $F_3 = M_i = 0$.

For further economy it is helpful to denote by $K_I(F_3,M_1,M_2,M_3)$ the class of solutions to the problem (P_1) and by $K_{II}(F_1,F_2)$ the class of solutions to the problem (P_2). We assume for the remainder of this chapter that the material is homogeneous and isotropic.

Let $\mathcal{D}$ denote the set of all equilibrium displacement fields $\underline{u}$ that satisfy the condition $\underline{s}(\underline{u}) = \underline{0}$ on the lateral boundary. The next theorem will be of future use.

Theorem 1.1. [79]. If $\underline{u} \in \mathcal{D}$ and $\underline{u}_{,3} \in C^1(\bar{B})$, then $\underline{u}_{,3} \in \mathcal{D}$ and

$$\underline{R}(\underline{u}_{,3}) = 0 , \quad H_\alpha(\underline{u}_{,3}) = \varepsilon_{\alpha\beta} R_\beta(\underline{u}) , \quad H_3(\underline{u}_{,3}) = 0. \tag{1.15}$$

Proof. The first assertion follows at once from the fact that $\underline{S}(\underline{u}_{,3}) = \partial \underline{S}(\underline{u})/\partial x_3$ and the proposition: if $\underline{u}$ is an elastic displacement field corresponding to null body forces, then so also is $\underline{u}_{,k} = \partial \underline{u}/\partial x_k$ (cf.[47],Sect.42). Next, with the aid of the equations of equilibrium (1.3), we find that

$$S_{3i}(\underline{u}_{,3}) = (S_{3i}(\underline{u}))_{,3} = -(S_{\rho i}(\underline{u}))_{,\rho} ,$$

$$\varepsilon_{\alpha\beta} x_\beta S_{33}(\underline{u}_{,3}) = -\varepsilon_{\alpha\beta} x_\beta (S_{\rho 3}(\underline{u}))_{,\rho} = -\varepsilon_{\alpha\beta}[(x_\beta S_{\rho 3}(\underline{u}))_{,\rho} - S_{\beta 3}(\underline{u})] ,$$

$$\varepsilon_{\alpha\beta} x_\alpha S_{3\beta}(\underline{u}_{,3}) = -\varepsilon_{\alpha\beta} x_\alpha (S_{\rho\beta}(\underline{u}))_{,\rho} = -\varepsilon_{\alpha\beta}(x_\alpha S_{\rho\beta}(\underline{u}))_{,\rho} + \varepsilon_{\alpha\beta} S_{\alpha\beta}(\underline{u}).$$

By (1.13), the divergence theorem, and the symmetry of $\underline{S}$ we arrive at

$$\underline{R}(\underline{u}_{,3}) = \int_\Gamma \underline{s}(\underline{u})\,ds,$$

$$H_\alpha(\underline{u}_{,3}) = \int_\Gamma \varepsilon_{\alpha\beta} x_\beta s_3(\underline{u})\,ds + \varepsilon_{\alpha\beta} R_\beta(\underline{u}), \tag{1.16}$$

$$H_3(\underline{u}_{,3}) = \int_\Gamma \varepsilon_{\alpha\beta} x_\alpha s_\beta(\underline{u})\,ds .$$

The desired result follows from (1.16) and hypothesis. □

Since $\underline{u}$ is an equilibrium displacement field, $\underline{u}$ is analytic (cf. [47], Sect.42). Theorem 1.1 has the following immediate consequences:

Corollary 1.1. If $\underline{u} \in K_I(F_3, M_1, M_2, M_3)$ and $\underline{u}_{,3} \in C^1(\bar{B})$, then $\underline{u}_{,3} \in \mathcal{D}$ and

$$\underline{R}(\underline{u}_{,3}) = \underline{0} , \quad \underline{H}(\underline{u}_{,3}) = \mathbf{0} .$$

Corollary 1.2. If $\underline{u} \in K_{II}(F_1, F_2)$ and $\underline{u}_{,3} \in C^1(\bar{B})$, then

$$\underline{u}_{,3} \in K_I(0, F_2, -F_1, 0).$$

Corollary 1.3. If $\underline{u} \in \mathcal{D}$ and $\partial^n \underline{u}/\partial x_3^n \in C^1(\bar{B})$, then $\partial^n \underline{u}/\partial x_3^n \in \mathcal{D}$ and

$$\underline{R}\left(\frac{\partial^n \underline{u}}{\partial x_3^n}\right) = \underline{0}, \quad \underline{H}\left(\frac{\partial^n \underline{u}}{\partial x_3^n}\right) = \underline{0} \quad \text{for } n \geq 2.$$

1.3. A Method of Construction of Saint-Venant's Solutions

Corollary 1.1 allows us to establish a simple method of deriving Saint-Venant's solution to the problem (P_1). Let Q be the class of solutions to the relaxed Saint-Venant's problem corresponding to $\underline{F} = \underline{0}$ and $\underline{M} = \underline{0}$. We note that if $\underline{u} \in K_I(F_3, M_1, M_2, M_3)$ and $\underline{u}_{,3} \in C^1(\bar{B})$, then by Corollary 2.1, $\underline{u}_{,3} \in Q$. Let us note that a rigid displacement field belongs to Q. It is natural to enquire whether there exists a solution $\underline{v}$ of the problem (P_1) such that $\underline{v}_{,3}$ is a rigid displacement field. This question is settled in the next theorem.

Theorem 1.2. Let $\underline{v} \in C^1(\bar{B}) \cap C^2(B)$ be a vector field such that $\underline{v}_{,3}$ is a rigid displacement field. Then $\underline{v}$ is a solution of the problem (P_1) if and only if $\underline{v}$ is Saint-Venant's solution.

Proof. Let $\underline{v} \in C^1(\bar{B}) \cap C^2(B)$ be a vector field such that

$$\underline{v}_{,3} = \underline{\alpha} + \underline{\beta} \times \underline{x}, \tag{1.17}$$

where $\underline{\alpha}$ and $\underline{\beta}$ are constant vectors. Then it follows that

$$\begin{aligned} v_\alpha &= -\frac{1}{2} a_\alpha x_3^2 - a_4 \varepsilon_{\alpha\beta} x_\beta x_3 + w_\alpha(x_1, x_2), \\ v_3 &= (a_1 x_1 + a_2 x_2 + a_3) x_3 + w_3(x_1, x_2), \end{aligned} \tag{1.18}$$

except for an additive rigid displacement field. Here $\underline{w}$ is an arbitrary vector field independent of x_3, and we have used the notations $a_\alpha = \varepsilon_{\rho\alpha}\beta_\rho$, $a_3 = \alpha_3$, $a_4 = \beta_3$. Let us prove that the functions w_i and the constants $a_s (s=1,2,3,4)$ can be determined so that $\underline{v} \in K_I(F_3, M_1, M_2, M_3)$. The stress-displacement relations imply that

$$\begin{aligned} S_{\alpha\beta}(\underline{v}) &= \lambda(a_\rho x_\rho + a_3)\delta_{\alpha\beta} + T_{\alpha\beta}(\underline{w}), \\ S_{3\alpha}(\underline{v}) &= \mu(w_{3,\alpha} - a_4 \varepsilon_{\alpha\rho} x_\rho), \quad S_{33}(\underline{v}) = (\lambda + 2\mu)(a_\rho x_\rho + a_3) + \lambda w_{\rho,\rho}. \end{aligned} \tag{1.19}$$

where

$$T_{\alpha\beta}(\underline{w}) = \mu(w_{\alpha,\beta} + w_{\beta,\alpha}) + \lambda\delta_{\alpha\beta}w_{\rho,\rho}. \tag{1.20}$$

The equations of equilibrium and the condition on the lateral boundary reduce to

$$(T_{\alpha\beta}(\underline{w}))_{,\beta} + f_\alpha = 0 \text{ on } \Sigma, \quad T_{\alpha\beta}(\underline{w})n_\beta = p_\alpha \text{ on } \Gamma, \tag{1.21}$$

$$\Delta w_3 = 0 \text{ on } \Sigma, \quad \frac{\partial w_3}{\partial n} = a_4\varepsilon_{\alpha\beta}n_\alpha x_\beta \text{ on } \Gamma, \tag{1.22}$$

where

$$f_\alpha = \lambda a_\alpha, \quad p_\alpha = -\lambda(a_\rho x_\rho + a_3)n_\alpha. \tag{1.23}$$

The relations (1.20),(1.21) and (1.23) constitute a two-dimensional boundary-value problem (cf. [47], Sect. 45). The necessary and sufficient conditions to solve this problem are

$$\int_\Sigma f_\alpha da + \int_\Gamma p_\alpha ds = 0, \quad \int_\Sigma \varepsilon_{\alpha\beta}x_\alpha f_\beta da + \int_\Gamma \varepsilon_{\alpha\beta}x_\alpha p_\beta ds = 0. \tag{1.24}$$

It follows from (1.23) and the divergence theorem that the conditions (1.24) are satisfied. We observe that the boundary-value problem (1.21) is satisfied if one chooses

$$T_{\alpha\beta}(\underline{w}) = -\lambda(a_\rho x_\rho + a_3)\delta_{\alpha\beta}.$$

Clearly, the above stresses satisfy the compatibility condition. It follows from (1.20) that

$$w_{1,1} = w_{2,2} = -\frac{\lambda}{2(\lambda+\mu)}(a_\rho x_\rho + a_3), \quad w_{1,2} + w_{2,1} = 0.$$

The integration of these equations yields

$$w_\alpha = a_1 w_\alpha^{(1)} + a_2 w_\alpha^{(2)} + a_3 w_\alpha^{(3)},$$

where

$$w_\alpha^{(\beta)} = \nu(\tfrac{1}{2}x_\rho x_\rho\delta_{\alpha\beta} - x_\alpha x_\beta), \quad w_\alpha^{(3)} = -\nu x_\alpha, \tag{1.25}$$

modulo a plane rigid displacement. Here ν designates Poisson's ratio.

It follows from (1.22) that $w_3 = a_4\varphi$, where the function φ is characterized by

$$\Delta\varphi = 0 \text{ on } \Sigma, \quad \frac{\partial\varphi}{\partial n} = \varepsilon_{\alpha\beta}n_\alpha x_\beta \text{ on } \Gamma. \tag{1.26}$$

Clearly, the vector field $\underline{v}$ can be written in the form

$$\underline{v} = \sum_{j=1}^{4} a_j \underline{v}^{(j)} \quad , \tag{1.27}$$

where the vectors $\underline{v}^{(j)}$ $(j=1,2,3,4)$ are defined by

$$\begin{gathered} v_\alpha^{(\beta)} = -\frac{1}{2} x_3^2 \delta_{\alpha\beta} + w_\alpha^{(\beta)} \, , \quad v_3^{(\beta)} = x_\beta x_3 \quad (\beta=1,2), \\ v_\alpha^{(3)} = w_\alpha^{(3)} \, , \quad v_3^{(3)} = x_3 \, , \quad v_\alpha^{(4)} = \varepsilon_{\beta\alpha} x_\beta x_3, \quad v_3^{(4)} = \varphi \, , \end{gathered} \tag{1.28}$$

We note that $\underline{v}^{(j)} \in \mathcal{D}$ $(j=1,2,3,4)$. The conditions on the terminal cross-section Σ_1 furnish the following system for the unknown constants

$$\begin{gathered} E(I_{\alpha\beta} a_\beta + A x_\alpha^0 a_3) = \varepsilon_{\alpha\beta} M_\beta \, , \\ EA(a_1 x_1^0 + a_2 x_2^0 + a_3) = - F_3 \quad , \\ \mu D a_4 = - M_3 \quad , \end{gathered} \tag{1.29}$$

where A is the area of the cross-section, x_α^0 are the coordinates of the centroid of Σ_1, E designates Young's modulus, and

$$I_{\alpha\beta} = \int_\Sigma x_\alpha x_\beta \, da \, , \qquad D = \int_\Sigma (x_\alpha x_\alpha + \varepsilon_{\alpha\beta} x_\alpha \varphi_{,\beta}) \, da \, . \tag{1.30}$$

If the rectangular Cartesian coordinate frame is chosen in such a way that the x_α-axes are principal centroidal axes of the cross-section Σ_1, then (1.27) and (1.29) lead to the well-known Saint-Venant solution. □

We present Saint-Venant's solutions, which are needed subsequently.

i) <u>Saint-Venant's extension solution</u>:

$$\begin{gathered} \underline{v} = a_3 \underline{v}^{(3)} \, , \qquad v_\alpha^{(3)} = -\nu x_\alpha \, , \qquad v_3^{(3)} = x_3 \quad , \\ S_{\alpha\beta}(\underline{v}) = 0 \, , \qquad S_{3\alpha}(\underline{v}) = 0 \, , \qquad S_{33}(\underline{v}) = E a_3 \, , \end{gathered} \tag{1.31}$$

where

$$F_3 = -EA \; a_3 \quad . \tag{1.32}$$

The relation (1.32) is known as Saint-Venant's formula for extension.

ii) Saint-Venant's bending solution:

$$\underline{v} = a_1 \underline{v}^{(1)} , \quad v_1^{(1)} = \frac{1}{2}(\nu x_2^2 - \nu x_1^2 - x_3^2),$$

$$v_2^{(1)} = -\nu x_1 x_2 , \quad v_3^{(1)} = x_1 x_3 , \tag{1.33}$$

$$S_{\alpha\beta}(\underline{v}) = 0 , \quad S_{3\alpha}(\underline{v}) = 0 , \quad S_{33}(\underline{v}) = E a_1 x_1 ,$$

where

$$M_2 = E I_{11} a_1 . \tag{1.34}$$

The relation (1.34) is called Saint-Venant's formula for bending.

iii) Saint-Venant's torsion solution:

$$\underline{v} = a_4 \underline{v}^{(4)} , \quad v_\alpha^{(4)} = \varepsilon_{\beta\alpha} x_\beta x_3 , \quad v_3^{(4)} = \varphi ,$$

$$S_{\alpha\beta}(\underline{v}) = 0 , \quad S_{33}(\underline{v}) = 0 , \quad S_{3\alpha}(\underline{v}) = \mu a_4 (\varphi_{,\alpha} - \varepsilon_{\alpha\rho} x_\rho), \tag{1.35}$$

where

$$M_3 = - \mu D a_4 . \tag{1.36}$$

The constant a_4 is known as specific angle of twist, and μD is called the torsional rigidity.
The relation (1.36) is known as Saint-Venant's formula for torsion.

Let us note that the vectors $\underline{v}^{(j)}$ (j=1,2,3,4) defined by (1.27) depend only on the cross-section and the elasticity field. Let $\hat{a}$ be the four-dimensional vector (a_1, a_2, a_3, a_4). We will write $\underline{v}\{\hat{a}\}$ for the displacement vector $\underline{v}$ defined by (1.27), indicating thus its dependence on the constants a_s (s=1,2,3,4).

In view of Corollaries 1.1, 1.2 and Theorem 1.2 it is natural to seek a solution of the problem (P_2) in the form

$$\underline{u}^o = \int_0^{x_3} \underline{v}\{\hat{b}\} dx_3 + \underline{v}\{\hat{c}\} + \underline{w}^o , \tag{1.37}$$

where $\hat{b} = (b_1, b_2, b_3, b_4)$ and $\hat{c} = (c_1, c_2, c_3, c_4)$ are two constant four-dimensional vectors, and $\underline{w}^o$ is a vector field independent of x_3 such that $\underline{w}^o \in C^1(\bar{\Sigma}) \cap C^2(\Sigma)$.

Theorem 1.3. The vector field $\underline{u}^o$ defined by (1.37) is a solution of