Graduate Texts in Mathematics

Gaisi Takeuti
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Axiomatic Set Theory

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Preface

This text deals with three basic techniques for constructing models of Zermelo-Fraenkel set theory: relative constructibility, Cohen's forcing, and Scott-Solovay's method of Boolean valued models. Our main concern will be the development of a unified theory that encompasses these techniques in one comprehensive framework. Consequently we will focus on certain fundamental and intrinsic relations between these methods of model construction. Extensive applications will not be treated here.

This text is a continuation of our book, "Introduction to Axiomatic Set Theory," Springer-Verlag, 1971; indeed the two texts were originally planned as a single volume. The content of this volume is essentially that of a course taught by the first author at the University of Illinois in the spring of 1969. From the first author's lectures, a first draft was prepared by Klaus Gloede with the assistance of Donald Pelletier and the second author. This draft was then revised by the first author assisted by Hisao Tanaka.

The introductory material was prepared by the second author who was also responsible for the general style of exposition throughout the text. We have included in the introductory material all the results from Boolean algebra and topology that we need. When notation from our first volume is introduced, it is accompanied with a definition, usually in a footnote. Consequently a reader who is familiar with elementary set theory will find this text quite self-contained.

We again express our deep appreciation to Klaus Gloede and Hisao Tanaka for their interest, encouragement, and hours of patient hard work in making this volume a reality. We also thank our typist, Mrs. Carolyn Bloemker, for her care and concern in typing the final manuscript.

Urbana, Illinois
March 23, 1972

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For information

A student approaching mathematical research is often discouraged by the sheer volume of the literature and the long history of the subject, even when the actual problems are readily understandable. The new series, Graduate Texts in Mathematics, is intended to bridge the gap between passive study and creative understanding; it offers introductions on a suitably advanced level to areas of current research. These introductions are neither complete surveys, nor brief accounts of the latest results only. They are textbooks carefully designed as teaching aids; the purpose of the authors is, in every case, to highlight the characteristic features of the theory.

Graduate Texts in Mathematics can serve as the basis for advanced courses. They can be either the main or subsidiary sources for seminars, and they can be used for private study. Their guiding principle is to convince the student that mathematics is a living science.

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Contents

Introduction

In this book, we present a useful technique for constructing models of Zermelo-Fraenkel set theory. Using the notion of Boolean valued relative constructibility, we will develop a theory of model construction. One feature of this theory is that it establishes a relationship between Cohen's method of forcing and Scott-Solovay's method of Boolean valued models.

The key to this theory is found in a rather simple correspondence between partial order structures and complete Boolean algebras. This correspondence is established from two basic facts; first, the regular open sets of any topological space form a complete Boolean algebra; and second, every Boolean algebra has a natural order. With each partial order structure **P**, we associate the complete Boolean algebra of regular open sets determined by the order topology on **P**. With each Boolean algebra **B**, we associate the partial order structure whose universe is that of **B** minus the zero element and whose order is the natural order on **B**.

If $\mathbf{B}_1$ is a complete Boolean algebra, if **P** is the associated partial order structure for $\mathbf{B}_1$, and if $\mathbf{B}_2$ is the associated Boolean algebra for **P**, then it is not difficult to show that $\mathbf{B}_1$ is isomorphic to $\mathbf{B}_2$ (See Theorem 1.40). This establishes a kind of duality between partial order structures and complete Boolean algebras; a duality that relates partial order structures, which have broad and flexible applications, to the very beautiful theory of Boolean valued models. It is this duality that provides a connecting link between the theory of forcing and the theory of Boolean valued models.

Numerous background results are needed for our general theory. Many of those results are well known and can be found in standard textbooks. However, to assist the reader who may not know all that we require, we devote §1 to a development of those properties of Boolean algebras, partial order structures, and topologies that will be needed later.

Throughout this text, we will use the following variable conventions. Lower case letters $a, b, c, \ldots$ are used only as set variables. Capital letters $A, B, C, \ldots$ will be used both as set variables and as class variables; in any given context, capital letters should be assumed to be set variables unless we specifically state otherwise.

Introduction

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1. Boolean Algebra

In preparation for later work, we begin with a review of the elementary properties of Boolean algebras.

Definition 1.1. A structure $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra with universe B iff $\mathbf{0}$ and $\mathbf{1}$ are two (distinct) elements of B; $+$ and $\cdot$ are binary operations on B; $^-$ is a unary operation on B; and $\forall a, b, c \in B$.

1. $a + b = b + a$	$ab = ba$	Commutative Laws.
2. $a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$	Associative Laws.
3. $a + bc = (a + b)(a + c)$	$a(b + c) = ab + ac$	Distributive Laws.
4. $\mathbf{0} + a = a$	$\mathbf{1}a = a$	Identity Laws.
5. $a + {}^-a = \mathbf{1}$	$a({}^-a) = \mathbf{0}$	Complementation Laws.

Remark. There are alternative definitions of a Boolean algebra. The reader might find it instructive to compare the definitions given in the standard texts.

Examples. 1. If $a \neq 0$ then $\langle \mathscr{P}(a)^*, \cup, \cap, ^-, 0, a\rangle$ is a Boolean algebra. If $a = 1$ we have a very special 2-element Boolean algebra that we denote by $\mathbf{2}$. Every 2-element Boolean algebra is isomorphic to $\mathbf{2}$.

2. If $a \neq 0$, $b \subseteq \mathscr{P}(a)$, $0 \in b$, $a \in b$, and if b is closed under set union, intersection, and relative complement then $\langle b, \cup, \cap, ^-, 0, a\rangle$ is a Boolean algebra. Such an algebra, i.e., one whose elements are sets and whose operations are union, intersection, and relative complement, we will call a *natural* Boolean algebra.

3. If for a first order logic whose language contains at least one predicate symbol we define an equivalence relation between sentences by

$$\phi \sim \psi \quad \text{iff} \quad \vdash [\phi \leftrightarrow \psi]$$

then the collection of equivalence classes is the universe for a Boolean algebra called the *Lindenbaum-Tarski algebra*. The operations are logical disjunction, conjunction, negation; $\vee$, $\wedge$, $\neg$, with the distinguished elements being truth and falsehood, i.e., $\mathbf{1}$ is the equivalence class of theorems and $\mathbf{0}$ is the equivalence class of contradictions.

Exercises. Prove the following for a Boolean algebra $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$:

1. $(\forall a)[a + b = a] \to b = \mathbf{0}$.
2. $(\forall a)[ab = a] \to b = \mathbf{1}$.

* $\mathscr{P}(a) = \{x | x \subseteq a\}$.

Notation: We will use the symbols $\mathbf{B}, \mathbf{B}', \mathbf{B}_1$ as variables on Boolean algebras. $|\mathbf{B}|$ is the universe of the Boolean algebra $\mathbf{B}$. When in a given context the symbols $\mathbf{0}$ and $\mathbf{1}$ appear it will be understood that they are the distinguished elements of whatever Boolean algebra is under discussion. If there are two or more Boolean algebras in the same discussion we will write $\mathbf{0_B}, \mathbf{1_B}, \mathbf{0_{B'}}, \mathbf{1_{B'}}$ to differentiate between the distinguished elements of the different spaces. If no confusion is likely the subscripts will be dropped. The same convention will be used in denoting Boolean operations.

Theorem 1.2. If $\langle B, +, \cdot, ^-, 0, 1\rangle$ is a Boolean algebra then $\forall a, b \in B$

1. $a + a = a \qquad aa = a$ — Idempotent Laws.
2. $a + ab = a \qquad a(a + b) = a$ — Absorption Laws.

Proof.

1. $a + a = (a + a)\mathbf{1} = (a + a)(a + {}^-a) = a + a({}^-a) = a + \mathbf{0} = a.$
2. $a + ab = a\mathbf{1} + ab = a(\mathbf{1} + b) = a({}^-b + b + b) = a({}^-b + b) = a\mathbf{1} = a.$

The proofs of the multiplicative properties are left to the reader.

Theorem 1.3. If $\langle B, +, \cdot, ^-, 0, 1\rangle$ is a Boolean algebra then

1. ${}^-\mathbf{0} = \mathbf{1}, {}^-\mathbf{1} = \mathbf{0}.$
2. $(\forall a \in B)[\mathbf{1} + a = \mathbf{1} \wedge \mathbf{0}a = \mathbf{0}].$

Proof.

1. ${}^-\mathbf{0} = \mathbf{0} + {}^-\mathbf{0} = \mathbf{1}.$
2. $\mathbf{1} + a = ({}^-a + a) + a = {}^-a + (a + a) = {}^-a + a = \mathbf{1}.$

The remaining proofs are left to the reader.

Theorem 1.4. If $\langle B, +, \cdot, ^-, 0, 1\rangle$ is a Boolean algebra then $\forall a, b \in B$

1. $a + b = \mathbf{1} \wedge ab = \mathbf{0} \rightarrow b = {}^-a.$
2. ${}^-({}^-a) = a.$
3. ${}^-(a + b) = ({}^-a)({}^-b), {}^-(ab) = {}^-a + {}^-b.$
4. $ab = a \leftrightarrow a + b = b.$

Proof.

1. $b = b\mathbf{1} = b(a + {}^-a) = ba + b({}^-a)$
   $= \mathbf{0} + b({}^-a) = a({}^-a) + b({}^-a)$
   $= (a + b)({}^-a) = \mathbf{1}({}^-a) = {}^-a.$
2. Since ${}^-a + a = \mathbf{1}$ and $({}^-a)a = \mathbf{0}$, we have from 1, ${}^-({}^-a) = a.$
3. $(a + b) + ({}^-a)({}^-b) = a + (b + {}^-a)(b + {}^-b)$
   $= a + (b + {}^-a) = \mathbf{1} + b = \mathbf{1}$

$$(a + b)({}^-a)({}^-b) = [a({}^-a) + b({}^-a)]({}^-b) = b({}^-a)({}^-b) = \mathbf{0}.$$

Hence by 1, ${}^-(a + b) = ({}^-a)({}^-b).$

4. If $ab = a$ then $a + b = ab + b = b$. If $a + b = b$ then $ab = a(a + b) = a.$

he proof of the other half of 3 we leave as an exercise for the reader.

Definition 1.5. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra then $\forall a, b \in B$

1. $(a - b) \triangleq a(^-b)$.
2. $(a \Rightarrow b) \triangleq {}^-a + b$.
3. $(a \Leftrightarrow b) \triangleq (a \Rightarrow b)(b \Rightarrow a)$.
4. $(a \leq b) \overset{\Delta}{\leftrightarrow} ab = a$.

Remark. We will refer to $\leq$ as the natural order on the Boolean algebra.

Theorem 1.6. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra with natural order $\leq$ then $\forall a, b, c \in B$

1. $a \leq a$.
2. $a \leq b \wedge b \leq a \rightarrow a = b$.
3. $a \leq b \wedge b \leq c \rightarrow a \leq c$.

Proof.

1. $aa = a$.
2. $a = ab = ba = b$.
3. If $a = ab \wedge b = bc$ then $a = ab = a(bc) = (ab)c = ac$.

Theorem 1.7. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra with natural order $\leq$ then $\forall a, b \in B$

1. $a \leq b \leftrightarrow {}^-b \leq {}^-a$.
2. $a \leq b \leftrightarrow a - b = \mathbf{0}$.
3. $a \leq b \leftrightarrow (a \Rightarrow b) = \mathbf{1}$.

Proof. 1. If $a \leq b$ then $a = ab$. Therefore ${}^-a = {}^-(ab) = {}^-a + {}^-b$. Then by Theorem 1.4.4 $(^-b)(^-a) = {}^-b$, i.e., ${}^-b \leq {}^-a$. Conversely if ${}^-b \leq {}^-a$ then

$$^-(^-a) \leq {}^-(^-b) \quad \text{i.e.,} \quad a \leq b.$$

2. If $a \leq b$ then $a = ab$. Therefore $a(^-b) = (ab)(^-b) = \mathbf{0}$. Conversely if $a(^-b) = \mathbf{0}$ then $a = a\mathbf{1} = a(b + {}^-b) = ab + a(^-b) = ab$ i.e., $a \leq b$.

3. If $a \leq b$ then $a = ab$ and ${}^-a = {}^-a + {}^-b$. Therefore $(a \Rightarrow b) = {}^-a + b = ({}^-a + {}^-b) + b = {}^-a + \mathbf{1} = \mathbf{1}$. Conversely if $(a \Rightarrow b) = \mathbf{1}$ then $a = a\mathbf{1} = a({}^-a + b) = ab$ i.e., $a \leq b$.

Theorem 1.8. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra with natural order $\leq$ then $\forall a, b, c, d \in B$

1. $\mathbf{0} \leq b \leq \mathbf{1}$.
2. $[a \leq b] \wedge [c \leq d] \rightarrow [ac \leq bd] \wedge [a + c \leq b + d]$.

Proof. 1. $\mathbf{0} = \mathbf{0}b \wedge b = b\mathbf{1}$.

2. If $a = ab$ and $c = cd$ then $(ac)(bd) = (ab)(cd) = ac$ and

$$(a + c)(b + d) = ab + ad + cb + cd = a + ad + cb + c = a + c.$$

Exercises. Prove the following for a Boolean algebra $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$:

1. $a \leq {}^-b \leftrightarrow ab = \mathbf{0}$.
2. $a \leq (a + b) \wedge b \leq (a + b)$.

3. $ab \leq a \wedge ab \leq b$.
4. $[a \leq c \wedge b \leq c] \rightarrow (a + b) \leq c$.
5. $[c \leq a \wedge c \leq b] \rightarrow c \leq ab$.

Definition 1.9. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra with natural order $\leq$, if $A \subseteq B$ and $b \in B$ then

1. $b = \sum_{a \in A} a \overset{\Delta}{\leftrightarrow} (\forall a \in A)[a \leq b] \wedge (\forall b' \in B)[(\forall a \in A)[a \leq b'] \rightarrow b \leq b']$.

2. $b = \prod_{a \in A} a \overset{\Delta}{\leftrightarrow} (\forall a \in A)[b \leq a] \wedge (\forall b' \in B)[(\forall a \in A)[b' \leq a] \rightarrow b' \leq b]$.

Definition 1.10. A Boolean algebra $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is complete iff

$$(\forall A \subseteq B)(\exists b, b' \in B)\left[b = \sum_{a \in A} a \wedge b' = \prod_{a \in A} a\right].$$

Example. If $a \neq 0$ then the Boolean algebra $\langle \mathscr{P}(a), \cup, \cap, ^-, 0, a\rangle$ is complete. Indeed if $A \subseteq \mathscr{P}(a)$ and $A \neq 0$, then

$$\sum_{b \in A} b = \bigcup (A) \wedge \prod_{b \in A} b = \bigcap (A).$$

Theorem 1.11. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra and $A \subseteq B$ then

1. $^-\sum_{a \in A} a = \prod_{a \in A} (^-a)$.

2. $^-\prod_{a \in A} a = \sum_{a \in A} (^-a)$.

Proof. 1. Since $(\forall b \in A)[b \leq \sum_{a \in A} a]$ we have $^-\sum_{a \in A} a \leq {^-b}$ and hence

$$^-\sum_{a \in A} a \leq \prod_{a \in A} (^-a).$$

Also $(\forall b \in A)[\prod_{a \in A} (^-a) \leq {^-b}]$. Therefore $b \leq {^-\prod_{a \in A}} (^-a)$, hence

$$\sum_{a \in A} a \leq {^-\prod_{a \in A}} (^-a)$$

i.e.,

$$\prod_{a \in A} (^-a) \leq {^-\sum_{a \in A}} a.$$

2. Left to the reader.

Theorem 1.12. If $\langle B, +, \cdot, ^-, \mathbf{0}, \mathbf{1}\rangle$ is a Boolean algebra, if $b, c \in B$, $A \subseteq B$, and

$$b = \sum_{a \in A} a$$

then

$$cb = \sum_{a \in A} ca.$$

Proof. If $a \in A$ then by Definition 1.9, $a \leq b$ and hence $ca \leq cb$. If for each $a \in A$, $ca \leq d$ then since $a = (^-c + c)a = {}^-ca + ca \leq {}^-c + d$ it follows from Definition 1.9 that $b \leq {}^-c + d$. Hence $cb \leq d$ and again from Definition 1.9 $\sum_{a\in A} ca = cb$.

Remark. Having now reviewed the basic properties of Boolean algebras we turn to the problem of characterizing complete Boolean algebras. As a first step in this direction we will show that the collection of regular open sets of a topological space is the universe of a Boolean algebra that is almost a natural algebra.

Definition 1.13. The structure $\langle X, T\rangle$ is a topological space iff $X \neq 0$,

1. $T \subseteq \mathscr{P}(X) \wedge 0 \in T \wedge X \in T$.
2. $A \subseteq T \to \bigcup (A) \in T$.
3. $(\forall N, N' \in T)[N \cap N' \in T]$.

T is a topology on X iff $\langle X, T\rangle$ is a topological space. If $a \in X$ and $N \in T$ then N is a neighborhood of a iff $a \in N$. If N is a neighborhood of a we write $N(a)$.

Theorem 1.14. $\mathscr{P}(X)$ is a topology on X.

Proof. Left to the reader.

Definition 1.15. T is the discrete topology on X iff $T = \mathscr{P}(X)$.

Definition 1.16. If T is a topology on X and $A \subseteq X$ then

1. $A^0 \triangleq \{x \in A \mid (\exists N(x))[N(x) \subseteq A]\}$.
2. $A^- \triangleq \{x \in X \mid (\forall N(x))[N(x) \cap A \neq 0]\}$.

Theorem 1.17. If T is a topology on X and $A \subseteq X$ then $A^0 \in T$.

Proof. If $B = \{N \in T \mid N \subseteq A\}$ then $B \subseteq T$. Furthermore

$$\begin{aligned} x \in A^0 &\leftrightarrow \exists N(x) \subseteq A \\ &\leftrightarrow \exists N(x) \in B \\ &\leftrightarrow x \in \textstyle\bigcup (B). \end{aligned}$$

Then $A^0 = \bigcup (B) \in T$.

Definition 1.18. T' is a base for the topology T on X iff

1. $T' \subseteq T$.
2. $(\forall A \subseteq X)[A = A^0 \to (\exists B \subseteq T')[A = \bigcup (B)]]$.

Theorem 1.19. If $X \neq 0$, if T' is a collection of subsets of X with the properties

1. $(\forall a \in X)(\exists A \in T')[a \in A]$.
2. $(\forall a \in X)(\forall A_1, A_2 \in T')[a \in A_1 \cap A_2 \to$ $(\exists A_3 \in T')[a \in A_3 \wedge A_3 \subseteq A_1 \cap A_2]]$.

Then T' is a base for a topology on X.

Proof. If $T = \{B \subseteq X \mid (\exists C \subseteq T')[B = \bigcup (C)]\}$ then $0 = \bigcup (0) \in T$ and from property 1, $X = \bigcup (T') \in T$. This establishes property 1 of Definition 1.13.

To prove 2 of Definition 1.13 we wish to show that $\bigcup (S) \in T$ whenever $S \subseteq T$. From the definition of T it is clear that if $S \subseteq T$ then $\forall B \in S, \exists C \subseteq T'$

$$B = \bigcup (C).$$

If

$$C_B = \{A \in T' \mid A \subseteq B\}$$

then

$$B = \bigcup (C_B)$$

and

$$\bigcup_{B \in S} B = \bigcup_{B \in S} \cup (C_B)$$

$$= \bigcup \Big(\bigcup_{B \in S} C_B \Big).$$

Since $\bigcup_{B \in S} C_B \subseteq T'$, $\bigcup (S) \in T$.

If $B_1, B_2 \in T$ then $\exists C_1, C_2 \subseteq T'$

$$B_1 = \bigcup (C_1) \wedge B_2 = \bigcup (C_2).$$

Therefore

$$B_1 \cap B_2 = \Big(\bigcup_{A_1 \in C_1} A_1 \Big) \cap \Big(\bigcup_{A_2 \in C_2} A_2 \Big)$$

$$= \bigcup_{\substack{A_1 \in C_1 \\ A_2 \in C_2}} (A_1 \cap A_2)$$

$$= \bigcup_{\substack{A_1 \in C_1 \\ A_2 \in C_2 \\ A_3 \subseteq A_1 \cap A_2}} A_3 \qquad \text{(By 2).}$$

Then $B_1 \cap B_2 \in T$; hence T is a topology on X. Clearly T' is a base for T.

Definition 1.20. If T is a topology on X and $A \subseteq X$ then

1. A is open iff $A = A^0$.
2. A is regular open iff $A = A^{-0}$.
3. A is closed iff $A = A^-$.
4. A is clopen iff A is both open and closed.
5. A is dense in X iff $A^- = X$.

Remark. From Theorem 1.17 we see that if T is a topology on X then T is the collection of open sets in that topology. A base for a topology is simply a collection of open sets from which all other open sets can be generated by unions.

For the set of real numbers R the intervals $(a, b) \triangleq \{x \in R \mid a < x < b\}$ form a base for what is called the natural topology on R. In this topology $(0, 1)$, and indeed every interval (a, b), is not only open but regular open. $[a, b] \triangleq \{x \in R \mid a \leq x \leq b\} = (a, b)^-$. Thus for example $[1, 2]$ is closed.

Furthermore $(0, 1) \cup (1, 2)$ is open but not regular open. The set of all rationals is dense in R. In this topology there are exactly two clopen sets 0 and R.

Theorem 1.21. 1. In any topology on X both 0 and X are clopen.

2. In the discrete topology on X every set is clopen and the collection of singleton sets is a base.

Proof. Left to the reader.

Remark. The next few theorems deal with properties that are true in every topological space $\langle X, T\rangle$. In discussing properties that depend upon X but are independent of the topology T, it is conventional to suppress reference to T and to speak simply of a topological space X. Hereafter we will use this convention.

Theorem 1.22. If $A \subseteq X$ and if $B \subseteq X$ then

1. $A^0 \subseteq A \subseteq A^-$.
2. $A^{00} = A^0 \wedge A^{--} = A^-$.
3. $A \subseteq B \to A^0 \subseteq B^0 \wedge A^- \subseteq B^-$.
4. $(X - A)^- = X - A^0 \wedge (X - A)^0 = X - A^-$.

Proof.

1. $$x \in A^0 \to \exists N(x) \subseteq A \to x \in A$$
$$x \in A \to (\forall N(x))[N(x) \cap A \neq 0] \to x \in A^-.$$

2. $$x \in A^0 \to \exists N(x) \subseteq A \to (\exists N(x))[x \in (N(x) \cap A^0) \wedge (N(x) \cap A^0) \in T \wedge (N(x) \cap A^0) \subseteq A^0] \to \exists N(x) \subseteq A^0 \to x \in A^{00}.$$

Since by 1, $A^{00} \subseteq A^0$ we conclude that $A^{00} = A^0$.

$$\begin{aligned} x \in A^{--} &\to (\forall N(x))[N(x) \cap A^- \neq 0] \\ &\to (\forall N(x))(\exists y)[y \in N(x) \wedge y \in A^-] \\ &\to (\forall N(x))(\exists y)(\exists N'(y))[N'(y) \cap A \neq 0 \wedge N'(y) \subseteq N(x)] \\ &\to (\forall N(x))[N(x) \cap A \neq 0] \\ &\to x \in A^-. \end{aligned}$$

Since by 1, $A^- \subseteq A^{--}$ it follows that $A^{--} = A^-$.

3. If $A \subseteq B$ then

$$\begin{aligned} x \in A^0 &\to \exists N(x) \subseteq A \\ &\to \exists N(x) \subseteq B \\ &\to x \in B^0 \\ x \in A^- &\to (\forall N(x))[N(x) \cap A \neq 0] \\ &\to (\forall N(x))[N(x) \cap B \neq 0] \\ &\to x \in B^-. \end{aligned}$$

4. $x \in (X - A)^- \leftrightarrow (\forall N(x))[N(x) \cap (X - A) \neq 0]$

$$\leftrightarrow (\forall N(x))[N(x) \not\subseteq A]$$

$$\leftrightarrow x \notin A^0$$

$$\leftrightarrow x \in X - A^0.$$

$$x \in (X - A)^0 \leftrightarrow \exists N(x) \subseteq (X - A)$$

$$\leftrightarrow (\exists N(x))[N(x) \cap A = 0]$$

$$\leftrightarrow x \notin A^-$$

$$\leftrightarrow x \in X - A^-.$$

Theorem 1.23. If $A \subseteq X$ and if $B \subseteq X$ then

1. A regular open implies A open.
2. A is open iff $X - A$ is closed.
3. A is closed iff $X - A$ is open.
4. $A \subseteq B$ and A dense in X implies B dense in X.

Proof.

1. If $A = A^{-0}$ then $A^0 = A^{-00} = A^{-0} = A$.
2. $A = A^0 \leftrightarrow (X - A) = (X - A^0)$
   $\leftrightarrow (X - A) = (X - A)^-$.
3. Left to the reader.
4. $A \subseteq B \rightarrow A^- \subseteq B^-$. But A dense in X implies $A^- = X$. Hence $B^- = X$.

Theorem 1.24. If C is a clopen set in the topological space X and $B^- - B^0 \subseteq C$ then $B^- - C$ is clopen.

Proof. If $x \in B^- - C$ then since $B^- - B^0 \subseteq C$

$$x \in B^0 \wedge x \notin C.$$

Since C is closed $X - C$ is open. Therefore $B^0 \cap (X - C)$ is open. Then $x \in B^0 \cap (X - C)$ implies $\exists N(x) \subseteq B^0 \cap (X - C) \subseteq B^- - C$. Thus $B^- - C$ is open.

If $(\forall N(x))[N(x) \cap (B^- - C) \neq 0]$ then

$$(\forall N(x))[N(x) \cap B^- \neq 0 \wedge N(x) \cap (X - C) \neq 0].$$

Since B^- is closed $x \in B^-$; since C is open $X - C$ is closed, hence $x \in X - C$. Therefore $x \in B^- - C$ and $B^- - C$ is closed.

Theorem 1.25. If C is a clopen set in the topological space X then $X - C$ is clopen.

Theorem 1.26. The clopen sets of a topological space form a natural Boolean algebra.

Proof. Left to the reader.

Theorem 1.27. If $A \subseteq X$ and $B \subseteq X$ then

1. $(A \cup B)^- = A^- \cup B^-$, $(A \cap B)^0 = A^0 \cap B^0$,
2. $(A \cap B)^- \subseteq A^- \cap B^-$, $A^0 \cup B^0 \subseteq (A \cup B)^0$.