Lecture Notes in Mathematics

Gitta Kutyniok

Affine Density in Wavelet Analysis

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Until September 2007

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Library of Congress Control Number: 2007928330
Mathematics Subject Classification (2000): 42C40, 42C15, 94A12

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN 978-3-540-72916-7 Springer Berlin Heidelberg New York

DOI 10.1007/978-3-540-72949-5

Springer is a part of Springer Science+Business Media
springer.com

Typesetting by the author and SPi using a Springer LaTeX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 12070794 41/SPi 5 4 3 2 1 0

Dedicated to

my Parents

Preface

During the last 20 years, wavelet analysis has become a major research area in mathematics, not only because of the beauty of the mathematical theory of wavelet systems (sometimes also called affine systems), but also because of its significant impact on applications, especially in signal and image processing. After the extensive exploration of orthonormal bases of classical affine systems that has occupied much of the history of wavelet theory, recently both *wavelet frames* — redundant wavelet systems — and *irregular wavelet systems* — wavelet systems with an arbitrary sequence of time-scale indices — have come into focus as a main area of research. Two main reasons for this are to serve new applications which require robustness against noise and erasures, and to derive a deeper understanding of the theory of classical affine systems. However, a comprehensive theory to treat irregular wavelet frames does not exist so far. The main difficulty consists of the highly sensitive interplay between geometric properties of the sequence of time-scale indices and frame properties of the associated wavelet system.

In this research monograph, we introduce the new notion of affine density for sequences of time-scale indices to wavelet analysis as a highly effective tool for studying irregular wavelet frames. We present many results concerning the structure of weighted irregular wavelet systems with finitely many generators, adding considerably to our understanding of the relation between the geometry of the time-scale indices of these general wavelet systems and their frame properties.

This book is the author's Habilitationsschrift in mathematics at the Justus-Liebig-Universität Gießen. It is organized as follows. The introduction presents a detailed overview of the recent developments in the study of irregular wavelet frames and of the already quite established theory of the relation between Beurling density and the geometry of sequences of time-frequency indices of Gabor systems. Furthermore, it explains our main results in an informal way. Chapter 2 reviews the terminology and notations from frame theory as well as from wavelet and time-frequency analysis employed in this book.

The notion of weighted affine density, which will turn out to be a most effective tool for studying the geometry of sequences of time-scale indices associated with weighted irregular wavelet systems, will be introduced in Chapter 3. We illustrate the new notion by giving several examples. We further compare this notion of affine density with the affine density that was independently and simultaneously introduced by Sun and Zhou [119] and point out the advantages of our notion.

In Chapter 4, we prove that the notion of weighted affine density leads to very elegant necessary conditions for the existence of general wavelet frames on the sequence of time-scale indices. The usefulness of this notion is emphasized by its utility for the study of a rather technical-appearing hypothesis known as the *local integrability condition (LIC)* of a characterization result for weighted wavelet Parseval frames by Hernández, Labate, and Weiss [77]. In fact, we show that under a mild regularity assumption on the analyzing wavelets, the LIC is in fact solely a density condition.

Chapter 5 is devoted to the study of a quantitative relation between frame bounds and affine density conditions, since the complexity of frame algorithms is strongly related to the values of the frame bounds. A striking result here is a fundamental relationship between the affine density of the sequence of time-scale indices, the frame bounds, and the admissibility constant of a weighted irregular wavelet frame with finitely many generators. Several implications of this result are outlined, among which is the revelation of a reason for the non-existence of a Nyquist phenomenon for wavelet systems and the uniformity of sequences of time-scale indices associated with tight wavelet frames. In addition, we also present the first result in which the existence of particular wavelet frames is completely characterized by density conditions. The non-existence of very general co-affine frames is then shown to follow as a corollary.

In Chapter 6, we show that most irregular wavelet frames (and even wavelet Schauder bases) satisfy a so-called *Homogeneous Approximation Property (HAP)*. This property not only implies certain invariance properties under time-scale shifts when approximating with wavelet frames, but is also shown to have impact on density considerations. In addition to these main results, our techniques introduce some very useful new tools for the study of wavelet systems, e.g., certain Wiener amalgam spaces and — related with these objects — a particular class of analyzing wavelets.

Chapter 7 is devoted to the study of shift-invariance, i.e., invariance under integer translations, which is a desirable feature for many applications, since this ensures that similar structures in a signal are more easily detectable. The oversampling theorems from wavelet analysis show that most classical affine systems can be turned into a shift-invariant wavelet system with comparable frame properties. Most interestingly, the process also leaves density properties invariant, and the question concerning necessity of this fact for irregular wavelet systems arises. In this chapter we study the analog of this problem in time-frequency analysis and give a complete answer for irregular Gabor systems. Along the way we introduce a new notion of weighted Beurling density and derive extensions of results from H. Landau [97], and Balan, Casazza, Heil, and Z. Landau [7]. The results obtained in this chapter are not only interesting by itself, but can also be regarded as an important step towards the study of similar questions in wavelet analysis.

Acknowledgments

I am first and foremost indebted to Christopher Heil, who initiated the research for this book by inviting me as a Visiting Assistant Professor to the School of Mathematics at the Georgia Institute of Technology for the Fall Term of 2001. He has been a great friend and wonderful collaborator since then. I am deeply grateful to him for agreeing to include generalizations to multiple generators of our joint papers [73, 75], joint results on the HAP for Schauder bases (see Chapter 6), and a result from our joint paper [74] in this book. I also owe special thanks to Guido Weiss for inviting me to the Department of Mathematics at Washington University in St. Louis several times and for numerous inspiring discussions about my work. Further, I am indebted to Pete Casazza for introducing me to frame theory, for inviting me to the Department of Mathematics at the University of Missouri, and for being a great collaborator in several joint research projects.

I would like to thank Akram Aldroubi, Radu Balan, Marcin Bownik, Maria Carro, Ole Christensen, Charles Chui, Wojciech Czaja, Hans Feichtinger, Karlheinz Gröchenig, Jeffrey Hogan, Mihail Kolountzakis, Ilya Krishtal, Demetrio Labate, Zeph Landau, Ursula Molter, Amos Ron, Zouwei Shen, Javier Soria, Darrin Speegle, Yang Wang, and Edward Wilson for their interest in my work and for their many useful suggestions and hints. Further thanks go to Martin Buhmann and Christopher Heil for careful reading of parts of this book.

I am thankful to the Institutes of Mathematics at the Universität Paderborn and the Justus–Liebig–Universität Gießen for creating a pleasant working atmosphere. I would also like to acknowledge support from the German Research Foundation (DFG) in the form of DFG Research Fellowship KU 1446/5-1. This enabled me to spent one year in the U.S. during which a large portion of research for this book was developed. In this regard I am also thankful to the Department of Mathematics at Washington University in St. Louis and the School of Mathematics at the Georgia Institute of Technology for their hospitality and support during this year. I am indebted to Martin Buhmann and Tomas Sauer for supporting this research stay and consenting to my absence from the Institute of Mathematics at the Justus–Liebig–Universität Gießen during that period of time.

I would like to thank the staff at Springer-Verlag Heidelberg for editorial advise and help. I am grateful to Fernanda Laezza, Demetrio and Francesca

Labate, and Shelby, Bernie, and Lindsay Johnson for accepting me as a family member during my times in St. Louis and Atlanta and thereby contributing to make these stays a wonderful experience. Finally, I wish to thank my parents Hildegard and Norbert Kutyniok for their endless support which I can always count on.

Gießen, June 2006 *Gitta Kutyniok*

Contents

1 Introduction ... 1
1.1 Irregular Wavelet and Gabor Frames ... 1
1.2 Density for Gabor Systems ... 3
1.3 Geometry of Time-Scale Indices ... 5
1.4 Overview of Main Results ... 7

2 Wavelet and Gabor Frames ... 11
2.1 Frame Theory ... 11
2.2 Wavelet Analysis ... 13
2.3 Time-Frequency Analysis ... 16
2.4 Amalgam Spaces ... 19

3 Weighted Affine Density ... 21
3.1 Definitions ... 21
3.2 Basic Properties ... 24
3.3 The Notion of Affine Density by Sun and Zhou ... 27
3.4 Comparison of Both Notions of Affine Density ... 32

4 Qualitative Density Conditions ... 35
4.1 Existence of an Upper Frame Bound ... 35
4.2 Existence of a Lower Frame Bound ... 37
4.3 Examples of Wavelet Systems ... 42
4.4 Density of Sequences in $\mathbb{R}^+$ ... 45
4.5 Affine Density and the Local Integrability Condition ... 48
4.5.1 Amalgam Spaces on $\mathbb{R} \setminus \{0\}$ ... 48
4.5.2 A Density Version of the Local Integrability Condition ... 50
4.5.3 A Characterization of Wavelet Parseval Frames ... 56

5 Quantitative Density Conditions ... 59
5.1 Outline and Comparison with Previous Work ... 59
5.2 Density of Product Sequences ... 61

5.3 A Fundamental Relationship 63
5.4 The Nyquist Phenomenon 75
5.5 Sufficient Density Conditions for Wavelet Frames 76
5.6 Existence of Special Weight Functions 81
5.7 Co-Affine Systems 82

6 Homogeneous Approximation Property 87
6.1 Amalgam Spaces and the Continuous Wavelet Transform 87
6.2 The Basic Class $\mathcal{B}_0$ 89
6.3 The Homogeneous Approximation Property for Wavelet Frames 93
6.4 The Comparison Theorem for Wavelet Frames 97
6.5 Density Results for Wavelet Schauder Bases 100

7 Weighted Beurling Density and Shift-Invariant Gabor Systems 105
7.1 Motivation and Outline of Chapter 105
7.2 Weighted Beurling Density 107
7.3 Littlewood–Paley Type Inequalities 110
7.4 Equivalent Definition of Weighted Beurling Density 115
7.5 Beurling Density, Frame Bounds, and Gabor Generators 119
7.6 Shift-Invariant Gabor Systems 121

References 127

Index 135

1

Introduction

1.1 Irregular Wavelet and Gabor Frames

Wavelet analysis has attracted rapidly increasing attention since Daubechies' groundbreaking book [41] in 1992 and is nowadays one of the major research areas in applied mathematics. The analyzing systems commonly used in wavelet analysis are the *classical affine systems*. Such a system consists of the collection of time-scale shifts of a function $\psi \in L^2(\mathbb{R})$, called the *analyzing wavelet*, associated with two parameters $a > 1$ and $b > 0$ and is given by

$$\{a^{-\frac{j}{2}}\psi(a^{-j}x - bk)\}_{j,k\in\mathbb{Z}}.$$

The origins of *time-frequency analysis* trace back to Gabor's article [59] on information theory, which appeared in 1946. This theory has also since become an important, independent branch of applied harmonic analysis. The function systems most often employed in this theory are the *regular Gabor systems*, which comprise the collection of time-frequency shifts determined by a function $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$, specifically

$$\{e^{2\pi ibnx}g(x - ak)\}_{k,n\in\mathbb{Z}}.$$

There exist extensions to the higher dimensional situation for both systems, but in this introduction we restrict our discussion to the one-dimensional case for simplicity.

Both wavelet and Gabor systems play important roles in signal processing and data compression, e.g., in developing JPEG 2000, in solving MRI problems, and for the FBI fingerprint database (see, for instance, the books by Benedetto and Ferreira [8], Chui [25, 26], Feichtinger and Strohmer [56, 57], and Mallat [100]). These two types of systems are also a source of many intriguing mathematical problems and a useful tool in other areas of mathematics, see, e.g., the applications of wavelets to the study of Navier-Stokes or Euler equations (see, for instance, the books authored by Debnath [44] and Hogan and Lakey [82]).

Until some years ago the focus of research in wavelet analysis had been mainly on the construction of orthonormal bases. But recently the theory of frames, which generalize the notion of bases by allowing redundancy yet still providing a reconstruction formula, has been growing rapidly, since several new applications have been developed. Due to their robustness not only against noise but also against losses, and due to their freedom in design, frames — especially tight frames — have proven themselves an essential tool for a variety of applications such as, for example, nonlinear sparse approximation, coarse quantization, data transmission with erasures, and wireless communications (see, for instance, Benedetto, Powell, and Yilmaz [10], Candès and Donoho [13], Goyal, Kovačević, and Kelner [60], and Strohmer and Heath [116]). Gabor frames have already been studied for a longer time (cf. the books by Feichtinger and Strohmer [56, 57]), but recently also wavelet frames have become a main area of research in the wavelet community. (See, for example, the various papers authored by Chan, Chui, Czaja, Daubechies, Gröchenig, Han, He, Hernández, Labate, Maggioni, Riemenschneider, Ron, L. Shen, Z. Shen, Shi, Stöckler, Q. Sun, and Weiss [109, 67, 34, 27, 28, 29, 77, 32, 42, 19, 30, 31, 111].)

However, most results concerning wavelet and Gabor frames are restricted to the special cases of classical affine systems and regular Gabor systems. Recently, general irregular wavelet and Gabor systems, which can be built by using arbitrary time-scale or time-frequency shifts, have attracted increasing attention (see, for instance, the papers by Aldroubi, Balan, Cabrelli, Casazza, Christensen, Deng, Favier, Feichtinger, Felipe, Heil, Kaiblinger, Kutyniok, Lammers, Z. Landau, Molter, Ramanathan, Steger, W. Sun, and Zhou [107, 22, 23, 18, 118, 15, 73, 92, 119, 120, 1, 55, 121, 7, 93, 94, 95, 117]). An *irregular wavelet system* is determined by an analyzing wavelet $\psi \in L^2(\mathbb{R})$ and a sequence of time-scale indices $\Lambda \subseteq \mathbb{R}^+ \times \mathbb{R}$, regarded as a sequence in the affine group $\mathbb{A}$, and is defined by

$$\mathcal{W}(\psi, \Lambda) = \{a^{-\frac{1}{2}}\psi(a^{-1}x - b)\}_{(a,b)\in\Lambda}.$$

Given a function $g \in L^2(\mathbb{R})$ and a sequence of time-frequency indices $\Lambda \subseteq \mathbb{R}^2$, an *irregular Gabor system* is given by

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi ibx}g(x - a)\}_{(a,b)\in\Lambda}.$$

The necessity of studying these general systems occurs since in practice a sequence of time-scale or time-frequency indices might be perturbed due to the impact of noise or other disturbances or may be directly imposed by the application at hand. Therefore results about the impact of properties of the sequence of time-scale or time-frequency indices on frame properties of the associated wavelet or Gabor system will turn out to be essentiell. Moreover, the study of irregular systems is very interesting from the mathematical point of view in deriving a deeper understanding of the theory of wavelet systems or

Gabor systems and, in particular, of the special case of classical affine systems or regular Gabor systems.

Later, it will become necessary to additionally equip the analyzing functions contained in the system with weights and also to consider systems with finitely many generators.

1.2 Density for Gabor Systems

Since time-scale and time-frequency indices associated with irregular wavelet and Gabor systems are initially completely arbitrary, we are led naturally to questions concerning the relation between their geometrical structure and the frame properties of the associated system. In order to put our results into perspective, let us review the density results that exist for the case of Gabor frames and the Heisenberg group.

Classical results are mostly concerned with regular Gabor systems, i.e., with rectangular lattices of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$. Baggett [4] and Daubechies [40] proved that if $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a complete subset of $L^2(\mathbb{R})$ then necessarily $ab \leq 1$. Since every frame is complete (but not conversely), it follows as a corollary that if $ab > 1$, then $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ cannot form a frame. Baggett's proof uses deep results from the theory of von Neumann algebras, while Daubechies provided a constructive proof of this result by using signal-theoretic methods (the Zak transform). However, her result is restricted to the case that ab is rational. Daubechies also noted that a proof for general ab can be inferred from results of Rieffel [108] on C^*-algebras. Another proof of this result based on von Neumann algebras was given by Daubechies, H. Landau, and Z. Landau in [43], and a new proof appears in Bownik and Rzeszotnik [12].

H. Landau [98] extended the result on Gabor frames to much more general sequences Λ in $\mathbb{R}^2$, deriving a necessary condition for $\mathcal{G}(g, \Lambda)$ to be a frame in terms of the Beurling density of Λ, but requiring some restrictions on g and Λ. For the rectangular lattice case, Janssen [86] gave an elegant direct proof that if $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a frame then $ab \leq 1$. This proof relies on the algebraic structure of the rectangular lattice $a\mathbb{Z} \times b\mathbb{Z}$ and the Wexler–Raz Theorem for Gabor frames. Perhaps the most elegant development along these lines was due to Ramanathan and Steger [107]. They proved that all Gabor frames $\mathcal{G}(g, \Lambda)$, without restrictions on $g \in L^2(\mathbb{R})$, but only for separated sequences $\Lambda \subseteq \mathbb{R}^2$, satisfy a certain *Homogeneous Approximation Property (HAP).* This is a fundamental result that is of independent interest, and in particular they deduced necessary density conditions on the sequence of time-frequency indices of irregular Gabor frames as a corollary. Ramanathan and Steger were also able to recover the completeness result of Rieffel by using this technique as a main tool. However, the proof by Ramanathan and Steger required Λ to be uniformly separated. Christensen, Deng, and Heil removed this hypothesis in [22]. Also, [22] extended the result to higher dimensions and to finitely many generators, and made several other contributions. Christensen, Deng, and Heil derived the following general result on the density of Gabor systems which we state for simplicity only in the one-dimensional, singly generated case.

Theorem 1.1. *Let $g \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}^2$ be given. Then the Gabor system $\mathcal{G}(g, \Lambda)$ has the following properties.*

(i) *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R})$, then $1 \leq D^-(\Lambda) \leq D^+(\Lambda) < \infty$.*
(ii) *If $\mathcal{G}(g, \Lambda)$ is a Riesz basis for $L^2(\mathbb{R})$, then $D^-(\Lambda) = D^+(\Lambda) = 1$.*

Here $D^{\pm}(\Lambda)$ are the upper and lower Beurling densities of Λ, which measure in some sense the largest and smallest number of points of Λ that lie on average in unit squares. The group structure employed for this notion of density is the one coming from the Heisenberg group modulo its center. The cutoff density 1 is called the *Nyquist density.* In the special case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, we have $D^{\pm}(a\mathbb{Z} \times b\mathbb{Z}) = \frac{1}{ab}$. Gröchenig and Razafinjatovo adapted the Ramanathan/Steger argument to prove an analogous result for windowed exponentials in [66].

Ramanathan and Steger conjectured in [107] that Theorem 1.1(i) should be improvable to say that if $\mathcal{D}^-(\Lambda) < 1$ then $\mathcal{G}(g, \Lambda)$ is incomplete in $L^2(\mathbb{R})$. However, Benedetto, Heil, and Walnut [9] showed that the Rieffel result does not extend to non-lattices: there exist complete (but non-frame) Gabor systems with upper Beurling density ε. The counterexample built fundamentally on the work of H. Landau on the completeness of exponentials in $L^2(S)$ where S is a finite union of intervals. Another counterexample, in which Λ is a subset of a lattice, appears in Y. Wang [123]. Moreover, Olevskii and Ulanovskii [103, 104] constructed a system consisting solely of translates which is complete in $L^2(\mathbb{R})$, but even the upper density of the set of indices regarded as a subset of $\mathbb{R}^2$ equals zero.

Recently, Balan, Casazza, Heil, and Z. Landau showed that such necessary density conditions, including the Nyquist density cutoff, apply to a much broader class of abstract frames called *localized frames* [6]. Localized frames were also independently introduced by Gröchenig [64] for quite different purposes.

Finally, we remark that density theorems for Gabor frames $\mathcal{G}(g, \Lambda)$ generated by Gaussian functions g are related to density questions in the Bargmann–Fock spaces, see, e.g., Seip [112]. We further mention that the notion of Beurling density was also employed by Heil and Kutyniok [74] to derive conditions on the existence of frames and Schauder bases of windowed exponentials, and an adapted notion of density and a new notion of dimension were the main tools to study wave packet frames and also Gabor pseudoframes for affine subspaces in the papers by Czaja, Kutyniok, and Speegle [36, 37].

For more details and extended references we refer to the recent survey paper on the history of the density theorem for Gabor systems by Heil [72].

1.3 Geometry of Time-Scale Indices

It is natural to ask whether wavelet systems share similar properties, and the immediate answer is that there is clearly no exact analogue of the Nyquist density for wavelet systems. In particular, consider the case of the classical affine systems $\mathcal{W}(\psi, \Lambda)$ with dilation parameter $a > 1$ and translation parameter $b > 0$, i.e.,

$$\Lambda = \{(a^j, bk)\}_{j,k \in \mathbb{Z}}.$$

It can be shown that for *each* $a > 1$ and $b > 0$ there exists a wavelet $\psi \in L^2(\mathbb{R})$ such that $\mathcal{W}(\psi, \Lambda)$ is a frame or even an orthonormal basis for $L^2(\mathbb{R})$. In fact, the wavelet set construction of Dai, Larson, and Speegle [39] shows that this is true even in higher dimensions: wavelet orthonormal bases in the classical affine form exist for any expansive dilation matrix. For additional demonstrations of the impossibility of a Nyquist density, even given constraints on the norm or on the admissibility condition of the wavelet, see the example of Daubechies in [40, Thm. 2.10] and the more extensive analysis by Balan in [5].

However, the more general question remains: for what sequences $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ and what weights $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \dots, L$ is it possible to construct wavelet frames of the form

$$\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell) = \bigcup_{\ell=1}^{L} \{w_\ell(a,b)^{\frac{1}{2}} a^{-\frac{1}{2}} \psi_\ell(a^{-1}x - b)\}_{(a,b)\in\Lambda_\ell}$$

with finitely many generators $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$? Two important examples of wavelet systems other than classical affine systems are the quasi-affine and co-affine systems.

Quasi-affine systems, introduced by Ron and Shen [109], are obtained by replacing the sequence Λ associated with a classical affine system by the new sequence

$$\Lambda = \{(a^j, bk)\}_{j<0, k\in\mathbb{Z}} \cup \{(a^j, a^{-j}bk)\}_{j\geq 0, k\in\mathbb{Z}},$$

and using the weight function

$$\begin{aligned} w(a^j, bk) &= 1, & j < 0,\ k \in \mathbb{Z}, \\ w(a^j, a^{-j}bk) &= a^{-j}, & j \geq 0,\ k \in \mathbb{Z}. \end{aligned}$$

In other words, "extra" elements are added to an affine system, and additionally the norms of the extra elements are adjusted. Ron and Shen proved that if a is an integer and $b = 1$ then an affine system is a frame if and only if the quasi-affine system is a frame. The utility of the quasi-affine system is that it is *shift-invariant*, i.e., integer translation-invariant, unlike the original classical affine system. Shift-invariance, i.e., invariance under integer translations, is a desirable feature for many applications, since it ensures that similar structures in a signal are more easily detectable. Quasi-affine systems were also studied in the papers by Bownik [11], Chui, Shi, and Stöckler [35], Gressman, Labate, Weiss, and Wilson [61], and Johnson [88].

Co-affine systems were studied recently by Gressman, Labate, Weiss, and Wilson [61]. If we write an affine system as $\{D_{a^j} T_k \psi\}_{j,k\in\mathbb{Z}}$, where D_{a^j} and T_k are the appropriate dilation and translation operators, then the associated co-affine system is $\{T_k D_{a^j}\psi\}_{j,k\in\mathbb{Z}}$. This amounts, in the terminology of this book, to taking

$$\Lambda = \{(a^j, a^{-j}k)\}_{j,k\in\mathbb{Z}},$$

and $w = 1$. It was shown in [61] that such a system $\mathcal{W}(\psi, \Lambda, w)$ can *never* form a frame for $L^2(\mathbb{R})$, and, moreover, this impossibility remains even when allowing weights of the form $w(a^j, a^{-j}k) = w(a^j)$. An extension of this result to higher dimensions was derived by Johnson [90].

Considering the Gabor situation we come to the conclusion that a notion of density for wavelet systems, despite the lack of a Nyquist density, should be exactly the right method to explain, for instance, the difference between affine/quasi-affine and co-affine systems, but even more to relate frame properties of a wavelet system with properties of the associated sequence of time-scale indices. It was already conjectured by Daubechies in [41, Sec. 4.1], that the value $\frac{1}{b \ln a}$ might play the role of a density for classical affine systems, since it is an ubiquitous constant in a variety of formulas in wavelet analysis. For example, if $\mathcal{W}(\psi, \{(a^j, bk)\}_{j,k\in\mathbb{Z}})$ is a tight frame for $L^2(\mathbb{R})$ and $\int_0^\infty |\hat{\psi}(\xi)|^2/|\xi| d\xi = 1$, then the frame bounds are exactly $\frac{1}{b \ln a}$. In [113], Seip introduced a notion of density for Bergman-type spaces on the unit disk, and it is possible to derive some density results for wavelet frames $\mathcal{W}(\psi, \Lambda)$ generated by certain wavelets ψ from those results. Some preliminary results relating density to wavelet frames also appeared in a paper by Olson and Seip [105], but until now there has been no general theory of density properties of wavelet frames; there did not even exist a notion of density for general irregular wavelet systems.

To build such a theory is a fascinating challenge, since the situation for wavelet systems is much more delicate than the one for Gabor systems due to the non-commutativity of the affine group. Results in this direction should lead to a much deeper understanding of the geometrical structure of the time-scale indices associated with a wavelet frame, thereby also delivering tools to construct irregular wavelet frames and to examine their stability.

Summarizing, this research monograph has the following aims.

- ▷ Derive a notion of weighted affine density for weighted sequences of time-scale indices of weighted irregular wavelet frames with finitely many generators in such a way that classical affine systems possess a uniform density equal to the ubiquitous constant $\frac{1}{b \ln a}$.
- ▷ Study whether the non-existence of co-affine frames is related to density properties.
- ▷ Derive necessary and sufficient density conditions for the existence of weighted irregular wavelet frames with finitely many generators.

- ▷ Relate the density of the weighted sequences of time-scale indices to the frame bounds of weighted irregular wavelet frames with finitely many generators.
- ▷ Reveal reasons why a Nyquist phenomenon does not exist for wavelet systems.
- ▷ Study the HAP for wavelet systems and its relation (or lack thereof) to density conditions.
- ▷ Study the affine density of a classical affine system and the weighted affine density of its associated quasi-affine system and examine whether their relation is enforced by the property that one system is a frame if and only if the other system is a frame.

1.4 Overview of Main Results

In the following we outline the organization of this book and present some of the highlights in an informal way.

In Chapter 2 we present some background and notation from frame theory and wavelet and time-frequency analysis which will be employed throughout. We further give a brief overview of Wiener amalgam spaces in the general setting of locally compact groups, since we will consider different group settings in this book. These spaces will serve as regularity conditions for analyzing wavelets as well as for Gabor generators.

We proceed in Chapter 3 to introduce the new notion of upper and lower weighted affine density for weighted irregular wavelet systems with finitely many generators and to study several of its properties. We show that it satisfies the property that classical affine systems possess a uniform affine density, i.e., upper and lower density coincide, and this density is exactly equal to the magical constant $\frac{1}{b \ln a}$. Moreover, we compare this density with another notion of density for wavelet systems simultaneously introduced by Sun and Zhou [119]. We show that for their density a weighted form has to be used to derive the same uniform density for the classical wavelet systems, thereby emphasizing our notion as more naturally in this sense.

In Chapter 4, we derive necessary conditions on the upper and lower weighted affine density for the existence of a weighted irregular wavelet frame with finitely many generators. These results only rely on conditions concerning finite upper and positive lower density, in this sense on *qualitative* density conditions. More precisely, we prove that if such a wavelet system possesses an upper frame bound, then necessarily the upper density has to be finite (Theorem 4.1). This result confirms the intuitive view of the density as the amount to which the time-scale indices are concentrated. We further show that provided that the wavelet system possesses a lower frame bound, then, under some hypotheses on the time-scale indices and with weights being equal to