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DIFFERENTIAL and INTEGRAL CALCULUS

PEACE PUBLISHERS
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ДИФФЕРЕНЦИАЛЬНОЕ И ИНТЕГРАЛЬНОЕ ИСЧИСЛЕНИЯ

На английском языке

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PREFACE

This text is designed as a course of mathematics for higher technical schools. It contains many worked examples that illustrate the theoretical material and serve as models for solving problems.

The first two chapters "Number. Variable. Function" and "Limit. Continuity of a Function" have been made as short as possible. Some of the questions that are usually discussed in these chapters have been put in the third and subsequent chapters without loss of continuity. This has made it possible to take up very early the basic concept of differential calculus—the derivative—which is required in the study of technical subjects. Experience has shown this arrangement of the material to be the best and most convenient for the student.

A large number of problems have been included, many of which illustrate the interrelationships of mathematics and other disciplines. The problems are specially selected (and in sufficient number) for each section of the course thus helping the student to master the theoretical material. To a large extent, this makes the use of a separate book of problems unnecessary and extends the usefulness of this text as a course of mathematics for self-instruction.

N. S. Piskunov

CHAPTER I

NUMBER. VARIABLE. FUNCTION

SEC. 1. REAL NUMBERS. REAL NUMBERS AS POINTS ON A NUMBER SCALE

Number is one of the basic concepts of mathematics. It originated in ancient times and has undergone expansion and generalisation over the centuries.

Whole numbers and fractions, both positive and negative, together with the number zero are called *rational numbers*. Every rational number may be represented in the form of a ratio, $\frac{p}{q}$, of two integers p and q; for example,

$$\frac{5}{7}, \quad 1.25=\frac{5}{4}.$$

In particular, the integer p may be regarded as a ratio of the integers $\frac{p}{1}$; for example,

$$6=\frac{6}{1}, \quad 0=\frac{0}{1}.$$

Rational numbers may be represented in the form of periodic terminating or nonterminating fractions. Numbers represented by nonterminating, but nonperiodic, decimal fractions are called *irrational numbers*; such are the numbers $\sqrt{2}$, $\sqrt{3}$, $5-\sqrt{2}$, etc.

The collection of all rational and irrational numbers makes up the set of *real numbers*. The real numbers are *ordered in magnitude*; that is to say, for each pair of real numbers x and y there is one, and only one, of the following relations:

$$x<y, \quad x=y, \quad x>y.$$

Real numbers may be depicted as points on a number scale. A *number scale* is an infinite straight line on which are chosen: 1) a certain point O called the origin, 2) a positive direction indicated by an arrow, and 3) a suitable unit of length. We shall usually make the number scale horizontal and take the positive direction to be from left to right.

If the number x_1 is positive, it is depicted as a point M_1 at a distance $OM_1=x_1$ to the right of the origin O; if the number x_2 is negative, it is represented by a point M_2 to the left of O at a distance $OM_2 = -x_2$ (Fig. 1). The point O represents the number zero. It is obvious that every real number is represented by a definite point on the number scale. Two different real numbers are represented by different points on the number scale.

The following assertion is also true: each point on the number scale represents only one real number (rational or irrational).

To summarise, all real numbers and all points on the number scale are in one-to-one correspondence: to each number there corresponds only one point, and conversely, to each point there corresponds only one number. This frequently enables us to regard "the number x" and "the point x" as, in a certain sense, equivalent expressions. We shall make wide use of this circumstance in our course.

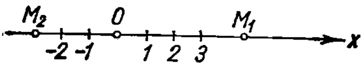

Fig. 1.

We state without proof the following important property of the set of real numbers: *both rational and irrational numbers may be found between any two arbitrary real numbers.* In geometrical terms, this proposition reads thus: *both rational and irrational points may be found between any two arbitrary points on the number scale.*

In conclusion we give the following theorem, which, in a certain sense, represents a bridge between theory and practice.

Theorem. *Every irrational number α may be expressed, to any degree of precision, with the aid of rational numbers.*

Indeed, let the irrational number $\alpha > 0$ and let it be required to evaluate α with an accuracy of $\frac{1}{n}$ $\left(\text{for example, } \frac{1}{10}, \frac{1}{100}, \text{ and so forth}\right)$.

No matter what α is, it lies between two integral numbers N and $N+1$. We divide the segment between N and $N+1$ into n parts; then α will lie somewhere between the rational numbers $N+\frac{m}{n}$ and $N+\frac{m+1}{n}$. Since their difference is equal to $\frac{1}{n}$, each of them expresses α to the given degree of accuracy, the former being smaller and the latter greater.

Example. The irrational number $\sqrt{2}$ is expressed by rational numbers:
1.4 and 1.5 to one decimal place,
1.41 and 1.42 to two decimal places,
1.414 and 1.415 to three decimal places, etc.

SEC. 2. THE ABSOLUTE VALUE OF A REAL NUMBER

Let us introduce a concept which we shall need later on: the absolute value of a real number.

Definition. *The absolute value (or modulus)* of a real number x (written $|x|$) is a nonnegative real number that satisfies the conditions

$$|x| = x \quad \text{if } x \geqslant 0;$$
$$|x| = -x \quad \text{if } x < 0.$$

Examples. $|2| = 2$; $|-5| = 5$; $|0| = 0$.

From the definition it follows that the relationship $x \leqslant |x|$ holds for any x.

Let us examine some of the properties of absolute values.

1. *The absolute value of an algebraic sum of several real numbers is no greater than the sum of the absolute values of the terms* $|x+y| \leqslant |x| + |y|$.

Proof. Let $x + y \geqslant 0$, then

$$|x+y| = x+y \leqslant |x| + |y| \text{ (since } x \leqslant |x| \text{ and } y \leqslant |y|).$$

Let $x + y < 0$, then

$$|x+y| = -(x+y) = (-x) + (-y) \leqslant |x| + |y|.$$

This completes the proof.

The foregoing proof is readily extended to any number of terms.

Examples.

$$|-2+3| < |-2| + |3| = 2+3 = 5 \text{ or } 1 < 5;$$
$$|-3-5| = |-3| + |-5| = 3+5 = 8 \text{ or } 8 = 8.$$

2. *The absolute value of a difference is no less than the difference of the absolute values of the minuend and subtrahend*:

$$|x-y| \geqslant |x| - |y|.$$

Proof. Let $x - y = z$, then $x = y + z$ and from what has been proved

$$|x| = |y+z| \leqslant |y| + |z| = |y| + |x-y|,$$

whence

$$|x| - |y| \leqslant |x-y|,$$

thus completing the proof.

3. *The absolute value of a product is equal to the product of the absolute values of the factors*:

$$|xyz| = |x|\,|y|\,|z|.$$

4. *The absolute value of a quotient is equal to the quotient of the absolute values of the dividend and the divisor*:

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

The latter two properties follow directly from the definition of absolute value.

SEC. 3. VARIABLES AND CONSTANTS

The numerical values of such physical quantities as time, length, area, volume, mass, velocity, pressure, temperature, etc., are determined by measurement. Mathematics deals with quantities divested of any specific content. From now on, when speaking of quantities, we shall have in view their numerical values. In various phenomena, the numerical values of certain quantities vary, while the numerical values of others remain fixed. For instance, in uniform motion of a point, time and distance change, while the velocity remains constant.

A *variable* is a quantity that takes on various numerical values. A *constant* is a quantity whose numerical values remain fixed. We shall use the letters x, y, z, u, ..., etc., to designate variables, and the letters a, b, c, ..., etc., to designate constants.

Note. In mathematics, a constant is frequently regarded as a special case of variable whose numerical values are the same.

It should be noted that when considering specific physical phenomena it may happen that one and the same quantity in one phenomenon is a constant while in another it is a variable. For example, the velocity of uniform motion is a constant, while the velocity of uniformly accelerated motion is a variable. Quantities that have the same value under all circumstances are called *absolute constants*. For example, the ratio of the circumference of a circle to its diameter is an absolute constant: $\pi = 3.14159$.

As we shall see throughout this course, the concept of a variable quantity is the basic concept of differential and integral calculus. In "Dialectics of Nature", Friedrich Engels wrote: "The turning point in mathematics was Descartes' variable magnitude. With that came *motion* and hence *dialectics* in mathematics, and *at once*, too, *of necessity* the differential and integral calculus."

SEC. 4. THE RANGE OF A VARIABLE

A variable takes on a series of numerical values. The collection of these values may differ depending on the character of the problem. For example, the temperature of water heated under ordinary conditions will vary from room temperature (15-18°C) to the boiling point, 100°C. The variable quantity $x = \cos\alpha$ can take on all values from -1 to $+1$.

The values of a variable are geometrically depicted as points on a number scale. For instance, the values of the variable $x = \cos\alpha$ for all possible values of α are depicted as the set of points of an interval on the number scale, from -1 to 1, including the points 1— and 1 (Fig. 2).