Differential Equations

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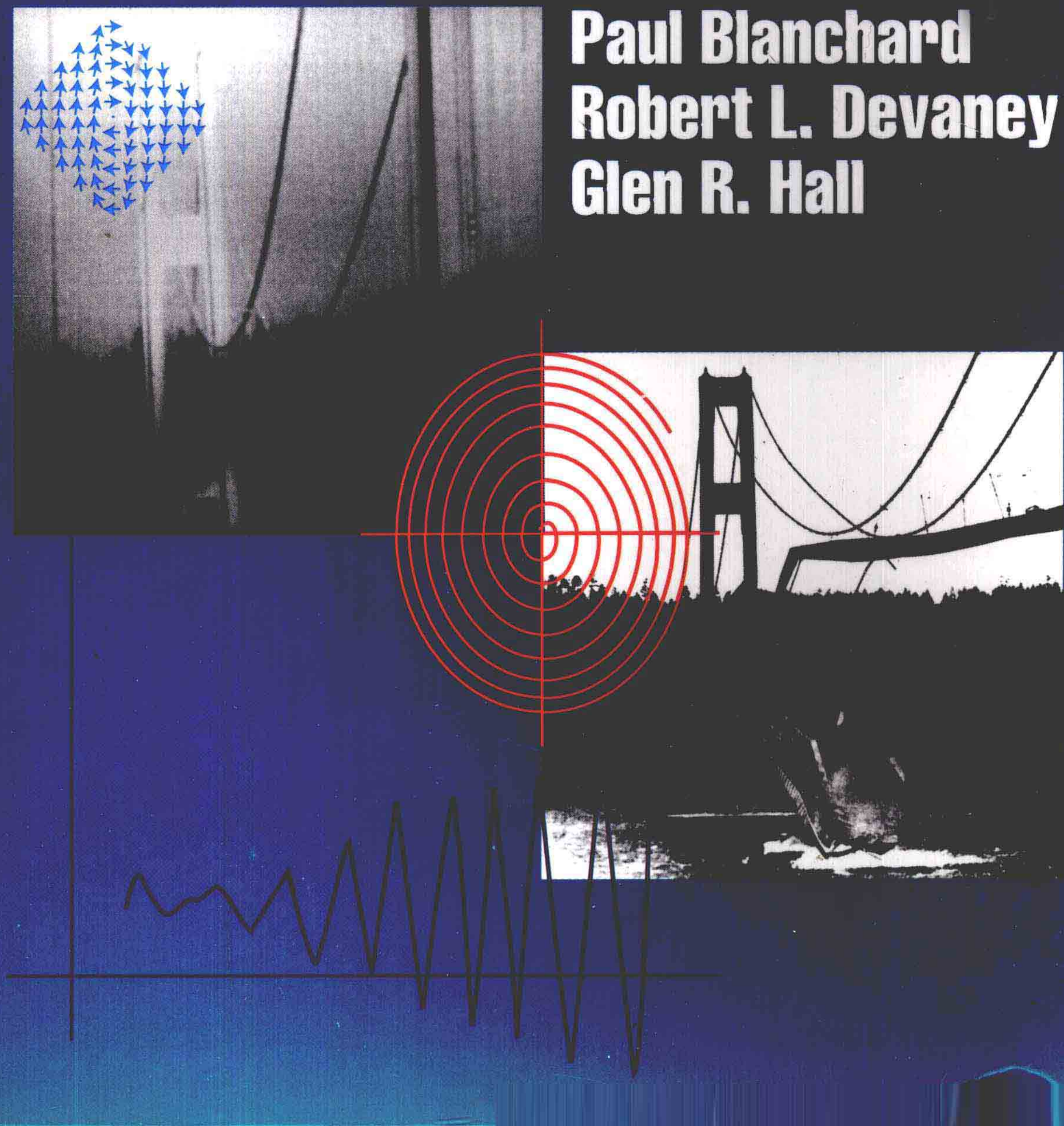

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Boston University

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PREFACE

The study of differential equations is a beautiful application of the ideas and techniques of calculus to our everyday lives. Indeed, it could be said that calculus was developed mainly so that the fundamental principles that govern many phenomena could be expressed in the language of differential equations. Unfortunately, it was difficult to convey the beauty of the subject in the traditional first course on differential equations because the number of equations that can be treated by analytic techniques is very limited. Consequently, the course tended to focus on technique rather than on concept.

This book is an outgrowth of our opinion that we are now able to effect a radical revision, and we approach our updated course with several goals in mind. First, the traditional emphasis on specialized tricks and techniques for solving differential equations is no longer appropriate given the technology that is readily available. Second, many of the most important differential equations are nonlinear, and numerical and qualitative techniques are more effective than analytic techniques in this setting. Finally, the differential equations course is one of the few undergraduate courses where it is possible to give students a glimpse of the nature of contemporary mathematical research.

The Qualitative, Numeric, and Analytic Approaches

Accordingly, this book is a radical departure from the typical "cookbook" differential equations text. We have eliminated most specialized techniques for deriving formulas for solutions, and we have replaced them with topics that focus on the formulation of differential equations and the interpretation of their solutions. To obtain an understanding of the solutions, we generally attack a given equation from three different points of view.

One major approach we adopt is qualitative. We expect students to be able to visualize differential equations and their solutions in many geometric ways. For example, we readily use slope fields, graphs of solutions, vector fields, and solution curves in the phase plane as tools to gain a better understanding of solutions. We also ask students to become adept at moving among these geometric representations and more traditional analytic representations.

Since differential equations are readily studied using the computer, we also emphasize numerical techniques. We assume that students have access to some sort of technology that approximates solutions and graphs these solutions easily. Even if we can find an explicit formula for a solution, we often work with the equation both numerically and qualitatively to understand the geometry and the long-term behavior of solutions. When we can find explicit solutions easily (such as in the case of separable first-order equations or constant-coefficient, linear systems), we do the calculations. But we never fail to examine the resulting formulas we obtain using qualitative and numerical points of view as well.

Specific Changes

There are several specific ways in which this book differs from other books at this level. First, we incorporate modeling throughout. We expect students to understand the meaning of the variables and parameters in a differential equation and to be able to interpret this meaning in terms of a particular model. Certain models reappear often as running themes, and they are drawn from a variety of disciplines so that students with various backgrounds will find something familiar.

We also adopt a dynamical systems point of view. Thus, we are always concerned with the long-term behavior of solutions of an equation, and using all of the appropriate approaches outlined above, we ask students to predict this long-term behavior of solutions. In addition, we emphasize the role of parameters in many of our examples, and we specifically address the manner in which the behavior of solutions changes as these parameters are varied.

Like other texts, we begin with first-order equations, but the only analytic technique we use to find closed-form solutions is separation of variables (and, at the end of the chapter, an integrating factor or two to handle certain linear equations). Instead, we emphasize the meaning of a differential equation and its solutions in terms of its slope field and the graphs of its solutions. If the differential equation is autonomous, we also discuss its phase line. This discussion of the phase line serves as an elementary introduction to the idea of a phase plane, which plays a fundamental role in subsequent chapters.

We then move directly from first-order equations to systems of first-order differential equations. Rather than consider second-order equations separately, we convert these equations to first-order systems. When these equations are viewed as systems, we are able to use qualitative and numerical techniques more readily. Of course, we then use the information about these systems gleaned from these techniques to recover information about the solutions of the original equation.

We also begin the treatment of systems with a general approach. We do not immediately restrict our attention to linear systems. Qualitative and numerical techniques work just as easily when a system is nonlinear, and one can proceed a long way toward understanding systems without resorting to algebraic techniques. However, qualitative ideas do not tell the whole story, and we are led naturally to the idea of linearization. With this background in the fundamental geometric and qualitative concepts, we then discuss linear systems in detail. As always, we not only emphasize the formula for the general solution of a linear system but also the geometry of its solution curves and of the related eigenvectors and eigenvalues.

While our study of systems requires the minimal use of some linear algebra, it is definitely not a prerequisite. As we deal primarily with two-dimensional systems, we easily develop all of the necessary algebraic techniques as we proceed. In the process, we give considerable insight into the geometry of such topics as eigenvectors and eigenvalues.

These topics form the core of our approach. However, there are many additional topics that one would like to cover in the course. Consequently, we have included discussions of forced second-order equations, nonlinear systems, Laplace transforms, numerical methods, and discrete dynamical systems. Although some of these topics are quite traditional, we always present them in a manner that is consistent with the philosophy developed in the first half of the text.

At the end of each chapter, we have included several "labs." Doing detailed numerical experimentation and writing reports has been our most successful modification of the traditional course at Boston University. Good labs are tough to write and to grade, but we feel that the benefit to students is extraordinary.

Pathways Through This Book

There are a number of possible tracks that instructors can follow in using this book. We feel that Chapters 1–3 form the core (with the possible exception of Sections 2.5 and 3.8, which cover systems in three dimensions). Most of the later chapters assume familiarity with this material. Certain sections such as Section 1.7 (bifurcations) and Section 1.9 (changing variables) may be skipped if some care is taken in choosing material from subsequent sections. However, the material on phase lines and phase planes, qualitative analysis, and solutions of linear systems is central.

A typical track for an engineering-oriented course would follow Chapters 1–3 (perhaps skipping Sections 1.9, 2.5, and 3.8). These chapters will take roughly two-thirds of a semester. The final third of the course might cover Sections 4.1–4.3 (forced, second-order linear equations and resonance), Section 5.1 (linearization of nonlinear systems), and Chapter 6 (Laplace transforms). Chapters 4 and 5 are independent of each other and can be covered in either order. In particular, Section 5.1 on linearization of nonlinear systems near equilibrium points forms an excellent capstone for the material on linear systems in Chapter 3.

Incidentally, it is possible to cover Sections 6.1 and 6.2 (Laplace transforms for first-order equations) immediately after Chapter 1. As we have learned from our colleagues in the College of Engineering at Boston University, some engineering programs teach a circuit theory course that uses the Laplace transform at an earlier point than is typically the case. Consequently, Sections 6.1 and 6.2 are written so that the differential equations course and such a circuits course could proceed in parallel. However, if possible, we recommend waiting to cover Chapter 6 entirely until after the material in Sections 4.1–4.3 has been discussed.

Instructors may wish to substitute material on discrete dynamics (Chapter 8) for Laplace transforms. A course for students with a strong background in physics might involve more of Chapter 5, including a treatment of Hamiltonian (Section 5.3) and gradient systems (Section 5.4). A course geared toward applied mathematics might include a more detailed discussion of numerical methods (Chapter 7).

Changes in the First Edition

We have been quite pleased with the reception that the preliminary edition of this book has enjoyed since its publication in 1995. We are especially indebted to the large number of readers and instructors who made comments about various points in the earlier edition. Accordingly, we have made some changes in this edition. The most significant changes include more thorough treatments of forcing and resonance for second-order equations and a revised treatment of Laplace transforms. The material in Chapter 2 has been extensively rewritten to follow more closely our intent to introduce analytic, qualitative, and numerical methods for systems at an early stage. Two appendices have been added. The first is an alternate treatment of first-order linear equations and can be used in place of Section of 1.8. The second appendix is a review of complex numbers and Euler's formula.

Most of the other changes involve only minor rearrangements of topics so that most instructors can avoid skipping sections within a chapter. As with any significant revision of an existing course, we anticipate that this book will continue to evolve in future editions. We encourage comments, suggestions, and criticism. The best way to comment is to send e-mail to **odes@math.bu.edu**. We'll do our best to acknowledge the e-mail, but we will definitely read and consider every comment.

Our Website and Ancillaries

Readers and instructors are invited to make extensive use of our web site

http://math.bu.edu/odes

At this site we have posted an on-line instructor's guide that includes discussions of how we use the text. We have also posted sample syllabi contributed by users at various institutions as well as information about workshops and seminars dealing with the teaching of differential equations. We also maintain a list of errata at this site. The *Instructor's Guide with Solutions*, available to instructors who have adopted the text for class use, contains a hardcopy of the on-line guide along with the solutions to all the problems. The *Student Solutions Manual* contains the solutions to all odd-numbered problems in the text.

Our publisher, Brooks/Cole, also maintains the DiffEQ Resource Center at

http://diffeq.brookscole.com

This site contains a wealth of information about the teaching and learning of differential equations, including an extensive array of laboratory and project ideas and links to a number of other sites related to the teaching of differential equations.

The Boston University Differential Equations Project

This book is a product of the now complete National Science Foundation Boston University Differential Equations Project sponsored by the National Science Foundation (NSF Grant DUE-9352833) and Boston University. The goal of that project was to rethink the traditional, sophomore-level differential equations course. We are especially thankful for that support.

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ACKNOWLEDGMENTS

As we move from the preliminary to the first edition, the list of people we are privileged to thank has grown exponentially. For this edition, we owe the greatest debt to **Gareth Roberts**. As project manager, he has overseen the production of the text and pictures. As mathematician and teacher he has been an invaluable critic and able assistant. As with his predecessor Sam Kaplan, the project manager for the preliminary edition, Gareth has left his mark on this text in many positive ways. Thanks, Gareth.

With the exception of a few professionally drawn figures, this book was entirely produced at Boston University's Department of Mathematics using Alex Kasman's ASTEX macro package in concert with LATEX2ε. Alex is a true TEX wizard, and anyone who is writing a textbook should consider his package. In fact, Alex's graphics macros are extremely useful in many contexts (see Alex's Web page available from the site **http://math.bu.edu**).

Much of the production work, solutions to exercises, accuracy checking and rendering of pictures was done by our team of graduate students: Bill Basener, Lee DeVille, and Stephanie Ruggiano. They spent many long days and nights in an alternately too-hot-or-too-cold windowless computer lab to bring this book to completion. We still rely on much of the work done by Adrian Iovita, Kinya Ono, Adrian Vajiac, and Nuria Fagella during the production of the preliminary edition.

Many other individuals at Boston University have made important contributions. In particular, our teaching assistants Duff Campbell, Michael Hayes, Eileen Lee, and Clara Bodelon had to put up with the headaches associated with our experimentation.

We received support from many of our colleagues at Boston University and at other institutions. Our chair, Marvin Freedman, encouraged us throughout. It was a special pleasure for us to work closely with colleagues in the College of Engineering—Michael Ruane (who coordinates the circuits course), Moe Wasserman (who permitted one of the authors to audit his course), and John Baillieul (a member of our advisory board). We also thank Donna Molinek (Davidson College), Carolyn Narasimhan (DePaul University), and James Walsh (Oberlin College) for organizing workshops for faculty on their campuses.

As is mentioned in the Preface, this book would not exist if our project had not received support from the National Science Foundation's Division of Undergraduate Education, and we thank the program directors at NSF/DUE for their enthusiastic support. We also thank the members of the advisory board—John Baillieul, Morton Brown, John Franks, Deborah Hughes Hallett, Philip Holmes, and Nancy Kopell. All have contributed their scarce time during workshops and trips to Boston University.

We were pleased that so many of our colleagues outside of Boston University were willing to help us with this project. Bill Krohn gave us valuable advice regarding our exposition of Laplace transforms, and Bruce Elenbogen did a thorough reading of early drafts of the beginning chapters. Preliminary drafts of our original notes were class tested in a number of different settings by Gregory Buck, Scott Sutherland, Kathleen Alligood, Diego Benedette, Jack Dockery, Mako Haruta, Jim Henle, Ed Packel, and Ben Pollina.

We have been pleased with the reception given to the preliminary edition of this text and are particularly grateful for the patience with which students and teachers alike have accepted our first attempt. Many have written us with excellent comments and suggestions—a few even caught the typo that we made. We thank them all. An updated list can be found at the web page address in the preface.

Thoughtful and insightful reviews have also been a tremendous help in the creation of both editions of the text. Reviews for the preliminary edition were done by Charles Boncelet, University of Delaware; Dean R. Brown, Youngstown State University; Michael Colvin, California Polytechnic State University; Peter Colwell, Iowa State University; James P. Fink, Gettysburg College; Michael Frame, Union College; Donnie Hallstone, Green River Community College; Stephen J. Merrill, Marquette University; LTC Joe Myers, U. S. Military Academy; Carolyn C. Narasimhan, DePaul University; Roger Pinkham, Stevens Institute of Technology; T. G. Proctor, Clemson University; Tim Sauer, George Mason University; Monty J. Strauss, Texas Tech University; and Paul Williams, Austin Community College.

Reviewers for this edition were David Arnold, College of the Redwoods; Steven H. Izen, Case Western Reserve University; Joe Marlin, North Carolina State University; Kenneth Meyer, University of Cincinnati; Joel Robbin, University of Wisconsin at Madison; Clark Robinson, Northwestern University; and Jim Walsh, Oberlin College.

Finally, as any author knows, writing a book requires significant sacrifices from one's family. Extra special thanks go to Lori, Kathy, and Dottie.

G.R.H., R.L.D., P.B.

A NOTE TO THE STUDENT

This book is probably different from most of your previous mathematics texts. If you thumb through it, you will see that there are few "boxed" formulas, no margin notes, and very few n-step procedures. We've written the book this way because we think that you are now at a point in your education where you should be learning to identify and work effectively with the mathematics inherent in everyday life. As you pursue your careers, no one is going to ask you to do all of the odd exercises at the end of some employee manual. They are going to give you some problem whose mathematical component may be difficult to identify and ask you to do your best with it. One of our goals in this book is to start preparing you for this type of work by avoiding artificial algorithmic exercises.

Our intention is that you will read this book as you would any other text, then work on the exercises, rereading sections and examples as necessary. Even though there are no template examples, you will find the discussions full of examples. Since one of our main goals is to demonstrate how differential equations are used to model physical systems, we often start with the description of a physical system, build a model, and then study the model to make conclusions and predictions about the original system. Many of the exercises ask you to produce or modify a model of a physical system, analyze it, and explain your conclusions. This is hard stuff, and you will need to practice. Since the days when you could make a living plugging and chugging through computations are over (computers do that now), you will need to learn these skills, and we hope that this book helps you develop them.

Another way in which this book may differ from your previous texts is that we expect you to make judicious use of a graphing calculator or a computer as you work the exercises and labs. The computer won't do the thinking for you, but it will provide you with numerical evidence that is essentially impossible for you to get in any other way. One of our goals is to give you practice as a sophisticated consumer of computer cycles as well as a skeptic of computer results.

Incidentally, one of the authors is known to have made a mistake or two in his life that the other two authors have overlooked. So we maintain a very short list of errata at our web site **http://math.bu.edu/odes**. Please check this page if you think that something you have read is not quite right.

Finally, you should know that the authors take the study of differential equations very seriously. However, we don't take ourselves very seriously (and we certainly don't take the other two authors seriously). We have tried to express both the beauty of the mathematics and some of the fun we have doing mathematics. If you think (some of?) the jokes are old or stupid, you're probably right.

All of us who worked on this book have learned something about differential equations along the way, and we hope that we are able to communicate our appreciation for the subject's beauty and range of application. We would enjoy hearing your comments. Feel free to send us e-mail at **odes@math.bu.edu**. We sometimes get busy and cannot always respond, but we do read and appreciate your feedback.

We had fun writing this book. We hope you have fun reading it.

G.R.H., R.L.D., P.B.

CONTENTS

1 FIRST-ORDER DIFFERENTIAL EQUATIONS

This book is about how to predict the future. To do so, all we have is a knowledge of how things are and an understanding of the rules that govern the changes that will occur. From calculus we know that change is measured by the derivative, and using the derivative to describe how a quantity changes is what the subject of differential equations is all about.

Turning the rules that govern the evolution of a quantity into a differential equation is called modeling, and in this chapter we study many models. Our goal is to use the differential equation to predict the future value of the quantity being modeled.

There are three basic types of techniques for making these predictions. Analytical techniques involve finding formulas for the future values of the quantity. Qualitative techniques involve obtaining a rough sketch of the graph of the quantity as a function of time as well as a description of its long-term behavior. Numerical techniques involve doing arithmetic (or having a computer do arithmetic) that yields approximations of the future values of the quantity. We introduce and use all three of these approaches in this chapter.

1.1 MODELING VIA DIFFERENTIAL EQUATIONS

The hardest part of using mathematics to study an application is the translation from real life into mathematical formalism. This translation is usually difficult because it involves the conversion of imprecise assumptions into very precise formulas. There is no way to avoid it. Modeling is difficult, and the best way to get good at it is the same way you get to play Carnegie Hall—practice, practice, practice.

What Is a Model?

It is important to remember that mathematical models are like other types of models. The goal is not to produce an exact copy of the "real" object but rather to give a representation of some aspect of the real thing. For example, a portrait of a person, a store mannequin, and a pig can all be models of a human being. None is a perfect copy of a human, but each has certain aspects in common with a human. The painting gives a description of what a particular person looks like; the mannequin wears clothes as a person does; and the pig is alive. Which of the three models is "best" depends on how we use the model—to remember old friends, to buy clothes, or to study biology.

The mathematical models we study are systems that evolve over time, but they often depend on other variables as well. In fact, real-world systems can be notoriously complicated—the population of rabbits in Wyoming depends on the number of coyotes, the number of bobcats, the number of mountain lions, the number of mice (alternative food for the predators), farming practices, the weather, any number of rabbit diseases, etc. We can make a model of the rabbit population simple enough to understand only by making simplifying assumptions and lumping together effects that may or may not belong together.

Once we've built the model, we should compare predictions of the model with data from the system. If the model and the system agree, then we gain confidence that the assumptions we made in creating the model are reasonable, and we can use the model to make predictions. If the system and the model disagree, then we must study and improve our assumptions. In either case we learn more about the system by comparing it to the model.

The types of predictions that are reasonable depend on our assumptions. If our model is based on precise rules such as Newton's laws of motion or the rules of compound interest, then we can use the model to make very accurate quantitative predictions. If the assumptions are less precise or if the model is a simplified version of the system, then precise quantitative predictions would be silly. In this case we would use the model to make qualitative predictions such as "the population of rabbits in Wyoming will increase ...." The dividing line between qualitative and quantitative prediction is itself imprecise, but we will see that it is frequently better and easier to make qualitative use of even the most precise models.

Some hints for model building

The basic steps in creating the model are

Step 1 Clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2 Completely describe the variables and parameters to be used in the model — "you can't tell the players without a program."

Step 3 Use the assumptions formulated in Step 1 to derive equations relating the quantities in Step 2.

Step 1 is the "science" step. In Step 1, we describe how we think the physical system works or, at least, what the most important aspects of the system are. In some cases these assumptions are fairly speculative, as, for example, "rabbits don't mind being overcrowded." In other cases the assumptions are quite precise and well accepted, such as "force is equal to the product of mass and acceleration." The quality of the assumptions determines the validity of the model and the situations to which the model is relevant. For example, some population models apply only to small populations in large environments, whereas others consider limited space and resources. Most important, we must avoid "hidden assumptions" that make the model seem mysterious or magical.

Step 2 is where we name the quantities to be studied and, if necessary, describe the units and scales involved. Leaving this step out is like deciding you will speak your own language without telling anyone what the words mean.

The quantities in our models fall into three basic categories: the **independent variable**, the **dependent variables**, and the **parameters**. In this book the independent variable is (almost) always time. Time is "independent" of any other quantity in the model. On the other hand, the dependent variables are quantities that are functions of the independent variable. For example, if we say that "position is a function of time," we mean that position is a variable that depends on time. We can vaguely state the goal of a model expressed in terms of a differential equation as "Describe the behavior of the dependent variable as the independent variable changes." For example, we may ask whether the dependent variable increases or decreases, or whether it oscillates or tends to a limit.

Parameters are quantities that don't change with time (or with the independent variable) but that can be adjusted (by natural causes or by a scientist running the experiment). For example, if we are studying the motion of a rocket, the initial mass of the rocket is a parameter. If we are studying the amount of ozone in the upper atmosphere, then the rate of release of fluorocarbons from refrigerators is a parameter. Determining how the behavior of the dependent variables changes when we adjust the parameters can be the most important aspect of the study of a model.

In Step 3 we create the equations. Most of the models we consider are expressed as differential equations. In other words, we expect to find derivatives in our equations. Look for phrases such as "rate of change of ..." or "rate of increase of ...," since rate of change is synonymous with derivative. Of course, also watch for "velocity" (derivative of position) and "acceleration" (derivative of velocity) in models from physics. The word *is* means "equals" and indicates where the equality lies. The phrase "A is proportional to B" means $A = kB$, where k is a proportionality constant (often a parameter in the model).

An important rule of thumb we use when formulating models is: *Always make the algebra as simple as possible.* For example, when modeling the velocity v of a cat falling from a tall building, we could assume:

- Air resistance increases as the cat's velocity increases.

This assumption says that air resistance provides a force that counteracts the force of gravity and that this force increases as the velocity v of the cat increases. We could