Arthur Jones
Sidney A. Morris
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Abstract Algebra and Famous Impossibilities

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Arthur Jones
Kenneth R. Pearson
Department of Mathematics
La Trobe University
Bundoora 3083
Australia

Sidney A. Morris
Faculty of Informatics
University of Wollongong
Wollongong 2500
Australia

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Preface

The famous problems of squaring the circle, doubling the cube and trisecting an angle captured the imagination of both professional and amateur mathematicians for over two thousand years. Despite the enormous effort and ingenious attempts by these men and women, the problems would not yield to purely geometrical methods. It was only the development of abstract algebra in the nineteenth century which enabled mathematicians to arrive at the surprising conclusion that these constructions are not possible.

In this book we develop enough abstract algebra to prove that these constructions are impossible. Our approach introduces all the relevant concepts about fields in a way which is more concrete than usual and which avoids the use of quotient structures (and even of the Euclidean algorithm for finding the greatest common divisor of two polynomials). Having the geometrical questions as a specific goal provides motivation for the introduction of the algebraic concepts and we have found that students respond very favourably.

We have used this text to teach second-year students at La Trobe University over a period of many years, each time refining the material in the light of student performance.

The text is pitched at a level suitable for students who have already taken a course in linear algebra, including the ideas of a vector space over a field, linear independence, basis and dimension. The treatment, in such a course, of fields and vector spaces as algebraic objects should provide an adequate background for the study of this book. Hence the book is suitable for Junior/Senior courses in North America and second-year courses in Australia.

Chapters 1 to 6, which develop the link between geometry and algebra, are the core of this book. These chapters contain a complete solution to the three famous problems, except for proving that π is a transcendental number (which is needed to complete the proof of the impossibility of squaring the circle). In Chapter 7 we give a self-contained proof that π is transcendental. Chapter 8 contains material about fields which is closely related to the topics in Chapters 2–4, although it is not required in the proof of the impossibility of the three constructions. The short concluding Chapter 9 describes some other areas of mathematics in which algebraic machinery can be used to prove impossibilities.

We expect that any course based on this book will include all of Chapters 1 – 6 and (ideally) at least passing reference to Chapter 9. We have often taught such a course which we cover in a term (about twenty hours). We find it essential for the course to be paced in a way that allows time for students to do a substantial number of problems for themselves. Different semester length (or longer) courses including topics from Chapters 7 and 8 are possible. The three natural parts of these are

(1) Sections 7.1 and 7.2 (transcendence of e),

(2) Sections 7.3 to 7.6 (transcendence of π),

(3) Chapter 8.

These are independent except, of course, that (2) depends on (1). Possible extensions to the basic course are to include one, two or all of these. While most treatments of the transcendence of π require familiarity with the theory of functions of a complex variable and complex integrals, ours in Chapter 7 is accessible to students who have completed the usual introductory real calculus course (first-year in Australia and Freshman/Sophomore in North America). However instructors should note that the arguments in Sections 7.3 to 7.6 are more difficult and demanding than those in the rest of the book.

Problems are given at the end of each section (rather than collected at the end of the chapter). Some of these are computational and others require students to give simple proofs.

Each chapter contains additional reading suitable for students and instructors. We hope that the text itself will encourage students to do further reading on some of the topics covered.

As in many books, exercises marked with an asterisk * are a good bit harder than the others. We believe it is important to identify clearly the end of each proof and we use the symbol ■ for this purpose.

We have found that students often lack the mathematical maturity required to write or understand simple proofs. It helps if students write down where the proof is heading, what they have to prove and how they might be able to prove it. Because this is not part of the formal proof, we indicate this exploration by separating it from the proof proper by using a box which looks like

(Include here what must be proved etc.)

Experience has shown that it helps students to use this material if important theorems are given specific names which suggest their content. We have enclosed these names in square brackets before the statement of the theorem. We encourage students to use these names when justifying their solutions to exercises. They often find it convenient to abbreviate the names to just the relevant initials. (For example, the name "Small Degree Irreducibility Theorem" can be abbreviated to S.D.I.T.)

We are especially grateful to our colleague Gary Davis, who pointed the way towards a more concrete treatment of field extensions (using residue rings rather than quotient rings) and thus made the course accessible to a wider class of students. We are grateful to Ernie Bowen, Jeff Brooks, Grant Cairns, Mike Canfell, Brian Davey, Alistair Gray, Paul Halmos, Peter Hodge, Alwyn Horadam, Deborah King, Margaret McIntyre, Bernhard H. Neumann, Kristen Pearson, Suzanne Pearson, Alf van der Poorten, Brailey Sims, Ed Smith and Peter Stacey, who have given us helpful feedback, made suggestions and assisted with the proof reading.

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A.J., S.A.M., K.R.P.

April 1991

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Contents

Introduction

0.1 Three Famous Problems

In this book we discuss three of the oldest problems in mathematics. Each of them is over 2,000 years old. The three problems are known as:

[I] doubling the cube (or duplicating the cube, or the Delian problem);

[II] trisecting an arbitrary angle;

[III] squaring the circle (or quadrature of the circle).

Problem I is to construct a cube having twice the volume of a given cube. Problem II is to describe how every angle can be trisected. Problem III is that of constructing a square whose area is equal to that of a given circle. In all cases, the constructions are to be carried out using only a ruler and compass.

Reference to Problem I occurs in the following ancient document supposedly written by Eratosthenes to King Ptolemy III about the year 240 B.C.:

To King Ptolemy, Eratosthenes sends greetings. It is said that one of the ancient tragic poets represented Minos as preparing a tomb for Glaucus and as declaring, when he learnt it was a hundred feet each way: "Small indeed is the tomb thou hast chosen for a royal burial. Let it be double [in volume]. And thou shalt not miss that fair form if thou quickly doublest each side of the tomb." But he was wrong. For when the sides are doubled, the surface [area] becomes four times as great, and the volume eight times. It became a subject of inquiry among geometers in what manner one might double the given volume without changing the shape. And this problem was called the duplication of the cube, for given a cube they sought to double it ...

The origins of Problem II are obscure. The Greeks were concerned with the problem of constructing regular polygons, and it is likely that the trisection problem arose in this context. This is so because the construction of a regular polygon with nine sides necessitates the trisection of an angle.

The history of Problem III is linked to that of calculating the area of a circle. Information about this is contained in the Rhind Papyrus, perhaps the best known ancient mathematical manuscript, which was brought by A.H. Rhind to the British Museum in the nineteenth century. The manuscript was copied by the scribe Ahmes about 1650 B.C. from an even older work. It states that the area of a circle is equal to that of a square whose side is the diameter diminished by one ninth; that is, $A = \left(\frac{8}{9}\right)^2 d^2$. Comparing this with the formula $A = \pi r^2 = \pi \frac{d^2}{4}$ gives

$$\pi = 4.\left(\frac{8}{9}\right)^2 = \frac{256}{81} = 3.1604\ldots.$$

The Papyrus contains no explanation of how this formula was obtained. Fifteen hundred years later Archimedes showed that

$$3\,\frac{10}{71} < \pi < 3\,\frac{10}{70} = 3\,\frac{1}{7}.$$

(Note that $3\frac{10}{71} = 3.14084\ldots$, $3\frac{1}{7} = 3.14285\ldots$, $\pi = 3.14159\ldots$.)

Throughout the ages these problems were tackled by most of the best mathematicians. For some reason, amateur mathematicians were also fascinated by them. In the time of the Greeks a special word was used to describe people who tried to solve Problem III – *τετραγωνιζειν* (tetragonidzein) which means *to occupy oneself with the quadrature*.

In 1775 the Paris Academy found it necessary to protect its officials from wasting their time and energy examining purported solutions of these problems by amateur mathematicians. It passed a resolution (Histoire de l'Académie royale, année 1775, p. 61) that no more solutions were to be examined of the problems of doubling the cube, trisecting an arbitrary angle, and squaring the circle and that the same resolution should apply to machines for exhibiting perpetual motion. (See [EH, p. 3].)

The problems were finally solved in the nineteenth century. In 1837, Wantzel settled Problems I and II. In 1882, Lindemann disposed of Problem III.

0.2 Straightedge and Compass Constructions

Construction problems are, and have always been, a favourite topic in geometry. Using only a ruler and compass, a great variety of constructions is possible. Some of these constructions are described in detail in Section 5.1:

> a line segment can be bisected; any angle can be bisected; a line can be drawn from a given point perpendicular to a given line; etc.

In all of these problems *the ruler is used merely as a straight edge, an instrument for drawing a straight line but not for measuring or marking off distances.* Such a ruler will be referred to as a *straightedge.*

Problem I is that of constructing with compass and straightedge a cube having twice the volume of a given cube. If the side of the given cube has length 1 unit, then the volume of the given cube is $1^3 = 1$. So the volume of the larger cube should be 2, and its sides should thus have length $\sqrt[3]{2}$. *Hence the problem is reduced to that of constructing, from a segment of length 1, a segment of length* $\sqrt[3]{2}$.

Problem II is to produce a construction for trisecting any given angle. While it is easy to give examples of particular angles which can be trisected, the problem is to give a construction which will work for every angle.

Problem III is that of constructing with compass and straightedge a square of area equal to that of a given circle. If the radius of the circle is taken as one unit, the area of the circle is π, and therefore the area of the constructed square should be π; that is, the side of the square should be $\sqrt{\pi}$. *So the problem is reduced to that of constructing, from a segment of length 1, a segment of length* $\sqrt{\pi}$.

0.3 Impossibility of the Constructions

Why did it take so many centuries for these problems to be solved? The reasons are (i) the required constructions are `impossible`, and (ii) a full understanding of these problems comes not from geometry but from abstract algebra (a subject not born until the nineteenth century). The purpose of this book is to introduce this algebra and show how it is used to prove the impossibility of these constructions.

A real number γ is said to be *constructible* if, starting from a line segment of length 1, we can construct a line segment of length $|\gamma|$ in a finite number of steps using straightedge and compass.

We shall prove, in Chapters 5 and 6, that a real number is constructible if and only if it can be obtained from the number 1 by successive applications of the operations of addition, subtraction, multiplication, division, and taking square roots. Thus, for example, the number

$$2 + \sqrt{3 + 10\sqrt{2}}$$

is constructible.

Now $\sqrt[3]{2}$ does not appear to have this form. Appearances can be deceiving, however. How can we be sure? The answer turns out to be that if $\sqrt[3]{2}$ did have this form, then a certain vector space would have the wrong dimension! This settles Problem I.

As for Problem II, note that it is sufficient to give just one example of an angle which cannot be trisected. One such example is the angle of 60°. It can be shown that this angle can be trisected only if $\cos 20°$ is a constructible number. But, as we shall see in Chapter 6, the number $\cos 20°$ is a solution of the cubic equation

$$8x^3 - 6x - 1 = 0$$

which does not factorize over the rational numbers. Hence it seems likely that *cube roots*, rather than square roots, will be involved in its solution, so we would not expect $\cos 20°$ to be constructible. Once again this can be made into a rigorous proof by considering the possible dimensions of a certain vector space.

As we shall show in Chapter 6, the solution of Problem III also hinges on the dimension of a vector space. Indeed, the impossibility of squaring the circle follows from the fact that a certain vector space (a different one from those mentioned above in connection with Problems I and II) is not finite-dimensional. This in turn is because the number π is "transcendental", which we shall prove in Chapter 7.

Additional Reading for the Introduction

More information about the background to the three problems can be found in the various books on the history of mathematics listed below. References dealing specifically with Greek mathematics include [JG], [TH] and [FLa]. A detailed history of Problem III is given in [EH].

The original solutions to Problems I and II by Wantzel are in [MW] and the original solution to Problem III by Lindemann is in [FL].

[EB1] E.T. Bell, *Men of Mathematics*, Simon and Shuster, New York, 1937.

[EB2] E.T. Bell, *The Development of Mathematics*, McGraw-Hill, New York, 1945.

[CB] C.B. Boyer, *A History of Mathematics*, Wiley, New York, 1968.

[RC] R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, New York, 1941.

[AD] A. De Morgan, *A Budget of Paradoxes*, Dover, New York, 1954.

[UD] U. Dudley, *A Budget of Trisections*, Springer-Verlag, New York, 1987.

[JG] J. Gow, *A Short History of Greek Mathematics*, Chelsea, New York, 1968.

[CH] C.R. Hadlock, *Field Theory and its Classical Problems*, Carus Mathematical Monographs, No. 19, Mathematical Association of America, 1978.

[TH] T.L. Heath, *A History of Greek Mathematics Vol.I & II*, Clarendon Press, Oxford, 1921.

[EH] E.W. Hobson, *Squaring the Circle*, Cambridge University Press, 1913; reprinted in *Squaring the Circle, and Other Monographs*, Chelsea, 1953.

[HH] H.P. Hudson, *Ruler and Compass*, Longmans Green, 1916; reprinted in *Squaring the Circle, and Other Monographs*, Chelsea, 1953.

[FK] F. Klein, *Famous Problems, and other monographs*, Chelsea, New York, 1962.

[MK] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.

[FLa] F. Lasserre, *The Birth of Mathematics in the Age of Plato*, Hutchinson, London, 1964.

[FL] F.L. Lindemann, "Über die Zahl π", *Mathematische Annalen*, 20 (1882), 213–225.

[DS] D.J. Struik, *A Concise History of Mathematics*, Dover, New York, 1967.

[VS] V. Sanford, *A Short History of Mathematics*, Harrap, London, 1958.

[HT] H.W. Turnbull, *The Great Mathematicians*, Methuen, London 1933.

[MW] M.L. Wantzel, "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas", *Journal de Mathématiques Pures et Appliquées*, 2 (1837), 366-372.