VOLUME TWO

# CALCULUS
## AND ANALYTIC GEOMETRY

FOBES AND SMYTH

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**MELCHER P. FOBES**

*Professor and Chairman*
*Department of Mathematics*
*The College of Wooster*

**RUTH B. SMYTH**

*Associate Professor*
*Department of Mathematics*
*The College of Wooster*

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*to our teachers:*

*EDWARD S. HAMMOND and CHESTER H. YEATON*



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*VOLUME TWO*

# CALCULUS
# AND ANALYTIC GEOMETRY

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*Albert A. Bennett, editor*

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# PREFACE

The present volume continues the plan of its predecessor by completing a two-year course in calculus and analytic geometry suitable as preparation for a standard, rigorous course in advanced calculus. Whereas the first volume furnished the basic plane analytic geometry and the elementary techniques and applications of differentiation and integration of functions of one argument, this extends the calculus of these functions and discusses their series expansion, and introduces partial differentiation, multiple integration, and the concomitant solid analytic geometry.

Though the text, as in the previous volume, tries to make careful, understandable, thorough discussion of the topics under consideration, the level of rigor and sophistication has increased somewhat, to go along with the growing mathematical maturity of the student. It is our hope that by this time he will be trained to care for precise statement of hypotheses and conclusion in theorems, and thoughtful analysis of the points in a proof at which various hypotheses are used.

A supplement to the chapters on infinite series provides a few carefully done $\epsilon$- and $\delta$-proofs of limit theorems to introduce the student to these techniques, though we feel that the complete and organized discussion of limits and continuity belongs to the realm of the advanced calculus course, in which the student who continues his mathematics can grasp it in half the time and with all the more understanding from having had repeated experience with such concepts as they develop throughout these two volumes.

The sections on "Things to Think About" continue to provide supplementary material for good students to ponder, and, while they include topics of widely varying degrees of difficulty, contain a good many significant ideas which perhaps do not belong to the body of the course but add to it, illuminate it, and form a bridge between it and the advanced calculus. Some such topics are the osculating circle and the involute (Chapter 26), rearrangement of series and Raabe's test (Chapter 28), convergence or divergence of the binomial series at endpoints of its interval of convergence (Chapter 29), osculating planes, principal normals, and binormals (Chapter 31), and a careful discussion of the differential (Chapter 32).

As with the previous volume, we owe real thanks to our colleagues who have taught this book, our students who have taught us, by their responses, what was good or bad about its original drafts, Prof. A. A. Bennett who read it critically and made most helpful suggestions, and all the people at Prentice-Hall who helped with the endless details attendant on the birth of a book. It is our hope that all these combined efforts may have a hand in the making of some good mathematicians.

M. P. F.

R. B. S.

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# 24

# POLAR COORDINATES

## 24.1 A NEW WAY OF LOCATING POINTS

We have now studied the differentiation and integration of the elementary functions of a single variable in some detail and have become acquainted with at least some of the principal simple uses to which these processes can be put in geometry and physics.

The next way in which we can broaden our viewpoint is by studying a few new and common ways in which curves and their corresponding functions can be represented, for there are some important curves and regions bounded by curves which are awkward to handle in rectangular coordinates. We start with a new way of locating points in a plane—by the use of *polar coordinates.*

Instead of a set of two mutually perpendicular lines as axes, we use in our plane a half-line, or *ray,* starting at a point $O$ and extending indefinitely in one direction from that point. It is

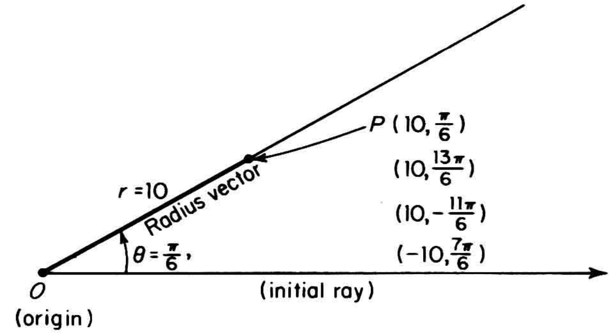


FIGURE 24.1

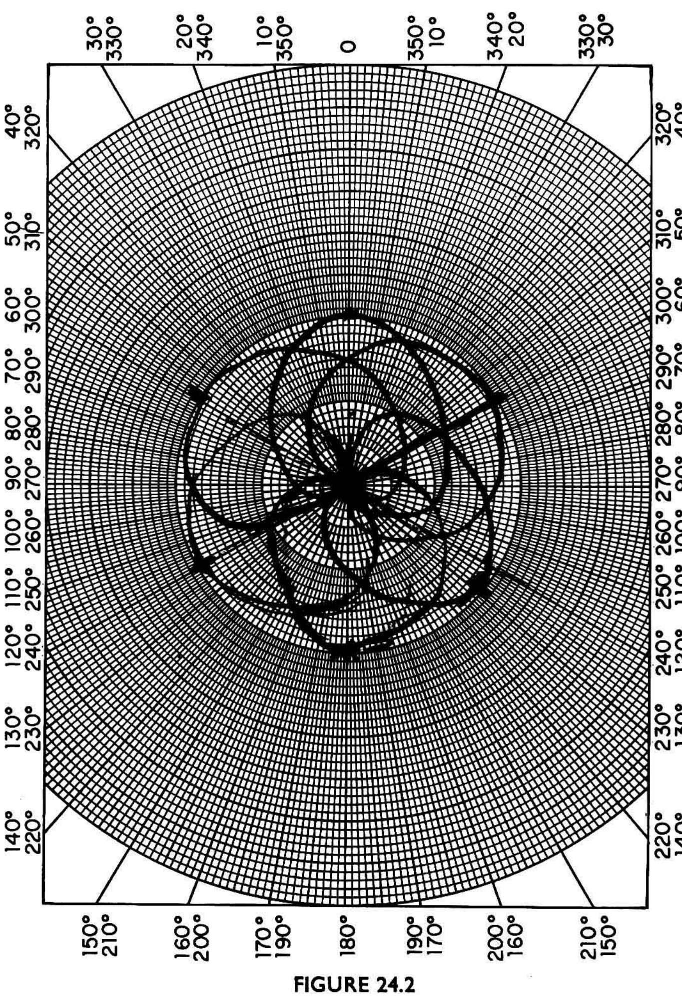

FIGURE 24.2

conventionally drawn to the right from $O$. $O$ is called the origin (or *pole*) and the ray is the *initial ray* (or *polar axis*) (Fig. 24.1). Any point $P$ in the plane may then be located by its distance $r$ from the origin and by an angle $\theta$ from the initial ray to $OP$. There are, of course, many such angles, positive and negative, and thus, inevitably though rather annoyingly, a point may be identified in more than one way. $OP$ is called the *radius vector* to $P$. An ordered pair of real numbers, $(r, \theta)$, is said to be a set of *polar coordinates* for a point $P$ (not the origin) if $r$ $(> 0)$ is the (undirected) length of $OP$ and $\theta$ is the radian measure of any angle from the initial ray to the ray $OP$. The origin is special: to it we assign as polar coordinates any pair $(0, \theta)$, $(-\infty < \theta < \infty)$. A point thus has many sets of polar coordinates; witness Fig. 24.1 in which $P$ has coordinates $(10, \pi/6)$, $(10, 13\pi/6)$, $(10, -11\pi/6)$, and, in general, $(10, 2n\pi + \pi/6)$ for any integer $n$. The point $O$ may be $(0, 0)$, $(0, \pi)$, $(0, -6)$, $(0, \pi/7)$, etc.

In assigning polar coordinates to identify a point, it is customary to allow $r$ also to be negative, in which case the distance $|r|$ is measured on the extension of the ray given by $\theta$ rather than on the ray itself. For example, in Fig. 24.1, the point $P$ may be thought of as on the extension of the ray $\theta = 7\pi/6$, and $P$ may thus have additional coordinates $(-10, 7\pi/6)$, $(-10, -5\pi/6)$, or, in general, $(-10, 2n\pi + 7\pi/6)$.

All the points for which $r$ has the constant value $c$ lie on the circle of radius $|c|$ with center at the origin, while all points for which $\theta = k$ are on a line through the origin making an angle of $k$ radians with the initial ray. Thus points given by their polar coordinates can be easily located on *polar coordinate graph paper* (Fig. 24.2), on which equally spaced sets of these circles and lines are drawn, just as points in rectangular coordinates are located on paper on which equally spaced sets of the curves $y = k$ and $x = c$ (horizontal or vertical lines) are printed.

## 24.2 FUNCTIONS AND THEIR POLAR GRAPHS

A function, you will recall, is a set of ordered pairs of numbers. Whether we write the rule of correspondence as $y = \sin x$ or $r = \sin \theta$ or $p = \sin f$, we always determine the same set of ordered pairs. In making the geometric picture of the function, however, we can interpret these pairs as rectangular coordinates (as we have been doing), or we can plot them as polar coordinates (or, actually, in a variety of other ways). Naturally, the ordered pairs will correspond to quite different points of the plane under these different interpretations, and a given function will have quite a different picture in rectangular coordinates from the one it has in polars. This is no more surprising than the fact that Greenland looks different on two different maps made up on two different projections, or that your reflection looks quite different in a teaspoon and in a mirror.

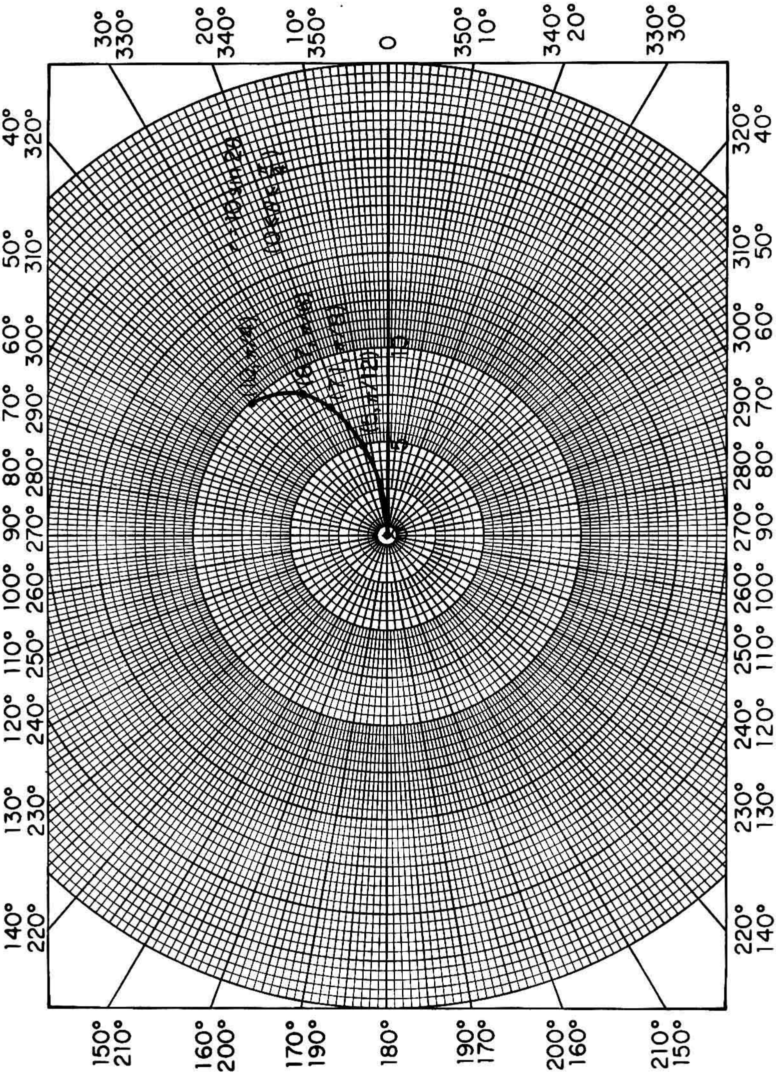

FIGURE 24.3

Let us plot the graph of a function whose number-pairs are interpreted as polar coordinates.

EXAMPLE 1. Graph $r = 10 \sin 2\theta$, the *four-leaved rose.**

SOLUTION. We begin by obtaining a few points. As $\theta$ goes from 0 to $\pi/4$, $2\theta$ goes from 0 to $\pi/2$, so we will start with this domain for $\theta$, obtaining the following table and the graph shown in Fig. 24.3.

| $\theta$ | $2\theta$ | $\sin 2\theta$ | $10 \sin 2\theta$ | point on graph |
|---|---|---|---|---|
| 0 | 0 | 0 | 0.0 | (0, 0) |
| $\pi/12$ | $\pi/6$ | $\frac{1}{2}$ | 5.0 | $(5, \pi/12)$ or $(5, 15°)$† |
| $\pi/8$ | $\pi/4$ | $\sqrt{2}/2$ | 7.1 | $(7.1, \pi/8)$ or $(7.1, 22.5°)$ |
| $\pi/6$ | $\pi/3$ | $\sqrt{3}/2$ | 8.7 | $(8.7, \pi/6)$ or $(8.7, 30°)$ |
| $\pi/4$ | $\pi/2$ | 1 | 10.0 | $(10, \pi/4)$ or $(10, 45°)$ |

Now as $\theta$ goes from $\pi/4$ to $\pi/2$, $2\theta$ goes from $\pi/2$ to $\pi$, and our knowledge of the sine function tells us that $r$ will go back through the values 10, 8.7, 7.1, 5, and 0 as $\theta$ goes through $\pi/4$, $\pi/3$, $3\pi/8$, $5\pi/12$, and $\pi/2$. That is, the graph will be symmetric in the ray $\theta = \pi/4$. Hence, as $\theta$ goes from 0 to $\pi/2$, we obtain the graph in solid line in Fig. 24.4.

Next, as $\theta$ goes from $\pi/2$ to $\pi$, $2\theta$ goes from $\pi$ to $2\pi$, and $\sin \theta$ is negative. Moreover, since $\sin (x + \pi) = -\sin x$, the points we obtain can be found from the calculations for the previous table to be $(0, \pi/2)$, $(-5, 7\pi/12)$, $(-7.1, 5\pi/8)$, $(-8.7, 2\pi/3)$, $(-10, 3\pi/4)$, $(-8.7, 5\pi/6)$, $(-7.1, 7\pi/8)$, $(-5, 11\pi/12)$, and $(0, \pi)$. These are pictured in the dotted portion of Fig. 24.4.

Finally as $\theta$ goes from $\pi$ to $2\pi$, $2\theta$ changes from $2\pi$ to $4\pi$, and $\sin 2\theta$ attains exactly the same values it has for $\theta$ between 0 and $\pi$. Thus the graph becomes that of Fig. 24.5 for $0 \leq \theta \leq 2\pi$, where the dotted portion distinguishes the part of the curve which came from negative values of $r$. If $\theta$ goes outside this domain, it is clear that the points merely repeat themselves and trace the same curve over and over. Notice that a given point of the locus may thus represent more than one of the ordered pairs which constitute the function—a situation different from that which occurs when we picture a function by rectangular coordinates.

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* We shall plot our graphs admitting both positive and negative values of $r$. There are some occasions, however, on which we are interested only in positive $r$. If we eliminate $r < 0$, the curve, of course, no longer necessarily pictures all the ordered pairs of the function, and may have fewer points on it. This is the case with this particular locus, which then becomes the "two-leaved rose." In our figures, we shall plot portions of the curve which come only from negative values of $r$ as dotted lines.

† For the actual plotting, it is sometimes easier to give the measure of $\theta$ in degrees, and we shall occasionally do this, without prejudice to our restriction that the polar coordinate $\theta$ is the radian measure of this angle. Polar coordinate paper is often printed with degree markings.