

Mathematical Papers



William Kingdon Clifford and Robert
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MATHEMATICAL PAPERS

BY

WILLIAM KINGDON CLIFFORD.

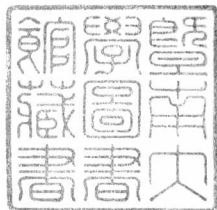
EDITED BY ROBERT TUCKER,

WITH AN INTRODUCTION BY H. J. STEPHEN SMITH.

"If he had lived we might have known something."

London:
MACMILLAN AND CO.

1882



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MY DEAR MRS CLIFFORD,

When towards the end of April 1879 I called upon you in answer to an urgent request to do so, it was with much surprise that I heard from you that it was one of your lamented Husband's last requests in regard to his work that I should be one of two persons to be asked to see the remaining Manuscript of his admirable book on *Dynamic* through the press. Though much gratified by his kind approval of what I had written upon the portion already in print, I could hardly look upon myself as qualified to undertake such a task—who indeed could hope to succeed in fitly editing fragments which had not received his last energizing touches? Whatever diffidence may have at first deterred me, yielded on further reflection to your pressing solicitation.

At this same interview you told me that it had been suggested, I believe by Mr Spottiswoode and Mr Macmillan, that a collected edition of Clifford's Mathematical Works, including the Memoirs already printed as well as such posthumous papers as might

be found in a state fit for publication, should be brought out under the auspices of a double editorship, one editor to be a resident at Cambridge and the other to be myself. If I hesitated to undertake the editing of the Manuscript of the *Dynamic*, you may believe I was even more afraid to undertake so great a responsibility as the editing of the Papers. Finding however at an interview I had with Dr Jack that it was desired that the papers should be brought out with all speed, and learning from Mr Spottiswoode that the plan of a double editorship had fallen through, I consented to do the best I could, feeling the less hesitation in making the attempt as I was able to obtain the assistance of Mr Spottiswoode himself and of Professors Cayley, Henrici and Smith.

Thus much of explanation I feel to be due from me to show that I did not engage in so great an undertaking of my own mere motion, but that I entered upon my task with no slight idea of its magnitude, although the actuality has far surpassed my expectation.

Your own assistance and kind interest in the progress of the work have been most valuable and most encouraging to me ; and I can now only wish that the result of my efforts may deserve your approval, and may be not unworthy of the illustrious mathematician, your much loved Husband.

" If I have done well, it is that which I desired : but if slenderly and meanly, it is that which I could attain unto."

I need not enter into any account of the causes which have operated to bring about so tardy a completion of my labours. Sufficient it is for me to know that you have understood and allowed for the many hindrances which have arisen from circumstances connected with the fragmentary condition of some of the papers, and from the pressing claims upon my own time and upon that of the eminent mathematicians, whose advice and assistance have so ungrudgingly been extended to me.

Thanking you, my dear Mrs Clifford, for the honour you have conferred upon me in entrusting to me the work of raising this monument to the memory of one you love so well,

I remain,

Yours faithfully,

R. TUCKER.

December 30th, 1881.

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Papers now published for the first time are distinguished by an * prefixed to the number indicating their order in this volume.

CORRIGENDA.

On p. 22, for xv. read xvii.

„ 114, 6 up, read convertical.

„ 134, in connexion with the formulæ, refer to *Quarterly Journal of Mathematics*, No. 25. For this reference I am indebted to Mr J. J. Walker, who also suggests the following corrections “on p. 135, 4 up, for κ read $\frac{1}{2}$; p. 136, omit $\left[\frac{1}{\kappa}\right]$, l. 10, - 2, l. 11.”

„ 205, r is v in Clifford's MS. The paper is printed as in the *London Math. Society's Proceedings*.

„ 621, l. 12, read vol. xxxv. p. 21.

PREFACE:

I. BIOGRAPHICAL.

II. BIBLIOGRAPHICAL.

BY

ROBERT TUCKER, M.A.,

HON. SEC. TO LONDON MATHEMATICAL SOCIETY,
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I.

WILLIAM KINGDON CLIFFORD,

Born at Exeter, May 4th, 1845. Died at Madeira, March 3rd, 1879.

AFTER the touching Biography prefixed to the *Lectures and Essays*, written with consummate literary skill by one who knew and loved Clifford well—"as his own soul,"—I feel it is unnecessary for me to say more than a few words as a preface to the present collection of his Mathematical Papers, and in the few following lines I shall confine myself to the simple statement of facts bearing upon his life and work as a Mathematician.

Clifford received his early education at Mr Templeton's school in Exeter, where in 1858 and 1859 he gained numerous distinctions in a very extended range of subjects at the University Local Examinations, both of Oxford and Cambridge. From this school he proceeded, in 1860, to King's College, London, where in like manner his efforts were crowned with success in more than one branch of study. Having in 1863 obtained a minor scholarship, he entered at Trinity College, Cambridge; at the end of his course in 1867 he graduated Second Wrangler and gained the Second Smith's Prize. In 1868, he was elected a Fellow of his College. He proceeded to M.A. in 1870; joined the Eclipse Expedition to Sicily of the same year, and was in 1871 elected Professor of Applied Mathematics and

Mechanics in University College, London, a post he occupied until the time of his death.

In June, 1874, Clifford was elected a Fellow of the Royal Society. He became a Member of the London Mathematical Society, June 18th, 1866, and served on its Council for every session from 1868—9 to 1876—7. Though he then ceased to be a member of the Council, he continued to take a warm interest in the proceedings of the Society, and on many a night he was present at the meetings when he was far from well and ought to have remained at home. I go back in imagination to the first meeting he attended—his was ever a welcome face in our room at University College and subsequently at Burlington House and Albemarle Street—it was on the evening of January 23rd, 1868, when Dr Hirst, our Treasurer, occupied the chair in the absence of our then President, Prof. Sylvester. Of the little band then present four have since been removed by death, viz., Clifford, Clerk Maxwell, Archibald Smith, and Thomas Cotterill. Clerk Maxwell made two communications: i. "on the construction of stereograms of surfaces," ii. "on the doctrine of Reciprocal Diagrams of Forces with the extension of Airy's Function of Stress from two dimensions to three." The former was made interesting by the fact that the author brought with him a real Image Stereoscope constructed after his own directions, and which invested the stereograms with a marvellous resemblance to the solids of which they were the plane presentments¹. It is possible that Clifford spoke upon both these papers, but I have no record of his having done so. Mr J. J. Walker read a third paper in connection with which Clerk Maxwell asked him if he could point out a method of determining in what cases all the possible parts of the impossible roots of an equation are negative. In studying the motion of governors for regulating machines, he had found that the stability of the motion depended on this

¹ These diagrams may be seen in Vol. ix of the *Quarterly Journal of Pure and Applied Mathematics*. The paper is entitled "On the Cyclide."

condition, which is easily obtained for a cubic but is more difficult to find in the case of equations of higher degrees. Clifford at once said that we obtain the condition required by forming an equation whose roots are the sums of the roots of the original equation taken in pairs and determining the condition that the real roots of this equation should be negative. This may serve to give an idea of his readiness and helpfulness at our meetings. He rarely rose to speak of his own accord, but when a direct appeal was made to him he was ever ready to contribute a few pertinent remarks of an eminently suggestive kind, often showing in the course of the discussion that he had carefully considered the subject for himself. In this way he often threw out hints which enabled others to advance still further the researches which they had hitherto almost looked upon as their own. His readiness is exemplified in the following extract from a letter I have received from his private tutor, the Rev. Percival Frost¹: "We were capital friends, yet I was so much engaged with a large number of pupils that I did not see very much of him except in a professional way. Even when he came to see me out of his working hours we used to get upon some mathematical curiosity, and both being fond of mathematics for their own sakes, we have often pursued our amusement into the small hours—once between 2 and 3—for which his tutor called him to account, good-naturedly excusing him when he heard of how he had been occupied. He often used to amuse me by solving in his head difficult problems, when some conversation like the following would take place. *Fr.* The men in the next room tell me this problem won't come out: there must be a mistake: just read it over and tell me where the setter has blundered. *Cl.* (reads it over and thinks a few minutes) I see how it is, he has, &c., &c."

Few anecdotes of Clifford's young days which show any foreshadowing of his future mathematical power have reached

¹ See also *Nature* (March 13th, 1879).

me. The following illustration, though trifling, may find a place here. His aunt (Mrs McLeod) writes: "In the year of the first Exhibition, 1851, his parents came to town, and I had the charge of him with my own child. When putting him to bed one night, I saw dear Willie looking very thoughtful. When I asked him what it was all about, he looked up in his smiling, loving way, saying, 'Aunt Annie, I don't think you would know.' But on my asking again, he said he was calculating 'how many edges (*sharp*) of a penknife it would take to go round the wheel of a coach.'

"We had a little talk about it, for it seemed impossible to me he could arrive at any conclusion. He then gave the figures and I daresay the size of the wheel: but all that I have long forgotten." The question and answer were submitted to an uncle, the late Mr Frank Kingdon, who was no mean mathematician, and he said the result was correct within a few figures.

A passage in Mr Pollock's Introduction to the *Lectures and Essays* (p. 41) would lead one to infer that Clifford turned his attention to kites at a somewhat late period of his life: it is however clear from a letter he himself wrote to Mr Miller that he had practised the flying of kites when a boy, for in 1863 he says "I have had in my mind, almost from the time I began to fly kites (I have not yet left off), the problem of finding the form of a kite-string under the action of the wind. On a rough trial, the other day, the intrinsic equation seemed not very difficult to obtain; if I get any result, I will send it to you hereafter¹."

One more anecdote which will give an idea of the quick-

¹ The question figures as 6009 in the *Educational Times*, July 1879, May 1880 (Reprint, vol. xxxiii. p. 59). In the same letter he adds, "I have been trying to construct a second interpretation of mechanical equations similar to that of tangential co-ordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong." Cf. *Lectures and Essays*, p. 41, "his thoughts often ran upon mechanical inventions."

ness and clearness of his perception of complicated relations in space, must close this brief sketch. Mr Frost, in his previously cited letter, goes on to say, "My brother A. H. Frost, who was in England for a short holiday from his missionary work in India, brought with him a complicated puzzle which was to be taken to pieces. It was not a two-dimension one, like many, but solid : about as big as a good snowball. My brother said, 'I have heard you talk of the wonderful capacity of Clifford, prove it to me by asking him to tea, and I will believe you if he can take my puzzle to pieces.' I accordingly asked him, and on my brother's giving him the thing, he, without fingering it but simply looking it over for a few minutes, put his head in his hands for some ten minutes and then took hold of the puzzle, and at once, to my brother's astonishment, dislocated it, and my brother believed in him ever afterwards."

These few personal details of one who stood in the front rank of the mathematicians of our time, though trivial, will not, I think, be deemed out of place in such a volume as this. Ever kindly and unselfish, Clifford maintained in his relations with his brother mathematicians the same amiable bearing which endeared him to a large circle of private friends.

II.

IF it were possible to ascertain the chronological order of creation of Clifford's papers, that order would have been followed in this work: but lacking the guidance of the 'vanished hand,' I soon found that I could not attempt such an arrangement and reconciled myself, unwillingly I must say to the publication of them in almost any order. In adopting this plan I received the approval of all the mathematicians whom I consulted upon the matter. In the following Bibliographical sketch, I have endeavoured to overcome the defects of this arrangement by assigning in all cases, to the best of my ability, the lower limit of publication or, in the case of posthumous papers, of composition. It is possible that some readers may be able to furnish data for correcting my statements: such information will be gratefully accepted.

When the number of a "Reprint" problem is given in black type, this shows that the *solution* is Clifford's, but not the *problem**. The titles of unpublished papers are printed in italics.

The MSS even the roughest, are all very neatly written and by the style of penmanship indicate four or five different epochs: it is by attention to this fact that I have been led to assign the respective dates to the posthumous papers.

* "Reprint" stands for "Mathematical Questions, with their Solutions, from the *Educational Times*," edited by W. J. C. Miller; "L. and E." stands for "Lectures and Essays," edited by Leslie Stephen and F. Pollock.

A lithographed specimen of the smallest handwriting faces the title-page*.

1863.

"On Jacobians and Polar Opposites," vi. pp. 23—33.

"Analogues of Pascal's Theorem," Nov. 20th, x. pp. 72—79.

Reprint, 1362, **1373**, 1378, **1379**, 1387, **1389**, **1393**, 1399, 1409, 4097 (1415)†, **1418**, 1423, 1448, 4143 (1459).

1864.

"Analytical Metrics" (see footnote, p. 80) xi. pp. 80—109.

"Geometrical Theorem," xlvii. pp. 410, 411.

Reprint: **1319**, **1394**, **1416**, **1421**, **1442**, **1443**, 1468, 1479, 1486, **1505**, 1507, **1514**, **1517**, **1519**, 2108 (1526), 1585, 1605.

1865.

"On Triangular Symmetry," xlviii. pp. 412—414.

Reprint: 1497, 1638, 1652, 1675, **1679**, **1680**, 1691, 1724, **1732**, **1733**, 1748, 1750, 1775, 1795, 1823.

1866.

"On some extensions of the fundamental proposition in M. Chasles's Theory of Characteristics," March, xlix. pp. 415—418.

"On the General Theory of Anharmonics," Nov. 22. xii. pp. 110—114.

"Bitangent Circles of a Conic," pp. 543—545, and

"Of Power-coordinates in General," pp. 546—555, may belong to this year.

Reprint: 1878, **1888**, 4199 (1907), 1918, 1929, 1962, 1996, 2220, 2229, 2253, 4696 (2281), 2301.

* Professor Henrici tells me that Clifford proposed to him that they should write in conjunction a series of books on Mathematics, beginning at the very commencement and carrying the subject in each case rapidly to the most advanced stages. The question was often discussed during the years in which Clifford was in full health, but this sheet is all that remains. Professor Henrici's occupation as Examiner to the London University prevented his actively commencing his part of the work. The short abstract xi. p. 650, is suggestive as to the place which some of the papers published in this volume were intended to take in a more extended scheme.

† The number in the bracket is that belonging to the first proposal of the question, the other number is that of the re-proposed question to which the solution was given.

On Question 2229, M. Chasles (Rapport sur les Progrès de la Géométrie, p. 353), writes :

"Ces surfaces anallagmatiques du quatrième ordre n'ont point tardé à fixer l'attention des géomètres. M. W. K. Clifford, notamment, en a fait connaître plusieurs propriétés (voir *Ed. Times*, Sept. 1866, p. 134)."

1867.

"On the Principal axes of a Rigid body," vii. pp. 34—37.
Reprint: 2343, 2383, 2446, 2510, 2522.

1868.

"On some of the conditions of Mental Development," March 6.
L. and E., Vol. i., pp. 75—108.
"On some Porismatic Problems," Nov. 9, iii. pp. 17—19.
"On a general investigation of the theory of Polars," Nov. 26. xiii.
pp. 115—118.
"On the Powers of Spheres" (?) *xxxiv. pp. 332—336. I assign the origin of this paper to this year, see my note, p. 332, though I am disposed to think that the paper was not actually written out long before 1876.
Reprint: 3885 (2674), 2732, 2748, 2776, 2793.

1869.

"On the Theory of Distances," xvi. pp. 130—164.
"On Syzygetic relations among the powers of Linear Quantics,"
Nov. 25. xiv. pp. 119—122.
"On Syzygetic relations connecting the powers of Linear Quantics."
*xv. pp. 123—129.
"*On the Umbilici of Anallagmatic surfaces.*" Title only given in the British Association Report for this year.
"*Lectures on Geometry,*" given to a Class of Ladies at S. Kensington.
Syllabus given, pp. 628—637.
"On Boundaries in General," printed at end of "Seeing and Thinking," see *infra*; also published in Macmillan's Magazine, August, 1879.
"Analysis of Cremona's Transformations" (?) pp. 538—542.
Reprint: 4236 (2817), 2858, 2923, 2924, 2932, 2942, 2960, 2979, 5626 (3000), 3021.

1870.

"On a case of Evaporation in the order of a Resulant," Feb. 10, xvii. pp. 165—167.

"On Theories of the Physical Forces," Feb. 18. L. and E., Vol. 1. pp. 100—123.

"Proof that every rational Equation has a root," Feb. 21. iv. p. 20.

"On the Space-theory of Matter," Feb. 21. v. pp. 21, 2.

"Synthetic Proof of Miquel's Theorem," March. viii. pp. 38—51.

"On an unexplained contradiction in Geometry." Title only in British

Association Report of this year.

"Lecture Notes," pp. 524—530.

Reprint : 3197, 3253, 3282.

1871.

"The History of the Sun: being an explanation of the nebular hypothesis and of recent controversies in regard to the time which can be allowed for the evolution of life." April 16. L. and E. Vol. 1. p. 68.

"On a Canonical form of Spherical Harmonics." August. xxv. pp. 234, 5.*

"Note on the Secular Cooling and the Figure of the Earth." Title only in British Association Report for this year.

Reprint, 4034 (3308).

1872.

"Atoms." Jan. 7. L. and E., Vol. 1. pp. 158—190.

"Either, the evidence for its Existence and the Phenomena it explains." April 14. L. and E., Vol. 1. p. 68.

"Remarks on a Theory of the Exponential Function derived from the equation $\frac{du}{dt} = pu$." May 9. xlv. p. 406.

"On Babbage's Calculating Machines." May 24. L. and E., Vol. 1. p. 69.

"On the Aims and Instruments of Scientific Thought." August. L. and E., Vol. 1. pp. 124—157.

"In Nature, Sept. 7, there is little more than the title of this paper given.