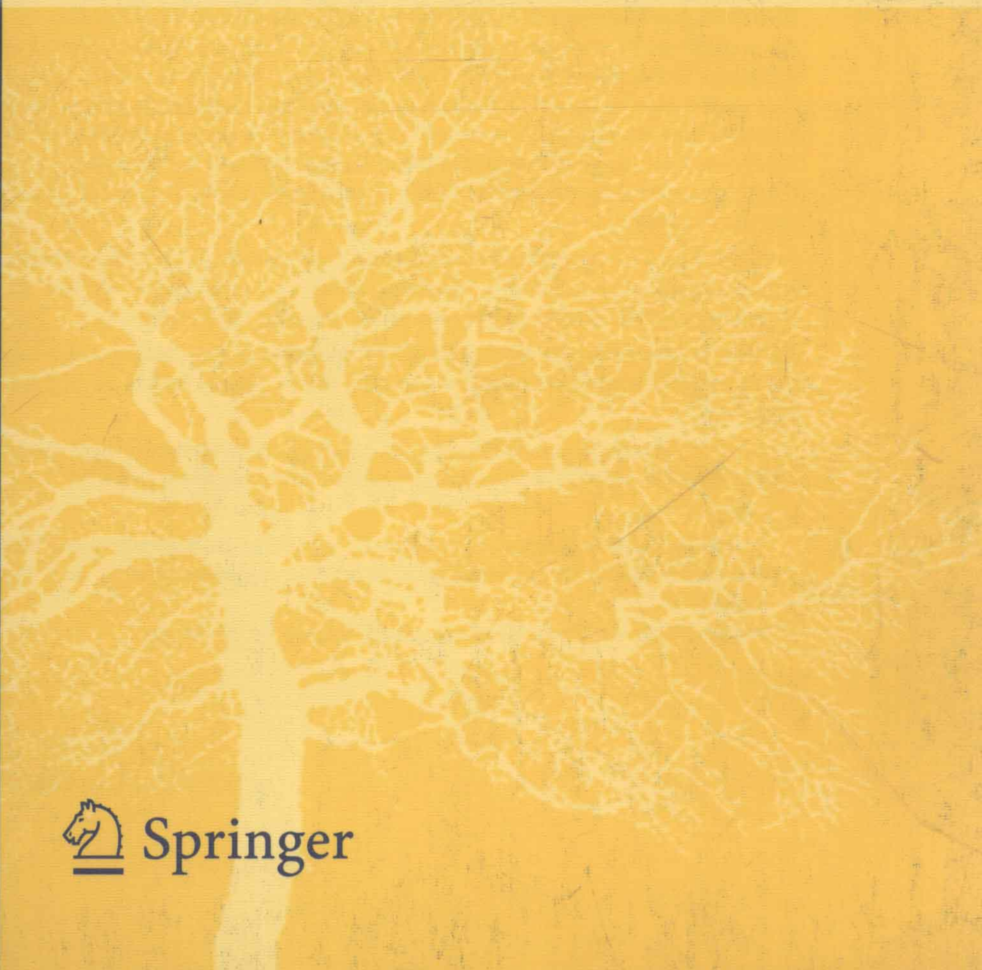
Lecture Notes in Mathematics

Charles Favre
Mattias Jonsson

The Valuative Tree

1853



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Mattias Jonsson

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Authors

Charles Favre

CNRS
Institut de Mathématique de Jussieu
Université Denis Diderot
Case 7012
2, place Jussieu
75251 Paris Cedex 05
France

e-mail: favre@math.jussieu.fr

Mattias Jonsson

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1109
U.S.A.

e-mail: mattiasj@umich.edu

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To Muriel and Johanna

Preface

This book grew out of a common passion for a beautiful natural object that we decided to call the "valuative tree". Motivated by questions stemming from complex dynamics and complex analysis, we realized that we needed to understand the link between valuations, which are purely algebraic objects, and more geometric or analytic constructions such as blowups or Lelong numbers. More precisely, we looked at the structure of a special set of valuations, and we found that this space had a very rich and delicate topological structure. We hope that the reader will share our enthusiasm while progressively exploring this space into its finer details all along this book.

This monograph has benefited from the help of many people. The first author wishes to warmly thank Bernard Teissier for his constant support and help, and Patrick Popescu-Pampu, Mark Spivakovsky and Michel Vaquié for fruitful discussions. The second author expresses his gratitude to Jean-François Lafont, Robert Lazarsfeld and Karen Smith. We both thank the referees for a number of useful suggestions.

Our work was done at several places, including the Institut de Mathématiques in Paris, the Department of Mathematics of the University of Michigan in Ann Arbor, RIMS in Kyoto and IMPA in Rio de Janeiro. We are grateful to these institutions for having provided us an excellent and motivating atmosphere for working on our project. During the writing period, the first author was supported by the CNRS, whereas the second author was supported by STINT and by NSF Grant No DMS-0200614.

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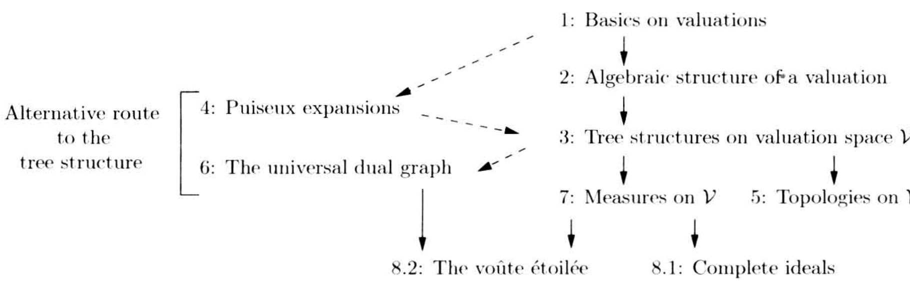
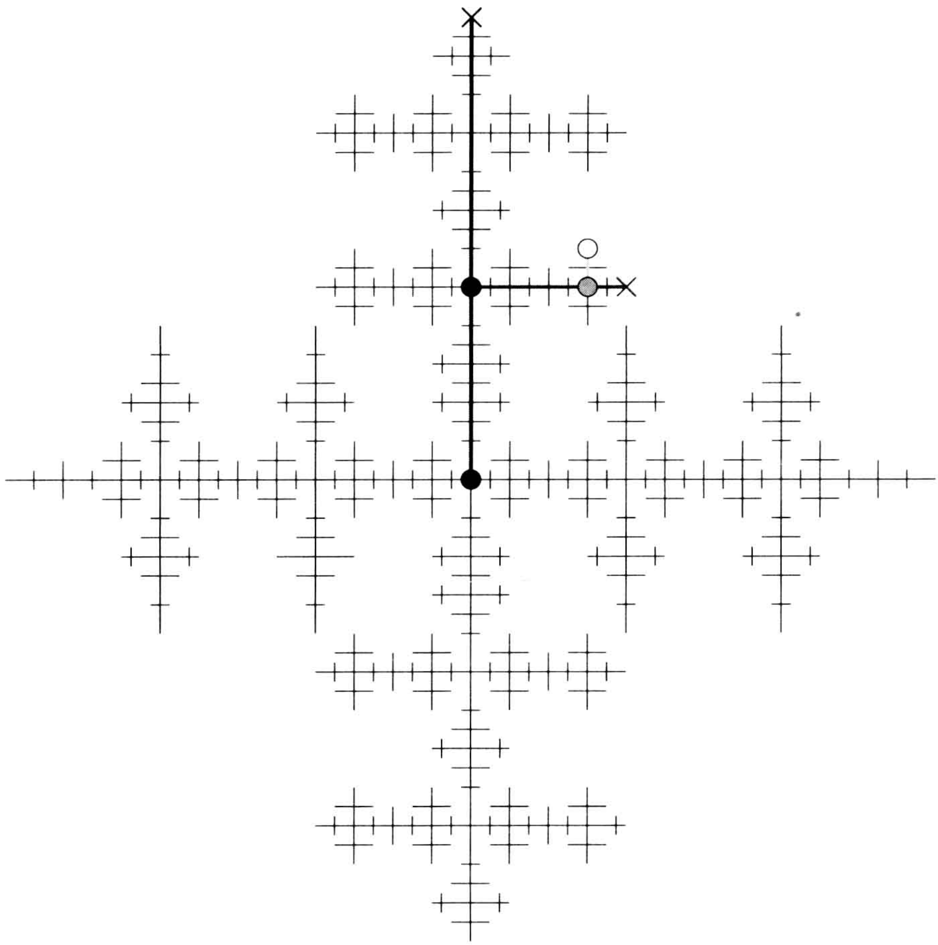
Structure of the Book

Before embarking to a journey into the valuative tree, we describe below the structure of the volume. A plain arrow linking chapter A to chapter B indicates that the understanding of B relies heavily on a previous lecture of A. A dashed arrow indicates a looser link between both chapters.



The Valuative Tree



- ● $\nu_{\mathfrak{m}}$
- ● $\nu_{y,3/2} = \nu_{\phi,3/2}$
- ◍ $\nu_{y^2-x^3,10/3} = \nu_{\phi,10/3}$
- ○ ν_{ϕ}
- × ν_y
- × $\nu_{y^2-x^3}$
- $m = 1$
- $m = 2$
- $m = 6$

$$\phi = (y^2 - x^3)^3 - x^{10}$$

Contents

metric trees; these are metric spaces in which any two points are joined by a unique path isometric to a real interval.[2]

The nonmetric tree structure on $\mathcal{V}$ arises as follows. For $\nu, \mu \in \mathcal{V}$, we declare $\nu \leq \mu$ when $\nu(\phi) \leq \mu(\phi)$ for all $\phi \in R$. Our normalization $\nu(\mathfrak{m}) = 1$ implies that the *multiplicity valuation* $\nu_{\mathfrak{m}}$ sending ϕ to its multiplicity $m(\phi)$ at the origin is dominated by any other valuation. This natural order defines a nonmetric tree structure on $\mathcal{V}$, rooted at $\nu_{\mathfrak{m}}$ (Theorem 3.14).

As for the other two tree structures, any irreducible (formal local) curve C defines a *curve valuation* $\nu_C \in \mathcal{V}$: $\nu_C(\phi)$ is the normalized intersection number between the curves C and $\{\phi = 0\}$. A curve valuation is a maximal element under $\leq$ and the segment $[\nu_{\mathfrak{m}}, \nu_C]$ is isomorphic, as a totally ordered set, to the interval $[1, \infty]$. We construct an increasing function $\alpha : \mathcal{V} \to [1, \infty]$ that restricts to a bijection of $[\nu_{\mathfrak{m}}, \nu_C]$ onto $[1, \infty]$ for any C; as a consequence, α defines a parameterization of $\mathcal{V}$. The number $\alpha(\nu)$ is called the *skewness* of ν.[3] It is defined by the formula $\alpha(\nu) = \sup_\phi \nu(\phi)/m(\phi)$.

In addition to the partial ordering and skewness parameterization just described, the valuative tree also carries an important *multiplicity* function. The multiplicity of a valuation ν is equal to the infimum of the multiplicity of all curves whose associated curve valuations dominate ν in the partial ordering. Thus the multiplicity function is an increasing function on $\mathcal{V}$ with values in $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. A second important parameterization of $\mathcal{V}$, *thinness*, can be defined in terms of skewness and multiplicity.[4] We shall refer loosely to the combination of the partial ordering, the parameterizations by skewness and thinness, and the multiplicity function as the *tree structure on* $\mathcal{V}$.

There are four types of inhabitants of the valuative tree $\mathcal{V}$. The interior points, i.e. the points that are not maximal in the partial ordering, are valuations that become monomial (i.e. determined by their values on a pair of local coordinates) after a finite sequence of blowups. We call them *quasimonomial*. They include all divisorial valuations but also all *irrational* valuations such as the monomial valuation defined by $\nu(x) = 1$, $\nu(y) = \sqrt{2}$. The other points in $\mathcal{V}$, i.e. the ends of the valuative tree, are curve valuations and *infinitely singular* valuations, which can be characterized as the valuations with infinite multiplicity.

There is in fact a fifth type of valuations. These valuations cannot be defined as real-valued functions, but define functions on R with values in $\mathbf{R}_+ \times \mathbf{R}_+$ (endowed with the lexicographic order). In fact they do have a natural place in the valuative tree, as tree tangent vectors at points corresponding to divisorial valuations (see Theorem B.1). Geometrically they are curve val-

[2] Metric trees are often called **R**-trees in the literature.

[3] The skewness is the inverse of the volume of a valuation as defined in [ELS] (see Remark 3.33).

[4] The thinness is also very precisely related to the Jacobian ideal of the valuation (see Remark 3.50).

uations where the curve is defined by an exceptional divisor. We hence call them *exceptional curve valuations.*

The valuative tree is a beautiful object which may be viewed in a number of different ways. Each corresponds to a particular interpretation of a valuation, and each gives a new insight into it. Some of them will hopefully lead to generalizations in a broader context. Let us describe four such points of views. See also the diagrams on page 7.

The first way consists of identifying valuations with balls of curves. For any two irreducible curves C_1, C_2 set $d(C_1, C_2) = m(C_1)m(C_2)/C_1 \cdot C_2$ where $m(C_i)$ is the multiplicity of C_i and $C_1 \cdot C_2$ is the intersection multiplicity of C_1 and C_2. It is a nontrivial fact that d defines an ultrametric on the set $\mathcal{C}$ of all irreducible formal curves (c.f. [Ga]). This fact allows us to associate to $(\mathcal{C}, d)$ a tree $\mathcal{T}_\mathcal{C}$ by declaring a point in $\mathcal{T}_\mathcal{C}$ to be a closed ball in $\mathcal{C}$. The tree structure on $\mathcal{T}$ is given by reverse inclusion of balls (partial ordering), inverse radii of balls (parameterization) and minimum multiplicity of curves in a ball (multiplicity). Theorem 3.57 states that the tree $\mathcal{T}_\mathcal{C}$ is isomorphic to the valuative tree $\mathcal{V}$ with its ends removed (i.e. to the set $\mathcal{V}_{\mathrm{qm}}$ of quasimonomial valuations).

A second way is through Puiseux series. Just as irreducible curves can be represented by Puiseux series, the elements in $\mathcal{V}$ are represented by valuations on the power series ring in one variable with Puiseux series coefficients. The set $\widehat{\mathcal{V}}_x$ of all such (normalized) valuations has a natural tree structure and a suitably defined restriction map from $\widehat{\mathcal{V}}_x$ to $\mathcal{V}$ recovers the tree structure on $\mathcal{V}$. In fact, $\mathcal{V}$ is naturally the orbit space of $\widehat{\mathcal{V}}_x$ under the action by the relevant Galois group (Theorem 4.17). This approach can also be viewed in the context of Berkovich spaces and Bruhat-Tits buildings. As a nonmetric tree, $\mathcal{V}$ embeds as the closure of a disk in the Berkovich projective line over the field of Laurent series in one variable. The metric on $\mathcal{V}_{\mathrm{qm}}$ induced by thinness then also arises from an identification of a subset of the Berkovich projective line with the Bruhat-Tits building of PGL_2 (see Section 4.6).

The third way is more algebraic in nature. The earliest systematic study of valuations in two dimensions was done in the fundamental work of Zariski in [Za1], [Za2] who, among other things, identified the set $\mathcal{V}_K$ of (not necessarily real-valued) valuations on R, vanishing on $\mathbf{C}^*$ and positive on the maximal ideal $\mathfrak{m} = (x, y)$, with sequences of infinitely near points. The space $\mathcal{V}_K$ carries a natural topology (the Zariski topology) and is known as the *Riemann-Zariski variety.* It is a non-Hausdorff quasi-compact space. The obstruction for $\mathcal{V}_K$ being Hausdorff stems from the fact that divisorial valuations do not define closed points. Namely, their associated valuation rings strictly contain valuation rings associated to exceptional curve valuations. One can then build a quotient space by identifying all valuations in the closure of a single divisorial one. This produces a compact Hausdorff space. Theorem 5.24 states that this space is precisely $\mathcal{V}$ (endowed with the topology of pointwise convergence).

The last way uses Zariski's identification of valuations with sequences of infinitely near points. We let Γ_π be the dual graph of a finite composition of blowups π. It is a simplicial tree whose set of vertices defines a poset Γ_π^*. When one sequence π contains another π', the poset Γ_π^* naturally contains $\Gamma_{\pi'}^*$. These posets therefore form an injective system whose injective limit (or, informally, union) Γ^* is a poset with a natural tree structure modeled on the rational numbers. By filling in the irrational points and adding all the ends to the tree we obtain a nonmetric tree Γ, the *universal dual graph.* This nonmetric tree can in fact be equipped with a parameterization and multiplicity function. These both derive from a combinatorial procedure that to each element in Γ^* attaches a vector $(a, b) \in (\mathbf{N}^*)^2$, the *Farey weight.*[5] A fundamental result (Theorem 6.22) asserts that the universal dual graph equipped with the Farey parameterization is canonically isomorphic to the valuative tree with the thinness parameterization, and that this isomorphism preserves multiplicity.

As we mentioned above, singularities can be understood through *functions* on the valuative tree. It is a remarkable fact that the information carried by these functions can also be described in terms of *complex measures* on $\mathcal{V}$. Let us be more precise. In the case of an ideal $I \subset R$, the function on $\mathcal{V}$ is given by $g_I(\nu) := \nu(I)$, and the measure ρ_I is a positive atomic measure supported on the Rees valuations of I. This decomposition into atoms of ρ_I corresponds exactly to the Zariski decomposition of I into simple complete ideals.

In [FJ1], we shall show that a plurisubharmonic function u also determines a function g_u on $\mathcal{V}$. The corresponding measure ρ_u, which is still positive but not necessarily atomic, captures essential information on the singularity of u. In particular, we shall show in [FJ2] that ρ_u determines the multiplier ideals of all multiples of u.

The identifications $g_I \leftrightarrow \rho_I$ and $g_u \leftrightarrow \rho_u$ are particular instances of a general correspondence between measures on $\mathcal{V}$ and certain functions on $\mathcal{V}_{\mathrm{qm}}$. In fact, this correspondence, being purely tree-theoretic in nature, is even more general, and extends the equivalence between positive measures and suitably normalized concave functions on the real line. By analogy, we thus write $\rho_I = \Delta g_I$, $\rho_u = \Delta g_u$ and speak about the *Laplace operator* Δ on the valuative tree.

There is a second instance where complex measures on $\mathcal{V}$ naturally appear, namely when we study the sheaf cohomology of the *voûte étoilée* $\mathfrak{X}$. In our setting, $\mathfrak{X}$ can be viewed as the total space of the set of all blowups above the origin. Elements of $H^2(\mathfrak{X}, \mathbf{C})$ naturally define functions on $\mathcal{V}_{\mathrm{qm}}$ whose Laplacians are atomic measures supported on divisorial valuations. The cup product on cohomology has a natural interpretation as an inner product on measures. This inner product is a bilinear extension of an inner product on the valuative tree itself, and ultimately derives from intersections of curves.

[5] We follow the terminology used in [HP].

We have tried to write this monograph with an eye towards applications, such as the study of singularities of plurisubharmonic functions and dynamics of fixed point germs. Our hope is that people who are new to valuation theory will be able to follow the exposition, which we have tried to make self-contained and elementary.

Experts on valuation theory will undoubtedly notice that we reproduce many known results and that we do not work in the most general setting possible. Indeed, the assumption that R be the ring of formal power series in two complex variables is unnecessarily restrictive. While we refer to Appendix E for a precise discussion, we mention here that our analysis goes through in two important cases: the ring of holomorphic germs at the origin in $\mathbf{C}^2$, and the local ring at a smooth (closed) point of an algebraic surface over an algebraically closed field. We decided to work in the concrete setting of formal power series; readers with a good background in algebra may easily adapt the arguments to more general situations.

It would certainly be interesting to investigate generalizations to the case when R is the local ring at a normal surface singularity, or any local ring of dimension at least three. However, not only would this require the introduction of a substantial amount of new material, but the corresponding valuation space would no longer be a tree in general. Thus we shall not consider these more general situations here.

We remark that a fair amount of the structure of the valuative tree is implicitly contained in the analysis by Spivakovsky [Sp]. In particular, his description of the dual graph of a valuation is closely related to the construction of the valuative tree as the universal dual graph. However, our approach is quite different from his; in particular we do not use continued fractions. A tree structure was described in a context similar to ours in [AA], but without any explicit reference to valuations.

The main applications of the tree structure on the valuative tree to analysis and dynamics will be explored in forthcoming papers: see [FJ1], [FJ2], and [FJ3].

We end this introduction by indicating the organization of the monograph, which is divided into eight chapters and an appendix.

In the first chapter we give basic definitions, examples and results on valuations. In particular we describe the relationship between valuations and sequences of infinitely near points (in dimension two).

Chapter 2 is technical in nature. We encode valuations by finite or countable pieces of data that we call sequences of key polynomials, or *SKP's*. This encoding is an adaptation of a method by MacLane [Ma], the possibility of which was indicated to us by B. Teissier. An SKP (or at least a subsequence of it) corresponds to generating polynomials and approximate roots in the language of Spivakovsky [Sp] and Abhyankar-Moh [AM], respectively. We are thus able to classify valuations on R. This classification is well-known to specialists (see [ZS2, Sp] for instance) but we feel that our concrete approach is of independent interest. The representation of valuations by SKP's is the key to the tree structure on the valuative tree.

The third chapter concerns trees. Our main goal is to visualize the encoding by SKP's in an elegant and coordinate free way. We first discuss different definitions of trees and the relations between them. Using SKP's we then show that valuation space $\mathcal{V}$ does carry an intricate tree structure that we later in the chapter analyze in detail.

As an alternative to SKP's, Chapter 4 contains an approach to the tree structure on $\mathcal{V}$ through Puiseux series. The results can be interpreted in the language of Berkovich. Specifically we indicate how the valuative tree embeds inside the Berkovich projective line and Bruhat-Tits building of PGL_2 over the local field of Laurent series in one variable. In fact, most of these results are at least implicitly contained in [Be] but we felt it was worthwhile to write down the details.

In Chapter 5 we analyze and compare different topologies on the valuative tree. The definition of a valuation as a function on R, as well as the tree structure on $\mathcal{V}$, gives rise to three types of topologies: the weak, the strong and the thin topology. In addition, we have two topologies on $\mathcal{V}_K$: the Zariski topology and the Hausdorff-Zariski (or HZ) topology. As mentioned above, the former gives rise to the weak topology on $\mathcal{V}$ through the quotient construction. The HZ topology is in fact equivalent to the weak tree topology induced by a natural *discrete* tree structure on $\mathcal{V}_K$.

In Chapter 6 we build the universal dual graph described above, and show how to identify it with the valuative tree. The fact that valuations can be simultaneously viewed algebraically as functions on R and geometrically in terms of blowups is extremely powerful and we spend a fair amount of time detailing some of the connections and implications. In particular, we show that the valuative tree has a natural self-similar, or fractal, structure, see Figure 6.12.

Chapter 7 is concerned with the relationship between measures on $\mathcal{V}$ and certain classes of functions on $\mathcal{V}_{\mathrm{qm}}$. The analysis is purely tree-theoretic and gives a connection between (complex) measures on a parameterized tree and functions on the (interior of the) tree satisfying certain regularity properties. Apart from being of independent interest, this analysis is fundamental to many applications.

In Chapter 8, we describe two instances where these measures appear naturally. First we reinterpret in our context Zariski's theory of simple complete ideals as explained in [ZS2]. This gives a new point of view on the decomposition of any integrally closed ideal as a product of simple ideals. We then construct Hironaka's "voûte étoilée" $\mathfrak{X}$ as the projective limit of the total spaces of all sequences of blowups above the origin. We use measures on $\mathcal{V}$ to understand the structure of the sheaf cohomology group $H^2(\mathfrak{X}, \mathbf{C})$.

Finally we conclude this monograph by an appendix containing a few results and discussions that did not find a natural home elsewhere in the monograph. Specifically, we discuss infinitely singular valuations; analyze the