IVAN NIVEN

# *CALCULUS*
## *An Introductory Approach*

**SECOND EDITION**

THE UNIVERSITY SERIES IN UNDERGRADUATE MATHEMATICS

# CALCULUS

# An Introductory Approach

*by*

IVAN NIVEN

*Professor of Mathematics*
*The University of Oregon*

*SECOND EDITION*

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## An Introductory Approach

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# PREFACE

The addition of new material is the primary change from the first edition of this book. Specifically, the second edition provides a fuller discussion of analytic geometry; the treatment of logarithmic functions has been expanded; formulas have been summarized at the ends of Chapters 3 to 7 for the convenience of the reader; and, most important of all, the number of problems has been increased considerably. The text has also been enhanced in many places by improved explanations.

My purpose, as in the first edition, is to present the ideas that lie at the heart of calculus, along with the necessary background material from analytic geometry. I restrict attention to a small collection of central concepts, and hence many of the topics offered in the larger books on calculus are omitted. I have aimed at a balance between theory and applications.

It is hoped that the book will be of use in situations where a brief course in calculus is wanted. For example, it should be suitable in a calculus course for liberal arts students, social science students, biological science students, or business administration students—a course designed to set forth the nature of the subject with an economy of time. Since the book grew out of lectures given at Stanford University in a special program for teachers sponsored by the General Electric Company, it should serve well courses designed for prospective and present teachers of high school mathematics. The reader should have a reasonably good knowledge of basic algebra and trigonometry, although discussions of a few topics—inequalities and radian measure, for example—have been included for convenience.

The section headings in the table of contents suffice to describe the topics considered. Special features of the book are as follows: an immediate discussion of actual problems in calculus, with a minimum of preliminaries; a simplicity of theory, relatively speaking, obtained on the one hand by restricting sharply the class of functions in the discourse and on the other hand by not attempting to state results in their most general form, or for that matter anywhere near their most general form; a development of the series expansions of the trigonometric, logarithmic, and exponential functions without the elaborate preparation that is ordinarily used; the postponing of the proof of the existence of the definite integral to an appendix, partly to break up the theory into smaller portions, and partly to keep more difficult ideas later in the presentation.

One of the most troublesome problems facing a writer of a beginning book on calculus is the matter of rigor. The difficulty is twofold: first that accuracy of statement can lead to overlengthy statements in which the reader may lose sight of the central idea in a welter of detail; second that since theorems in calculus are propositions about real numbers, ideally there should be a preamble on the logical foundations of the real number system. I have tried to avoid the first of these difficulties by a restriction, mentioned already, on the scope of the material. As to the second difficulty, I take two fundamental propositions about real numbers as axiomatic. It is assumed that a bounded sequence of increasing real numbers has a limit. The mean value theorem is also assumed without a strict proof, although a heuristic argument is given to show the plausibility of this result. These results cannot be established without a rather thorough analysis of the real number system, which is precluded by the self-imposed limitation on the length of this book.

The more difficult problems are starred. Answers to odd-numbered problems are given at the end of the book.

I was fortunate in having the manuscript read by two friendly critics of quite different backgrounds, one a university student, the other an experienced mathematician. First, I am indebted to my son Scott Niven who, in addition to making helpful suggestions about details drew my attention to certain obscure passages. Second, I am grateful to Professor Herbert S. Zuckerman for pointing out possible arrangements of the material; among other things, the final chapter was expanded considerably along lines he suggested.

IVAN NIVEN

# TABLE OF CONTENTS

# CHAPTER 1

# WHAT IS CALCULUS?

**1.0** A short definition of calculus, in a sentence or two for example, is likely to be meaningless except to persons already familiar with the subject. The reason for this is that such a definition necessarily refers to certain mathematical operations whose nature can be known thoroughly only by prolonged examination and study. Even this book in its entirety is only a partial answer to the question "What is calculus?", because a complete answer cannot be given in a short book. A brief volume cannot encompass the elegant general results of calculus. For example, some of the topics that are treated in the later chapters of this book are approached in special ways as individual results, whereas in larger books on calculus these results are obtained as byproducts of broad sweeping theories.

In this first chapter we pose a few simple problems of calculus, with solutions given at once in some cases, but solutions postponed until later chapters in others. In diving right in, we assume that the reader has a rudimentary knowledge of inequalities and functions; these topics are elaborated in Chapter 2, but in more detail than necessary for the present chapter. Prior to the study of actual problems from calculus, there is one preliminary discussion on the concept of slope.

**1.1 Slope.*** The slope of a straight line is simply a measure of the steepness of rise or fall of the line, viewed from left to right. Thus the direction of a line is indicated by its slope, defined as follows. Suppose that the line makes an angle $\alpha$ with the $x$-axis, the orientation of $\alpha$ being from the positive end of the $x$-axis counterclockwise around to the line. Then the slope of the line is defined as $\tan \alpha$. If the line is

* Any reader with a knowledge of the concept of slope should bypass this section.

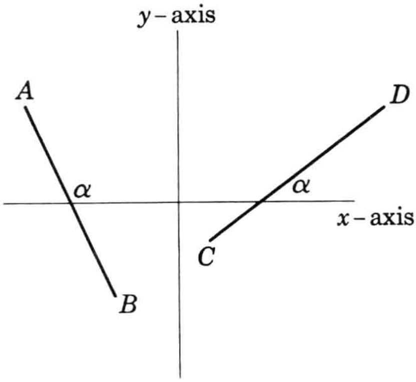


FIG. 1.1

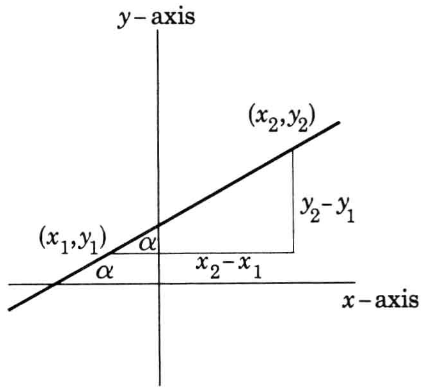


FIG. 1.2

falling from left to right, as in the case AB in Figure 1.1, then the slope is negative, whereas if the line is rising from left to right, as in the case CD in Figure 1.1, then the slope is positive. These cases correspond to the angle $\alpha$ being obtuse and acute, respectively. If any two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ are selected on the line, then the definition of the tangent function in trigonometry gives the well-known formula for the slope $m$,

$$(1) \qquad m = \tan \alpha = \frac{y_2 - y_1}{x_2 - x_1}.$$

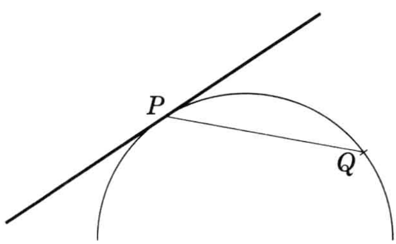


FIG. 1.3

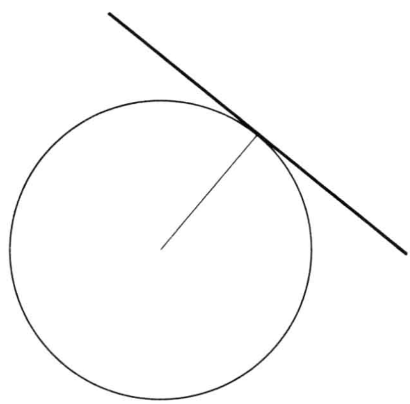

FIG. 1.4

Any line parallel to the $x$-axis has $\alpha = 0°$ and $y_2 = y_1$, and the slope of such a line is zero. Any line parallel to the $y$-axis has $\alpha = 90°$, and so such a line has no slope because tan 90° does not exist as a real number. Another way of seeing that a vertical line has no slope is to observe that in such a case $x_2 = x_1 = 0$, and so equation (1) involves division by zero.

The slope of a curve at any point is defined as the slope of the tangent line to the curve at P. The tangent line at P, illustrated in Figure 1.3, can be described as the limiting position of the straight line PQ as the point Q is moved along the curve towards the point P. That is to say, we regard P as a fixed point and observe the nature of the straight line PQ as Q approaches P, moving along the curve. When Q coincides with P there is no line PQ, but as Q moves towards P there is a limiting position of PQ which we will be able to determine by our analysis. In the special case where the curve is a circle, a well-known theorem from elementary geometry states that the tangent line at any point is perpendicular to the radius drawn to the point of tangency, as illustrated in Figure 1.4. For curves other than the circle there is no corresponding theorem, but we shall be able with the use of calculus to determine the tangent lines of many types of curves. From these considerations we can observe that whereas a straight line has one slope, a curve has a slope at each point, and the slope is generally different from point to point.

## Problems

1. Prove that the formula (1) can be written

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

2. Find the slopes of the following lines:
   (a) through (0, 0) and (5, 7);
   (b) through $(-1, 2)$ and $(4, -6)$;
   (c) through (1, 2) and $(a, b)$, presuming that $a \neq 1$;
   (d) through $(2, -3)$ and $(-3, -3)$.

3. Find the numerical value of $x$ so that the line joining $(x, -4)$ and (6, 1) shall have slope 2.

4. Prove that the line joining A$(5, -7)$ and B(6, 1) has the same slope as the line joining B(6, 1) and C(8, 17), and thus establish that the three points A, B, C are collinear.

5. Prove that the points $(4, -9)$, $(-1, -6)$, and $(-16, 3)$ are collinear.

6. Find the numerical value of $y$ so that the point $(7, y)$ shall lie on the line joining (1, 3) and (9, 19).

7. Prove that the line through (4, 2) and (7, 1) is parallel to the line through (5, 6) and (−1, 8). Suggestion: two lines (not parallel to the $y$-axis) are parallel if and only if their slopes are equal.

8. Find the numerical value of $y$ required so that the line through (−2, 3) and (8, 1) shall be parallel to the line through (1, −4) and (−4, $y$).

9. Prove that the four points (1, −2), (3, −5), (8, 4), and (6, 7) are the vertices of a parallelogram.

10. Prove that the points (1, −2), (3, −5), (8, 4), and (10, 1) are the vertices of a parallelogram.

*11. Find a point, other than (6, 7) and (10, 1), which forms a parallelogram along with (1, −2), (3, −5), and (8, 4).

12. Prove that the line through (7, 0) and (10, $\sqrt{3}$) is inclined to the $x$-axis at an angle of 30°. (Recall that $\tan 30° = 1/\sqrt{3}$.)

13. Prove that the line through (1, 5) and (8, 12) is inclined to the $x$-axis at an angle of 45°. (Recall that $\tan 45° = 1$.)

14. Find the slope of the line joining $(a, b)$ and $(c, d)$ presuming that $a$ is not equal to $c$.

15. Find the slope of the line joining $(a, b)$ and $(a+c, b+d)$, presuming that $c$ is not equal to zero.

**1.2 An Example.** We now give a sample problem from differential calculus. What should be the dimensions of a cylindrical tin can of fixed volume $18\pi$ cubic inches, so that the surface area is a minimum? Since the surface area is roughly proportional to the total amount of metal in the can, the solution of the problem will give almost the dimensions for minimum cost of the metal. The figure $18\pi$ for the volume was chosen so that the arithmetic calculations would work out simply. Actually $18\pi$ cubic inches is very close to a volume of one quart.

Let $r$, $h$, $S$, and $V$ denote, respectively, the radius, the height, the total surface area, and the volume of a circular cylinder. Then it will be recalled that

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi rh.$$

The symbol $V$ can be replaced by the constant $18\pi$, and so by simple algebra we can eliminate $h$ in the formula for $S$,

$$18\pi = \pi r^2 h, \quad h = 18/r^2, \quad S = 2\pi r^2 + 2\pi rh,$$

$$S = 2\pi\left(r^2 + \frac{18}{r}\right).$$

Thus we have obtained $S$ as a function of $r$. (It may be noted that we could have eliminated $r$ in the algebraic process, and arrived at a formula giving $S$ as a function of $h$. But square roots enter in, and so the above procedure is preferred).

* The more difficult problems are starred.

The graph of the function $S = 2\pi(r^2 + 18/r)$ is shown in Figure 1.5, for a reasonable set of positive values of $r$. To find the minimum value of $S$, we need some technique for locating the low point on the curve, the point labeled P in Figure 1.5. It is the point where the slope of the curve is zero. To the right of the point P the slope of the curve is positive, whereas to the left of the point P the slope of the curve is negative. The question of determining the coordinates of the point P can be settled easily by the methods of differential calculus. We are not at present in a position to finish the above problem of minimizing $S$, and we will return to the question later. But the example shows that some questions of maxima and minima could be handled if we knew how to calculate the slope of a curve, and so find where the slope is zero.

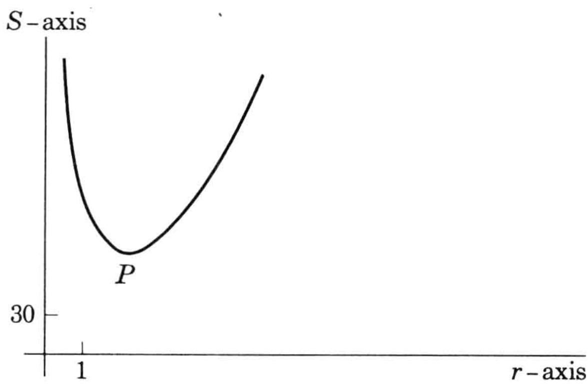


FIG. 1.5 Graph of $S = 2\pi[r^2 + 18/r]$.
(Note the different units of length on the axes.)

We have referred to the equation $S = 2\pi(r^2 + 18/r)$ as a "function" Most of the functions used in this book can be expressed as simple equations in this way, because we will work with a restricted class of functions. Furthermore in most cases our function will have a graph as in Figure 1.5, for example, or in Figure 1.6 below.

The commonly used variables in mathematics are of course $y$ and $x$ rather than $S$ and $r$, so the functions we deal with will have equations like

$$y = x^2, \quad y = x^2 - 3x + 2, \quad y = \sqrt{x^2 - 2}.$$

When we say that $y$ is a function of $x$, as in these examples, we mean simply that to each assigned numerical value of $x$ there is a corresponding numerical value of $y$. In the three examples above, if we assign to $x$ the value 3, then we get $y = 9$, $y = 2$, and $y = 7$, respectively. It should be noted that for many functions there are limitations on the values of $x$ that are to be assigned. In the case of the function $y = \sqrt{x^2 - 2}$ for instance, the value $x = 0$ would not be assigned because this gives $y = \sqrt{-2}$, which is a number of a type not considered in this book. We limit our attention to real numbers, which are described briefly at the start of Chapter 2. Another example of a limitation of this sort can be seen in Figure 1.5; here we would not assign negative values to $r$ because the radius $r$ must be a positive number.

The graph of a function is a pictorial representation of the function by use of a coordinate system. It is easier to conceive a function in its graphical form than in a more abstract way as a correspondence between sets of numbers. For this reason we will approach functions through their graphs wherever possible.

It will also be necessary to think of functions in a general sense. Just as the symbol $x$ is used to denote an arbitrary number, so the notation $f(x)$ is used to denote an arbitrary function. Thus $f(x)$, read "$f$ of $x$", is a general notation for a function of $x$. The corresponding equation is $y = f(x)$. For a specific function like $y = x^2 - 3x + 2$, we can also write $f(x) = x^2 - 3x + 2$. Then in turn $x$ can be specified, for example

$$f(5) = 5^2 - 3 \cdot 5 + 2 = 12, \quad f(9) = 9^2 - 3 \cdot 9 + 2 = 56.$$

These sketchy remarks on the idea of a function will suffice for this chapter. A fuller discussion is given in § 2.5.

## Problems

1. In the analysis of the cylindrical tin can a formula was derived for $S$ as a function of $r$. Derive a formula for $S$ as a function of $h$.
2. Draw the graph of $y = 4x - 3$ on a coordinate axis system.
3. Sketch the graph of $y = x^2 - 1$, using a succession of integer values of $x$ from $x = -5$ to $x = 5$. (The integers are the numbers $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$.)
4. Sketch the graph of $y = 7 - x^2$.
5. Given $f(x) = x^2 - 3x + 2$, compute $f(7), f(-7), f(0)$ and $f(-3)$.
6. In the case $f(x) = 12 + x - x^2$, find the values of $f(0), f(1), f(-4), f(5/2)$, and $f(-8)$.
7. For the function $f(x) = x^3 - 3$, evaluate $f(2)$, $f(-2)$, $f(0)$, $f(-3)$, and $f(1/2)$.

**1.3 The Slope of a Special Curve.** We turn to a much simpler equation than $S = 2\pi(r^2 + 18/r)$ to begin our treatment of the slope of a curve. The problem now is to find the slope of the curve $y = x^2$

at the point (3, 9). The parabolic equation $y = x^2$ is about the simplest among non-linear equations, and so provides a good starting point. As in Figure 1.6, let us denote the point (3, 9) by A, and let P be any nearby point on the curve. Since the point P, unlike A, is not a fixed point, we give it coordinates $(x, x^2)$. This is nothing but the general coordinates $(x, y)$ with $y$ replaced by $x^2$ in accordance with the presumption that P is a point on the curve $y = x^2$. The slope of the chord AP, by formula (1) of § 1.1, is $(x^2-9)/(x-3)$. The tangent line to the curve $y = x^2$ at A is the limiting position of the chord AP as the point P approaches the point A. Thus the slope of the tangent line at A, which by definition is the same as the slope of the curve at A, is the limiting value of $(x^2-9)/(x-3)$ as $x$ approaches 3. We cannot simply substitute $x = 3$, because (i) geometrically the points P and A coincide and there is no chord AP, and (ii) algebraically the expression $(x^2-9)/(x-3)$ becomes 0/0, which has no meaning.

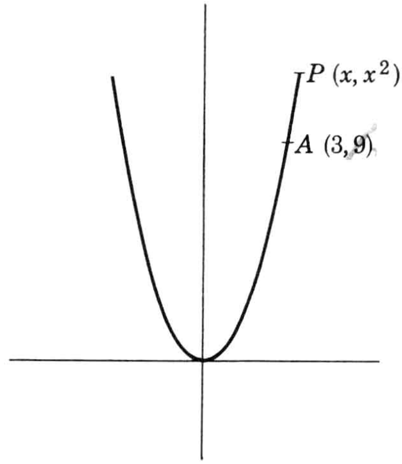


FIG. 1.6 Graph of $y = x^2$.

Now by elementary algebra we have

$$\frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3} = x+3,$$

and $(x^2-9)/(x-3)$ is indeed equal to $x+3$ for all values of $x$ except one, namely $x = 3$. The function $f(x) = x+3$ has a straight line graph as shown in Figure 1.7 when plotted in the usual fashion with a horizontal $x$-axis and a vertical $y$-axis. The function $F(x) = (x^2-9)/(x-3)$

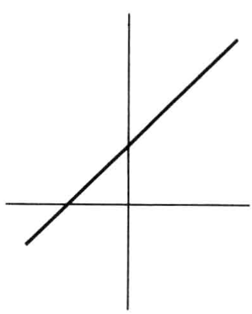

FIG. 1.7 Graph of $f(x) = x+3$.

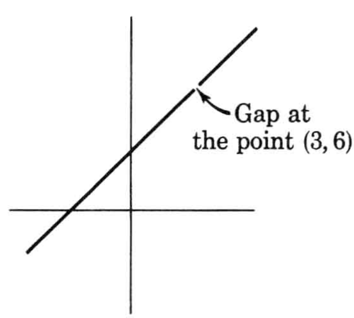


FIG. 1.8 Graph of $f(x) = (x^2-9)/(x-3)$.

has virtually the same graph but with a point missing, a gap at (3, 6). It is clear intuitively that the limit of $(x^2-9)/(x-3)$ as $x$ approaches 3 is the same as the limit of $x+3$ as $x$ approaches 3. We shall spell out a precise definition of limit later. In the meantime we write the standard notation for limits

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6,$$

where "lim" stands for "limit", and "$x \to 3$" is short for "as $x$ approaches 3". We notice that while $(x^2-9)/(x-3)$ has no value at $x = 3$, nevertheless it has a limit as $x$ approaches 3. This limiting value 6 is thus the slope of the curve $y = x^2$ at the point (3, 9).

It is instructive also to consider a set of values of $(x^2-9)/(x-3)$ as $x \to 3$:

| $x$ | $(x^2-9)/(x-3)$ |
|---|---|
| 3.1 | 6.1 |
| 3.01 | 6.01 |
| 3.001 | 6.001 |
| 3.0001 | 6.0001 |
| $3+10^{-10}$ | $6+10^{-10}$ |

This table of values suggests the germ of the idea of limit, namely that

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = 6$$

means that $(x^2-9)/(x-3)$ gets closer and closer to 6 as $x$ approaches 3. We shall formulate this technically in the next chapter.

## Problems

1. Find the slope of the curve $y = 2x^2$ at the point (3, 18).
2. Find the slope of the curve $y = x^2$ at the point (2, 4).
3. Find the slope of the curve $y = x^2$ at the point (1, 1).
4. Find the slope of the curve $y = x^2$ at the point (−3, 9).
5. Solve the preceding problem by using the information that the slope of the curve $y = x^2$ is 6 at the point (3, 9), and the fact that the graph of the curve is symmetric about the $y$-axis.
6. Find the slope of the curve $y = -x^2$ at the point (5, −25).
7. What is the slope of the curve $y = x^2+6$ at the point where $x = 6$?
8. What would be the expected value of the slope of $y = x^2$ at (0, 0)? Check this value by the limit procedure.
9. Find the slope of $y = x^2+x$ at (1, 2).

*10. What is the slope of the curve $y = x^3$ at the point (3, 27)?