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Edited by A. Dold and B. Eckmann

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Julian Musielak

Orlicz Spaces and Modular Spaces

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PREFACE

The first version of these notes was published in Polish by the A. Mickiewicz University in 1978, under the title "Modular spaces". It contained Sections 1,3,4,15-22 of the present text, which is enlarged by the theory of generalized Orlicz spaces. We limit ourselves to the case of spaces of scalar-valued functions, which is most fundamental for the applications. The general, not necessarily convex case is stressed as is modular convergence, not only norm convergence.

The reader who intends to study generalized Orlicz spaces only, may restrict himself to Sections 1,2 and 5 of Chapter I and proceed then directly to Chapter II.

The author would like to acknowledge the help of all who have contributed to the present text. Doz. Dr. A. Kozek, the referee of "Modular spaces" to "Wiadomości Matematyczne", was so kind as to communicate to the author his detailed critical remarks. "Modular spaces" was also read critically by Dr. H. Hudzik and Dr. A. Kamińska, who contributed their valuable remarks. The content of some parts of the present Section 12 was communicated to the author by Doz. Dr. L. Drewnowski, and Dr. H. Hudzik is the author of the first draft of Section 14. The author would like also to express his gratitude to the many others who helped in any way in preparation of this book.

Julian Musielak

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continuation on page 225

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continuation on page 225

CONTENTS

CHAPTER I

MODULAR SPACES

§ 1. Modular spaces

1.1. Definition. Let X be a real or complex vector space. A functional $\varrho : X \to [0,\infty]$ is called a pseudomodular, if there holds for arbitrary $x, y \in X$:

1° $\varrho(0) = 0$,

2° $\varrho(-x) = \varrho(x)$ in case of X real

$\varrho(e^{it}x) = \varrho(x)$ for every real t in case of X complex,

3° $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

If in place of 3° there holds

$3^{\circ\prime}$ $\varrho(\alpha x + \beta y) \leq \alpha^{s} \varrho(x) + \beta^{s} \varrho(y)$ for $\alpha, \beta \geq 0$, $\alpha^{s} + \beta^{s} = 1$

with an $s \in (0,1]$, then the pseudomodular ϱ is called s-convex. 1-convex pseudomodulars are called convex. If besides 1° there holds also

$1^{\circ\prime}$ $\varrho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$,

then ϱ is called a semimodular. If moreover,

$1^{\circ\prime\prime}$ $\varrho(x) = 0$ implies $x = 0$,

then ϱ is called a modular.

1.2. Examples. I. If X is an s-normed space with an s-homogeneous norm $\| \|^{s}$, then $\varrho(x) = \|x\|^{s}$ is an s-convex modular in X. Similarly, if $\| \|^{s}$ is an s-homogeneous pseudonorm in X, then $\varrho(x) = \|x\|^{s}$ is an s-convex pseudomodular in X. However, in general a modular is not a norm; among others, it may assume the value $+\infty$.

II. Let $X = L^{p}$ over the interval $[a,b]$, then $\varrho(x) = \int_a^b |x(t)|^{p} dt$ is a p-convex modular on X for $0 < p < 1$ and a convex modular on X for $p \geq 1$.

III. Let X be the space of real numbers and let $\varrho(x) \geq 0$ for every $x \in X$, $\varrho(0) = 0$. Then 2° means that ϱ is an even function, 3° is equivalent to the statement that the function ϱ is nondecreasing for $x \geq 0$, and $3^{\circ\prime}$ with $s = 1$ means that ϱ is a convex function.

1.3. Properties of pseudomodulars:

§. $\varrho(\alpha x) \leq \varrho(x)$ for $|\alpha| \leq 1$,

II. $\varrho\left(\sum_{i=1}^{n}\alpha_i x_i\right) \leqslant \sum_{i=1}^{n}\varrho(x_i)$ for $\alpha_i \geqslant 0$, $\sum_{i=1}^{n}\alpha_i = 1$,

III. if ϱ is s-convex, $0 < s \leqslant 1$, then $\varrho\left(\sum_{i=1}^{n}\alpha_i x_i\right) \leqslant \sum_{i=1}^{n}\alpha_i^s\,\varrho(x_i)$ for $\alpha_i \geqslant 0$, $\sum_{i=1}^{n}\alpha_i^s = 1$.

1.4. Definition. If ϱ is a pseudomodular in X, then

$$X_\varrho = \{x \in X:\ \lim_{\lambda \to 0}\varrho(\lambda x) = 0\}$$

is called a modular space.

Obviously, X_ϱ is a vector subspace of X.

In the following, we shall need the notions of an F-norm (F-pseudonorm) and of an s-norm (s-pseudonorm). Let us recall that a functional $|\ |: X \to [0,\infty)$ in a vector space X is called an F-pseudonorm, if it satisfies the following conditions: (a) $|0| = 0$, (b) $|-x| = |x|$ (or $|e^{it}x| = |x|$ for real t in case of a complex space X), (c) $|x+y| \leqslant |x| + |y|$, (d) if $\alpha_k \to \alpha$ and $|x_k - x| \to 0$, then $|\alpha_k x_k - \alpha x| \to 0$. If, moreover, (e) $|x| = 0$ implies $x = 0$, then $|\ |$ is called an F-norm. Every F-norm (F-pseudonorm) $|\ |$ such that $|\alpha x|$ is a nondecreasing function of $\alpha > 0$ for every $x \in X$ is a modular (pseudomodular) in X. It should be noted that for every F-norm $|\ |$ one can define an equivalent F-norm satisfying the above condition (see S.Rolewicz [2], p.16). If $|\ |: X \to [0,\infty)$ satisfies the above conditions (a)-(c) and the condition (d') $|\alpha x| = |\alpha|^s|x|$, $0 < s \leqslant 1$, then $|\ |$ is called an s-homogeneous pseudonorm or briefly, an s-pseudonorm, and adding (e), we obtain an s-norm in X. An s-norm we shall usually denote $\|\ \|^s$. If $s = 1$, we obtain a norm which will be denoted $\|\ \|$. A space X with an F-norm (s-norm, norm) is a metric vector space with distance $d(x,y) = |x-y|$.

1.5. Theorem. If ϱ is a pseudomodular in X, then

$$|x|_\varrho = \inf\left\{u > 0:\ \varrho\left(\frac{x}{u}\right) \leqslant u\right\}$$

is an F-pseudonorm in X_ϱ, having the following properties:

I. if $\varrho(\lambda x_1) \leqslant \varrho(\lambda x_2)$ for every $\lambda > 0$, where $x_1, x_2 \in X_\varrho$, then $|x_1|_\varrho \leqslant |x_2|_\varrho$,

II. if $x \in X_\varrho$, then $|\alpha x|_\varrho$ is a nondecreasing function of $\alpha \geqslant 0$,

III. if $|x|_\varrho < 1$, then $\varrho(x) \leqslant |x|_\varrho$.

If ϱ is a semimodular in X, then $|\ |_\varrho$ is an F-norm in X . If ϱ is an s-convex pseudomodular, $0 < s \leq 1$, then

$$\|x\|_\varrho^s = \inf\{u>0:\ \varrho\left(\frac{x}{u^{1/s}}\right) \leq 1\}$$

is an s-pseudonorm in X, and the properties I-III remain valid if we replace $|\ |_\varrho$ by $\|\ \|_\varrho^s$. If ϱ is an s-convex semimodular, then $\|\ \|_\varrho^s$ is an s-homogeneous norm in X_ϱ (if s = 1, we shall write also $\|\ \|_\varrho = \|\ \|_\varrho^1$).

Proof. It is clear that the set $\{u>0:\ \varrho\left(\frac{x}{u}\right) \leq u\}$ is nonempty for $x \in X_\varrho$, and so $0 \leq |x|_\varrho < \infty$ and $|0|_\varrho = 0$. Also the symmetry condition $|-x|_\varrho = |x|_\varrho$ (or $|e^{it}x|_\varrho = |x|_\varrho$) is obvious. Moreover, if ϱ is a semimodular and $|x|_\varrho = 0$, then $\varrho\left(\frac{x}{u}\right) \leq u$ for every $u>0$, and so $\varrho(\lambda x) \leq \varrho\left(\frac{\lambda x}{u}\right) \leq \frac{u}{\lambda}$ for $0<u<\lambda$ and every $\lambda>0$. Thus $\varrho(\lambda x) = 0$ for all $\lambda > 0$, and so $x = 0$. Now, let $x,y \in X_\varrho$ and $a>0$, $u = |x|_\varrho + a$, $v = |y|_\varrho + b$. Then $\varrho\left(\frac{x}{u}\right) \leq u$ and $\varrho\left(\frac{y}{v}\right) \leq v$, whence

$$\varrho\left(\frac{x+y}{u+v}\right) = \varrho\left(\frac{u}{u+v}\,\frac{x}{u} + \frac{v}{u+v}\,\frac{y}{v}\right) \leq \varrho\left(\frac{x}{u}\right) + \varrho\left(\frac{y}{v}\right) \leq u+v.$$

Hence $|x+y|_\varrho \leq u+v+2a$, and we obtain $|x+y|_\varrho \leq |x|_\varrho + |y|_\varrho$, because $a>0$ is arbitrary. Properties I and II follow from the definition of $|\ |_\varrho$ and from the Property 1.3, immediately. Now, let $\alpha_k \to \alpha$ and $|x_k - x|_\varrho \to 0$. Writing $a_k = \alpha_k - a$ and $y_k = x_k - x$, we have $a_k \to 0$ and $|y_k|_\varrho \to 0$. Now, given an $\varepsilon > 0$, we get $\varrho\left(a_k \frac{x}{\varepsilon}\right) \to 0$ as $k \to \infty$, and so $\varrho\left(\frac{a_k x}{\varepsilon}\right) \leq \varepsilon$ for sufficiently large k. Hence $|a_k x|_\varrho \to 0$. Consequently, taking as N a positive integer such that $|\alpha_k| \leq N$ for all k and $|a| \leq N$, we have

$$|\alpha_k x_k - \alpha x|_\varrho \leq |\alpha_k (x_k - x)|_\varrho + |(\alpha_k - \alpha)x|_\varrho \leq |N y_k|_\varrho + |a_k x|_\varrho \leq N|y_k|_\varrho + |a_k x|_\varrho \to 0$$

as $k \to \infty$. We prove now that $\varrho(x) \leq |x|_\varrho$ for $|x|_\varrho < 1$. Indeed, let $|x|_\varrho < u < 1$, then $\varrho\left(\frac{x}{u}\right) \leq u$, and so $\varrho(x) = \varrho\left(u\frac{x}{u}\right) \leq \varrho\left(\frac{x}{u}\right) \leq u$. Since u is arbitrary, this implies $\varrho(x) \leq |x|_\varrho$.

The proof of the second part of the theorem with an s-convex pseudomodular ϱ runs along similar lines to the above; taking $\alpha > 0$, we have

$$\|\alpha x\|_\varrho^s = \inf\{u>0:\ \varrho\left(\frac{\alpha x}{u^{1/s}}\right) \leq 1\} = \alpha^s \inf\{\frac{u}{\alpha^s} > 0:\ \varrho\left(\frac{x}{(u/\alpha^s)^{1/s}}\right) \leq 1\} = \alpha^s \|x\|_\varrho^s.$$

1.6. Theorem. Let ϱ be a pseudomodular in X. If $x \in X_\varrho$ and $x_k \in X_\varrho$ for k = 1,2,..., then the condition $|x_k - x|_\varrho \to 0$ as $k \to \infty$ is equivalent to the condition $\varrho(\lambda(x_k - x)) \to 0$ as $k \to \infty$ for every $\lambda > 0$. If $x_k \in X_\varrho$ for k = 1,2,..., then (x_k) is a Cauchy sequence in the space X_ϱ with respect to the F-pseudonorm $|\ |_\varrho$, if and only if, $\varrho(\lambda(x_k-x_l))\to 0$ as $k,l\to\infty$ for every $\lambda>0$. If ϱ is an s-convex pseudomodular in X, then the same statement holds, replacing $|\ |_\varrho$ by $\|\ \|_\varrho^s$.

Proof. We limit ourselves to the proof of the first part of the statement, with $x = 0$. Let $\varrho(\lambda x_k)\to 0$ for every $\lambda>0$, then for an arbitrary $\lambda>0$ there exists an index k_0 such that $\varrho(\lambda x_k)<\frac{1}{\lambda}$ for $k\geq k_0$. Hence $|x_k|_\varrho\leq\frac{1}{\lambda}$ for $k\geq k_0$, and so $|x_k|_\varrho\to 0$. Conversely, let $|x_k|_\varrho\to 0$, then $|\lambda x_k|_\varrho\to 0$ for every $\lambda>0$. Taking $0<\varepsilon<1$, there exists a k_1 such that $|\lambda x_k|_\varrho<\varepsilon$ for $k\geq k_1$. Hence $\varrho(\lambda x_k)\leq|\lambda x_k|_\varrho<\varepsilon$ for $k\geq k_1$, and so $\varrho(\lambda x_k)\to 0$.

1.7. Definition. A pseudomodular ϱ in X will be called

a) right-continuous, if $\lim_{\lambda\to 1+}\varrho(\lambda x)=\varrho(x)$ for all $x\in X_\varrho$,

b) left-continuous, if $\lim_{\lambda\to 1-}\varrho(\lambda x)=\varrho(x)$ for all $x\in X_\varrho$,

c) continuous, if it is both right-continuous and left-continuous.

1.8. Theorem. Let ϱ be an s-convex pseudomodular in X. If ϱ is right-continuous, then the inequalities $\|x\|_\varrho^s<1$ and $\varrho(x)<1$ are equivalent for every $x\in X_\varrho$. If ϱ is left-continuous, then the inequalities $\|x\|_\varrho^s\leq 1$ and $\varrho(x)\leq 1$ are equivalent for every $x\in X_\varrho$.

This theorem follows from the above definition, immediately. Let us still remark that the equivalences mentioned in 1.8 are not generally valid without continuity assumptions on ϱ, as show the examples where X is the space of real numbers and $\varrho_1(x)=0$ for $0\leq x\leq 1$, $\varrho_1(x)=\infty$ for $x>1$, or $\varrho_2(x)=0$ for $0\leq x<1$, $\varrho_2(x)=\infty$ for $x\geq 1$, because $\|x\|_{\varrho_1}^s=\|x\|_{\varrho_2}^s=|x|^s$.

1.9. Examples. A function φ defined in the interval $[0,\infty)$, nondecreasing and continuous for $u\geq 0$ and such that $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$, $\varphi(u)\to\infty$ as $u\to\infty$, is called a φ-function. Let (Ω,Σ,μ) be a measure space, i.e. Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω and μ is a nonnegative, complete measure in Σ, which does not vanish identically.

I. Let X be the space of all real-valued (or complex-valued) Σ-measurable and μ-almost everywhere finite functions on Ω, with equality μ-almost everywhere. Then

$$\varrho(x)=\int_\Omega\varphi(|x(t)|)\,d\mu\quad\text{for}\quad x\in X$$

is a continuous modular in X. If φ is additionally an <u>s-convex function</u>, i.e. $\varphi(\alpha u+\beta v) \leqslant \alpha^s\varphi(u)+\beta^s\varphi(v)$ for $u,v \geqslant 0$, $\alpha,\beta \geqslant 0$, $\alpha^s+\beta^s=1$, where $0<s\leq 1$, then ϱ is an s-convex modular in X. The respective modular space X is called an <u>Orlicz space</u> and denoted $L^{\varphi}(\Omega,\Sigma,\mu)$ or briefly L^{φ}; we shall deal with Orlicz spaces and their generalizations in Chapter II.

II. Let Σ be a σ-algebra of subsets of a nonempty set Ω and let $\mathcal{T}$ be a nonempty set, which will be called the set of parameters. Let us suppose that to every $\tau\in\mathcal{T}$ there corresponds a nonnegative, complete measure μ_τ on Σ, which does not vanish identically. Let X be the space of all real-valued (or complex-valued) Σ-measurable functions, μ_τ-almost everywhere finite for every $\tau\in\mathcal{T}$, with equality μ_τ-almost everywhere for all $\tau\in\mathcal{T}$. Then

$$\varrho(x)=\sup_{\tau\in\mathcal{T}}\int_\Omega \varphi(|x(t)|)\,d\mu_\tau$$

is a modular in X. If φ is additionally s-convex, then ϱ is an s-convex modular in X. In case when the set $\mathcal{T}$ consists of one element only, Example II is reduced to Example I.

III. Let us yet consider the following special case of Example II. We take both Ω and $\mathcal{T}$ as the set of all positive integers, and as Σ the σ-algebra of all subsets of Ω. Given an infinite matrix $[a_{ni}]$ of nonnegative numbers such that for every i there exists an n for which $a_{ni}\neq 0$. For any $A\subset\Omega$ we define $\mu_n(A)=\sum_{i\in A} a_{ni}$. Then the modular from Example II becomes

$$\varrho(x)=\sup_n \sum_{i=1}^{\infty} a_{ni}\varphi(|t_i|) \quad \text{for} \quad x=(t_i).$$

If $a_{ni}=1$ for all i and n, we obtain

$$\varrho(x)=\sum_{i=1}^{\infty}\varphi(|t_i|),$$

and the resulting modular space X_ϱ is the <u>Orlicz sequence space</u> l^{φ}.

IV. Let $\mathcal{T}$ and X be defined as in Example II. Moreover, let $\mathcal{T}$ be a Hausdorff topological space, $\tau_0\notin\mathcal{T}$ and $\mathcal{T}_0=\mathcal{T}\cup\{\tau_0\}$. A topology in $\mathcal{T}_0$ is defined, taking as neighbourhoods of τ_0 the complements with respect to $\mathcal{T}_0$ of compact sets in $\mathcal{T}$. Then

$$\varrho(x)=\overline{\lim_{\tau\to\tau_0}}\int_\Omega\varphi(|x(t)|)\,d\mu_\tau = \inf_U \sup_{\tau\in U\setminus\{\tau_0\}}\int_\Omega \varphi(|x(t)|)\,d\mu_\tau,$$

where U runs over all neighbourhoods of τ_0 in $\mathcal{T}_0$, is a pseudomodular in X. If φ is additionally s-convex, then ϱ is an s-convex pseudomodular in X.

V. Let X be the space of all real-valued (or complex-valued) functions defined in an interval $[a,b]$ and vanishing at t = a. The value (possibly also ∞)

$$V_\varphi(x) = \sup_{\Pi} \sum_{i=1}^{m} \varphi(|x(t_i) - x(t_{i-1})|),$$

where the supremum is taken over all partitions $\Pi : a=t_0 < t_1 < \ldots < t_m = b$ of the interval $[a,b]$, is called the φ-variation of the function $x \in X$. The φ-variation V_φ is a modular in X, and in case of φ s-convex, is an s-convex modular in X.

VI. Turning back to Example 1.2.I of an s-normed space X with an s-norm $\|\ \|^s$, $0 < s \leq 1$, the s-norm $\|\ \|_\varrho^s$ defined by means of the modular $\varrho(x) = \|x\|^s$ as in Theorem 1.5 is equal to $\|\ \|^s$, i.e. $\|x\|_\varrho^s = \|x\|^s$ for all $x \in X_\varrho$.

We go now back to the definition of the s-pseudonorm $\|\ \|_\varrho^s$ in X_ϱ, given in 1.5, and we show that there exists another s-pseudonorm in X_ϱ equivalent to $\|\ \|_\varrho^s$.

1.10. Theorem. If ϱ is an s-convex pseudomodular, then the functional

$$|||x|||_\varrho^s = \inf_{\xi>0} \frac{1+\varrho(\xi^{1/s}x)}{\xi} = \inf_{u>0} u\left(1+\varrho\left(\frac{x}{u^{1/s}}\right)\right)$$

is an s-pseudonorm in X_ϱ and

$$\|x\|_\varrho^s \leq |||x|||_\varrho^s \leq 2\|x\|_\varrho^s$$

for every $x \in X_\varrho$.

Proof. In order to prove the triangle inequality for $|||\ |||_\varrho^s$, let $x,y \in X_\varrho$ and a number $a > 0$ be given. Then there exist $u,v > 0$ such that

$$\varrho\left(\frac{x}{u^{1/s}}\right) < \frac{|||x|||_\varrho^s + a}{u} - 1, \quad \varrho\left(\frac{y}{v^{1/s}}\right) < \frac{|||y|||_\varrho^s + a}{v} - 1.$$

Hence

$$\varrho\left(\frac{x+y}{(u+v)^{1/s}}\right) \leq \frac{u}{u+v}\varrho\left(\frac{x}{u^{1/s}}\right) + \frac{v}{u+v}\varrho\left(\frac{y}{v^{1/s}}\right) < \frac{|||x|||_\varrho^s + |||y|||_\varrho^s + 2a}{u+v} - 1,$$

and so

$$|||x+y|||_\varrho^s \leq (u+v)\left[1 + \varrho\left(\frac{x+y}{(u+v)^{1/s}}\right)\right] < |||x|||_\varrho^s + |||y|||_\varrho^s + 2a.$$

Thus $|||x+y|||_\varrho^s \leq |||x|||_\varrho^s + |||y|||_\varrho^s$. The condition $|||\alpha x|||_\varrho^s = |\alpha|^s |||x|||_\varrho^s$ is verified similarly as in the proof of 1.5. In order to prove the inequalities for norms let us first observe that $\varrho(z) > 1$ implies $\varrho(z) \geq \|z\|_\varrho^s$

for any $z \in X_\varrho$. Indeed, in the converse case we would have

$$1 < \varrho\left(\frac{x}{\varrho(x)^{1/s}}\right) \leqslant \frac{1}{\varrho(x)}\ \varrho(x) = 1,$$

a contradiction. Thus, supposing $\varrho\left(\frac{x}{u^{1/s}}\right) > 1$, $u > 0$, we obtain

$$\varrho\left(\frac{x}{u^{1/s}}\right) \geqslant \left\|\frac{x}{u^{1/s}}\right\|_\varrho^s = \frac{1}{u}\|x\|_\varrho^s,$$

and so

$$u\left(1 + \varrho\left(\frac{x}{u^{1/s}}\right)\right) \geqslant u\left(1 + \frac{1}{u}\|x\|_\varrho^s\right) \geqslant \|x\|_\varrho^s.$$

Moreover, if $\varrho\left(\frac{x}{u^{1/s}}\right) \leqslant 1$, then $\|x\|_\varrho^s \leqslant u$, and again

$$u\left(1 + \varrho\left(\frac{x}{u^{1/s}}\right)\right) \geqslant u \geqslant \|x\|_\varrho^s.$$

Hence $\|x\|_\varrho^s \leqslant |||x|||_\varrho^s$. Now, let us take an arbitrary $a > 0$ and let us write $u = \|x\|_\varrho^s + a$, then $\varrho\left(\frac{x}{u^{1/s}}\right) \leqslant 1$, and so

$$u\left(1 + \varrho\left(\frac{x}{u^{1/s}}\right)\right) \leqslant 2u = 2\|x\|_\varrho^s + 2a.$$

Hence $|||x|||_\varrho^s \leqslant 2\|x\|_\varrho^s + 2a$, and since a is arbitrary, we obtai $|||x|||_\varrho^s \leqslant 2\|x\|_\varrho^s$.

In case of $s = 1$, the norm $\|\ \|_\varrho = \|\ \|_\varrho^1$ in X_ϱ is called the <u>Luxemburg norm</u> and the norm $|||\ |||_\varrho = |||\ |||_\varrho^1$ in X_ϱ is called the <u>Amemiya norm</u>. Applying the notion of the conjugate modular ϱ^* we shall introduce in § 2 still another norm $\|\ \|_\varrho'$, called the Orlicz norm.

§ 2. Conjugate modulars

Let ϱ be a convex pseudomodular in a (real or complex) vector space and let X_ϱ^* be the space of all linear, continuous functionals (real or complex) over the normed space $\langle X_\varrho, \|\ \|_\varrho\rangle$ with $\|x\|_\varrho = \inf\{u > 0: \varrho\left(\frac{x}{u}\right) \leqslant 1\}$. Following H.Nakano [1], we shall define a semimodular ϱ^* on X_ϱ^* and investigate the resulting modular space. First, we prove that

2.1.<u>Theorem</u>.A linear functional x^* over X_ϱ is continuous with respect to the norm $\|\ \|_\varrho$, if and only if, there exists a constant $\gamma > 0$ such that $|x^*x| \leqslant \gamma(\varrho(x)+1)$ for every $x \in X_\varrho$.

<u>Proof</u>. Supposing the above condition to be satisfied,we have for

every $\varepsilon > 0$

$$\left|x^*\left(\frac{x}{\|x\|_\varrho + \varepsilon}\right)\right| \leq \gamma\left(\varrho\left(\frac{x}{\|x\|_\varrho + \varepsilon}\right) + 1\right) \leq 2\gamma ,$$

and so $x^* \in X^*_\varrho$. Conversely, if $x^* \in X^*_\varrho$, then taking $\|x\|_\varrho \leq 1$ and writing $\gamma = \|x^*\|$, we have $|x^*x| \leq \gamma \|x\|_\varrho \leq \gamma \leq \gamma(\varrho(x)+1)$. Taking $\|x\|_\varrho > 1$ we see, as in the proof of 1.10, that $\varrho(x) \geq \|x\|_\varrho$. Hence $|x^*x| \leq \gamma \| x \|_\varrho \leq \gamma\varrho(x) \leq \gamma(\varrho(x)+1)$.

2.2. <u>Definition</u>. The <u>conjugate</u> <u>modular</u> ϱ^* to ϱ is defined by the formula

$$\varrho^*(x^*) = \sup_{x \in X_\varrho} \left(|x^*x| - \varrho(x)\right) \quad \text{for} \quad x^* \in X^*_\varrho .$$

2.3. <u>Theorem</u>. If ϱ is a convex pseudomodular over X, then ϱ^* is a convex, left-continuous semimodular over X^*_ϱ.

<u>Proof</u>. It is easily seen that $\varrho^*(0) = 0$ and $\varrho^*(-x) = \varrho^*(x)$, $\varrho^*(x) \geq 0$ for every $x^* \in X^*_\varrho$. Moreover, taking $x^*, y^* \in X^*_\varrho$ and $\alpha, \beta \geq 0, \alpha+\beta = 1$, we have

$$\varrho^*(\alpha x^* + \beta y^*) \leq \sup_{x \in X_\varrho}\left(\alpha|x^*x| - \alpha\varrho(x) + \beta|y^*x| - \beta\varrho(x)\right) \leq \alpha\varrho^*(x^*) + \beta\varrho^*(y^*) .$$

Finally, let $\varrho^*(\lambda x^*) = 0$ for every $\lambda > 0$. Because $|y^*y| \leq \varrho(y) + \varrho^*(y^*)$ for all $y \in X_\varrho$, $y^* \in X^*_\varrho$, so taking an arbitrary $x \in X_\varrho$ and an $\eta > 0$ such that $\varrho(\eta x) < \infty$, we have

$$\lambda\eta|x^*x| = |\lambda x^*(\eta x)| \leq \varrho(\eta x) + \varrho^*(\lambda x^*) = \varrho(\eta x).$$

Hence $|x^*x| \leq \varrho(\eta x)/\lambda\eta$. Taking $\lambda \to \infty$, we obtain $x^*x = 0$. Hence $x^* = 0$. The left-continuity of ϱ^* follows from the fact that $\lim_{\lambda \to 1-} \varrho^*(\lambda x^*) = \sup_{0<\lambda<1} \sup_{x \in X_\varrho} \left(\lambda|x^*x| - \varrho(x)\right) = \varrho^*(x^*)$.

From the above theorem it follows that one may define in X^*_ϱ a norm generated by means of the semimodular ϱ^*:

$$\|x^*\|_{\varrho^*} = \inf\left\{u > 0 : \ \varrho^*\left(\frac{x^*}{u}\right) \leq 1\right\} \quad \text{for} \quad x^* \in X^*_\varrho .$$

Besides this norm, supposing ϱ to be a convex semimodular in X, one may define in X^*_ϱ a norm

$$\|x^*\|^*_\varrho = \sup\{|x^*x| : \|x\|_\varrho \leq 1, \ x \in X_\varrho\} .$$

In order to investigate the connection between norms $\|x^*\|^*_\varrho$ and $\|x^*\|_{\varrho^*}$ we prove first the following

2.4. <u>Lemma</u>. If ϱ is a convex pseudomodular in X, then $\|x\|_\varrho > 1$ implies $\varrho(x) \geq \|x\|_\varrho$.

<u>Proof.</u> Let $1 < k < \|x\|_\varrho$, then $\varrho\left(\frac{x}{k}\right) > 1$ and we obtain $\frac{1}{k}\varrho(x) \geq$

$\varrho\left(\frac{x}{k}\right) > 1$, i.e. $\varrho(x) > k$. Hence we conclude $\varrho(x) \geq \|x\|_\varrho$.

2.5. Theorem. If ϱ is a convex, left-continuous semimodular in X, then $\|x^*\|_{\varrho^*} \leq \|x^*\|_\varrho^* \leq 2\,\|x^*\|_{\varrho^*}$ for every $x^* \in X_\varrho^*$.

Proof. By Theorem 1.8, the inequalities $\|x\|_\varrho \leq 1$ and $\varrho(x) \leq 1$ are equivalent. Hence $\|x^*\|_\varrho^* \leq \sup\limits_{\varrho(x)\leq 1, x\in X_\varrho} (\varrho(x) + \varrho^*(x^*)) = 1 + \varrho^*(x^*)$. Supposing $\|x^*\|_{\varrho^*} \leq u$, we have $\varrho^*\left(\frac{x^*}{u}\right) \leq 1$ and so $\|x^*\|_\varrho^* \leq 2u$, which implies the right-hand side inequality. In order to prove the left-hand one, let us first remark that if $x^* \in X_\varrho^*$ and $\|x^*\|_\varrho^* \leq 1$, then $\varrho^*(x^*) = \sup\limits_{\varrho(x)\leq 1, x\in X_\varrho} (|x^*x| - \varrho(x))$. Indeed, we have

$$\sup_{\varrho(x)>1} (|x^*x| - \varrho(x)) \leq \sup_{\varrho(x)>1} (\|x^*\|_\varrho^* \|x\|_\varrho - \varrho(x)) \leq \sup_{\varrho(x)>1} (\|x\|_\varrho - \varrho(x)) \leq 0,$$

by Lemma 2.4. Hence

$$\varrho^*(x^*) = \max\left[\sup_{\varrho(x)\leq 1} (|x^*x| - \varrho(x)), \sup_{\varrho(x)>1} (|x^*x| - \varrho(x))\right] = \sup_{\varrho(x)\leq 1} (|x^*x| - \varrho(x)).$$

Now, since $\|x^*/\|x^*\|_\varrho^*\|_\varrho^* = 1$ for $x \neq 0$, we have

$$\varrho^*\left(\frac{x^*}{\|x^*\|_\varrho^*}\right) = \sup_{\varrho(x)\leq 1} \left(\frac{|x^*x|}{\|x^*\|_\varrho^*} - \varrho(x)\right) \leq \sup_{\varrho(x)\leq 1} \left(\frac{\|x^*\|_\varrho^* \|x\|_\varrho}{\|x^*\|_\varrho^*} - \varrho(x)\right) \leq \sup_{\varrho(x)\leq 1} (\|x\|_\varrho - \varrho(x)) \leq 1.$$

But this implies the required inequality.

Applying the conjugate modular ϱ^* we define now the Orlicz norm $\|\ \|_\varrho'$ in X_ϱ by means of the formula given below:

2.6. Theorem. If ϱ is a convex pseudomodular in X, then

$$\|x\|_\varrho' = \sup\{|x^*x| : x^* \in X_\varrho^*, \ \varrho^*(x^*) \leq 1\}$$

is a pseudonorm in X_ϱ and $\|x\|_\varrho' \leq |||x|||_\varrho$ for every $x \in X_\varrho$. If ϱ is a convex semimodular, then $\|\ \|_\varrho'$ is a norm in X_ϱ.

Proof. Let $x \in X_\varrho$. Since $|x^*x| \leq \varrho^*(x^*) + \varrho(x)$ for every $x^* \in X_\varrho^*$, we obtain $\|x\|_\varrho' \leq 1 + \varrho(x)$ and so $0 \leq \|x\|_\varrho' < \infty$. Moreover, it is easily seen that $\|\alpha x\|_\varrho' = |\alpha|\,\|x\|_\varrho'$ for every number α. Hence, taking $\xi > 0$, we have

$$\xi\,\|x\|_\varrho' = \|\xi x\|_\varrho' \leq 1 + \varrho(\xi x).$$

Thus $\|x\|_\varrho' \leq \xi^{-1}(1 + \varrho(\xi x))$ for every $\xi > 0$, and consequently, $\|x\|_\varrho' \leq |||x|||_\varrho$. Since the triangle inequality for $\|\ \|_\varrho'$ is obvious, we conclude that $\|\ \|_\varrho'$ is a pseudonorm in X_ϱ. Now, let us suppose that $\|x\|_\varrho' = 0$. Then $x^*x = 0$ for every $x^* \in X_\varrho^*$ satisfying $\varrho^*(x^*) \leq 1$. Let us take an $x^* \in X_\varrho^*$ with $\varrho^*(x^*) > 1$, then taking $y^* = x^*/\varrho^*(x^*)$, we have $\varrho^*(y^*) \leq 1$, by convexity of ϱ^*. Hence $y^*x = 0$, but this implies $x^*x = 0$. Consequently, we obtain $x = 0$.

§ 3. F-modular spaces

As we have seen in Theorem 1.5, the F-pseudonorm $|\ |_\varrho$ in case of a general pseudomodular ϱ and the s-norm $\|\ \|_\varrho^s$ in case of an s-convex modular are given by different formulae and both theories are going parallel, but not identical lines. There will be given now a unified theory, embracing both cases. For this purpose we define an operation on the real halfline $R_+ = [0,\infty)$, which will be denoted by means of the symbol $F(u,v)$ or $u \oplus v$ for $u,v \in R_+$.

3.1. Definition. An F-operation in R_+ will be defined as a function $F: R_+ \times R_+ \to R_+$ having the following properties for $u,v \in R_+$:

1° $F(u,v) = F(v,u)$, 2° $F(u,F(v,w)) = F(F(u,v),w)$, 3° $F(u,0) = F(0,u) = u$,

4° $F(u,v)$ is nondecreasing as a function of each of the variables, separately,

5° F is continuous from $R_+ \times R_+$ to R_+.

We shall write also $u \oplus v = F(u,v)$. The operation $\oplus$ may be extended to any finite number of terms by means of the formulae

$$\bigoplus_{i=1}^{1} u_i = u_1, \quad \bigoplus_{i=1}^{2} u_i = u_1 \oplus u_2, \quad \bigoplus_{i=1}^{n} u_i = F\left(\bigoplus_{i=1}^{n-1} u_i, u_n\right) \text{ for } n>2.$$

3.2. Properties. Obviously, we have $F(u,v) \geqslant 0$ and $F(0,0) = 0$. The following properties of F-operations are immediate:

I. If $0 \leqslant u_1 \leqslant u_2$ and $0 \leqslant v_1 \leqslant v_2$, then $F(u_1,v_1) \leqslant F(u_2,v_2)$.

II. If F_n are F-operations for $n = 1,2,\ldots$ and $F_n(u,v) \to F(u,v)$ as $n \to \infty$ uniformly in $R_+ \times R_+$, then F is an F-operation.

3.3. Examples. I. $F_1(u,v) = u+v$, II. $F_p(u,v) = (u^p+v^p)^{1/p}$, $p \geqslant 1$,

III. $F_\infty(u,v) = \max(u,v)$,

IV. $$F(u,v) = \begin{cases} u+v & \text{for } u+v<1,\ u<1,\ v<1 \\ 1 & \text{for } u+v\geqslant 1,\ u<1,\ v<1 \\ \max(u,v) & \text{for } u\geqslant 1 \text{ or } v \geqslant 1. \end{cases}$$

3.4. Remark. F_∞ is the smallest possible F-operation, i.e. $F(u,v) \geqslant F_\infty(u,v)$ for any F-operation F and all $u,v \geqslant 0$. Indeed, we have $F(u,v) \geqslant F(u,0) = u$ and $F(u,v) \geqslant F(0,v) = v$, and so $F(u,v) \geqslant \max(u,v) = F_\infty(u,v)$.

3.5. Definition. Let $h: R_+ \to R_+$ be a homeomorphism of R_+ onto itself and let $G = h^*F$ be defined by $G(u,v) = h^{-1}F(h(u),h(v))$ for $u,v \in R_+$, where h^{-1} is the inverse function to h. If such a homeomorphism h