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Combinatorial Set Theory

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COMBINATORIAL SET THEORY

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PREFACE

Combinatorial theory is largely the study of properties that a set or family of sets may have by virtue of its cardinality, although this may be widened to consider related properties held by sets carrying a simple structure, such as ordered or well ordered sets. These properties are relevant to either finite or infinite sets, although frequently the questions that pose interesting problems for finite sets are either meaningless or trivial for infinite sets, and vice-versa. There has been a recent great upsurge in the study of finite combinatorial problems, and a significant, though more manageable, increase in interest in the combinatorial properties of infinite sets.

This book deals solely with combinatorial questions pertinent to infinite sets.

Some results have arisen purely in the context of infinite sets. One of the early results in the subject is the following: given an infinite set S of power κ then there is a family of more than κ subsets of S any two of which intersect in a set of size less than κ. Chapter 1 looks at questions related to this. Problems of a different nature for families of sets are studied in Chapter 4, firstly a decomposition problem and then delta- and weak delta-systems. As a special case is the following: given any family $\mathscr{A}$ of $\aleph_2$ denumerable sets there is a subfamily $\mathscr{B}$ of $\mathscr{A}$ of size $\aleph_2$ such that the intersection of two sets from $\mathscr{B}$ is the same for all pairs from $\mathscr{B}$. Chapter 3 is devoted to the study of set mappings, that is, functions f defined on a set S such that given any x in S then $f(x)$ is a subset of S for which $x \notin f(x)$. Conditions are placed on the family $\{f(x); x \in S\}$ which ensure the existence of a large free set T (a subset of S such that $x \notin f(y)$ and $y \notin f(x)$ for all x, y from T).

Other results stem from problems that have been extensively studied for finite sets, and have been found to yield interesting questions when reformulated to apply to infinite sets. For example, in Chapter 5 we study infinite graphs, and in particular show that for any infinite cardinal κ there is a graph with chromatic number κ which contains no triangle, or indeed no pentagon; however any graph which contains no quadrilateral has chromatic number at most $\aleph_0$. Chapters 2, 4 and 7 are devoted to various extensions of Ramsey's classical theorem, which (in its finite form) states: given integers n, k, r if the

n-element subsets of a finite set S are divided into r classes then provided that S is sufficiently large there will be some k-element subset of S all the n-element subsets of which fall in the one class. Chapters 2 and 4 cover ordinary partition relations, polarized partition relations and square bracket partition relations for cardinal numbers, whilst Chapter 7 is concerned with ordinary partition relations for ordinal numbers.

Familiarity with the standard notions of set theory has been assumed throughout. An Appendix summarizes those properties of cardinal and ordinal numbers and their arithmetic which are basic to a study of this book.

The development of infinitary combinatorial theory over the past twenty years or so has been greatly stimulated by associates of the Hungarian school, under the encouragement of Paul Erdös in particular. A glance at the list of references gives some indication of how many of the results in this book show his influence.

To all those people who have created this subject, I here acknowledge my debt and record my gratitude.

Brisbane, 1976. Neil H. Williams

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FOREWORD ON NOTATION

The following is a brief summary of standard notation in use throughout this book. More technical notation is introduced throughout the text as the need for it becomes apparent. The Index of notation (pp. 206–208) provides a ready reference to the page on which a symbol is first defined.

Set membership is denoted by $\in$, and $\subseteq$ is the inclusion relation with $\subset$ denoting proper inclusion. The set of all subsets of a set is $\mathfrak{P}x$, so $\mathfrak{P}x = \{y; y \subseteq x\}$. The union of all the sets in x is written $\bigcup x$ and the intersection $\bigcap x$, so

$$\bigcup x = \{z; \exists y \in x(z \in y)\}; \bigcap x = \{z; \forall y \in x(z \in y)\} .$$

Set difference is written $x - y$, so $x - y = \{z \in x; z \notin y\}$. The set of unordered pairs, one member from A and the other from B is $A \otimes B$, so

$$A \otimes B = \{\{x, y\}; x \in A, y \in B \text{ and } x \neq y\}.$$

Ordered pairs are written $\langle x, y\rangle$, and sequences as $\langle x_\alpha; \alpha < \beta\rangle$. The *length* of the sequence $\langle x_\alpha; \alpha < \beta\rangle$ is β, written $\ell n\langle x_\alpha; \alpha < \beta\rangle = \beta$. If $\boldsymbol{x}, \boldsymbol{y}$ are two sequences then $\boldsymbol{x}\char`^\boldsymbol{y}$ is the *concatenation* of $\boldsymbol{x}$ and $\boldsymbol{y}$, that is, the sequence obtained by placing the entries from $\boldsymbol{y}$ in order after those from $\boldsymbol{x}$. If A is any set, then the *domain* and the *range* of A are defined by

$$\mathrm{dom}(A) = \{x; \exists y(\langle x, y\rangle \in A)\}; \qquad \mathrm{ran}(A) = \{y; \exists x(\langle x, y\rangle \in A)\} .$$

That f is a function with domain A and range contained in B is indicated briefly by writing $f: A \to B$. The set of all such functions is ${}^{A}B$, so

$${}^{A}B = \{f; f: A \to B\} .$$

If $f: A \to B$, then the value of f at x is $f(x)$. The restriction of f to a set X is written $f \upharpoonright X$, so

$$f \upharpoonright X = \{\langle x, y\rangle \in f; x \in X\} ;$$

and $f[X]$ is the range of $f \upharpoonright X$, so

$$f[X] = \{f(x); x \in X\} .$$

The words *set* and *family* are used synonymously. However an *indexed family*, written $(A_i; i \in I)$, stands for that function A with domain I and

$A(i) = A_i$ for each i in I. The cartesian product of an indexed family is written $\times(A_i; i \in I)$. A *decomposition* of a set A is a family $\mathbf{\Delta}$ of sets such that $A = \bigcup\mathbf{\Delta}$; this is the same as a *partition* of A. The partition $\mathbf{\Delta}$ is *disjoint* if the sets in $\mathbf{\Delta}$ are pairwise disjoint. For elements a, b of A and a partition $\mathbf{\Delta}$ of A, the notation

$$a \equiv b (\text{mod } \mathbf{\Delta})$$

means that there is some Δ_k in $\mathbf{\Delta}$ such that $a, b \in \Delta_k$.

The cardinality of a set X is written $|X|$, and $[X]^\kappa$, $[X]^{<\kappa}$, ... denote $\{Y \subseteq X; |Y| = \kappa\}$, $\{Y \subseteq X; |Y| < \kappa\}$, ... The operations of cardinal addition, multiplication and exponentiation are written $\eta \dot{+} \theta$, $\eta \cdot \theta$ and η^θ, while the corresponding ordinal operations are written $\alpha + \beta$, $\alpha\beta$ and α^β. The infinite cardinal sum and product of an indexed family $(\eta_i; i \in I)$ of cardinals are written $\Sigma(\eta_i; i \in I)$ and $\Pi(\eta_i; i \in I)$, whereas the ordinal sum and product of a well ordered sequence $\langle\alpha_\nu; \nu < \beta\rangle$ of ordinal numbers are written $\Sigma_0(\alpha_\nu; \nu < \beta)$ and $\Pi_0(\alpha_\nu; \nu < \beta)$.

Let X be a set ordered by a relation $\prec$. By $\text{tp}(X, \prec)$ or $\text{tp}(X)$ is meant the order type of X under $\prec$. For subsets A, B of X, write $A \prec B$ to mean that $a \prec b$ for all a in A and b in B. The notation $\{x_1, x_2, ..., x_n\}_\prec$ refers to the set $\{x_1, ..., x_n\}$ and further indicates that $x_1 \prec x_2 \prec ... \prec x_n$. A subset A of X is *cofinal* in X if, for all x in X, there is a in A with $x \preccurlyeq a$. If α is an ordinal,

$$[X]^\alpha = \{Y \subseteq X; \text{tp}(Y, \prec) = \alpha\} .$$

The *ordinal numbers* are defined so that if α is an ordinal, then

$$\alpha = \{\beta; \beta \text{ is an ordinal and } \beta < \alpha\} .$$

If A is a set of ordinals then sup A is the supremum of A, so sup $A = \bigcup A$. *Cardinal numbers* are identified with the initial ordinals. The sequence of infinite cardinals is $\aleph_0, \aleph_1, \aleph_2, ..., \aleph_\alpha, ...$. The cardinal successor to a cardinal κ is denoted by κ^+; the iteration of this n times by $\kappa^{(n+)}$. The cofinality of a cardinal κ is written κ', so κ' is the least cardinal such that κ can be written as a sum of κ' cardinals all less than κ. The cardinal κ is *regular* if $\kappa' = \kappa$, and otherwise κ is *singular.* A cardinal of the form λ^+ for some λ is a *successor cardinal*; other cardinals are *limit cardinals.* The cardinal κ is a *strong limit cardinal* if $2^\lambda < \kappa$ whenever $\lambda < \kappa$. Regular limit cardinals are *weakly inaccessible*; regular strong limit cardinals are *strongly inaccessible.* The *cardinal beths* (starting from κ) are defined by induction:

$$\beth_0(\kappa) = \kappa , \qquad \beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$$

The *Generalized Continuum Hypothesis* (GCH) is the statement: $2^\kappa = \kappa^+$

for all infinite cardinals κ. The GCH has been assumed throughout this book whenever it leads to a simplification in the statement or proof of a result. Theorems reached with its aid have the letters GCH appended to their result number (as in Theorem 1.6 (GCH)). In many cases the full strength of the GCH is not required, or the result could have been reformulated in a more involved form so as to avoid GCH all together. It is left to the interested reader to observe when this is so.

The Greek letters κ, λ, ι, η, θ are used throughout to stand for cardinal numbers, and usually κ, λ are infinite. Other small Greek letters denote ordinal numbers, as do k, l (except that ω is always the least infinite ordinal). The letters m, n always stand for non-negative integers.

CONTENTS

CHAPTER 1

ALMOST DISJOINT FAMILIES OF SETS

§1. Almost disjoint families

One of the early results in combinatorial set theory was the following theorem of Sierpinski [84]. He proved that any infinite set of power κ can be decomposed into a family of more than κ infinite subsets in such a way that $|A \cap B| < |A|, |B|$ for any two different subsets A, B from the family. Two sets A and B will be called *almost disjoint* when $|A \cap B| < |A|, |B|$. Almost disjoint systems of sets were examined in more detail by Tarski [94]. It seems appropriate to start this book with an investigation of this and related problems.

Definition 1.1.1. The *degree of disjunction* $\delta(\mathscr{A})$ of a family $\mathscr{A}$ is the least cardinal θ such that $|A_1 \cap A_2| < \theta$ for all pairs A_1, A_2 in $\mathscr{A}$.

We shall be concerned with finding families $\mathscr{A}$ of subsets of a given set of infinite power κ with degree of disjunction at most θ for fixed cardinal θ. We need consider only the case when $\theta \leqslant |A|$ for all A in in $\mathscr{A}$ (and so certainly $\theta \leqslant \kappa$) since otherwise the condition $\delta(\mathscr{A}) \leqslant \theta$ imposes no restriction on $\mathscr{A}$. Clearly one can always find such a family of power κ, consisting in fact of pairwise disjoint subsets, so the problem is to find when there is a family $\mathscr{A}$ with $|\mathscr{A}| > \kappa$ and $\delta(\mathscr{A}) \leqslant \theta$. An upper bound for $|\mathscr{A}|$ is given by the following theorem.

Theorem 1.1.2. *If an infinite set of power κ can be decomposed into a family $\mathscr{A}$ with $|\mathscr{A}| = \lambda$ and $\delta(\mathscr{A}) \leqslant \theta$, then $\lambda \leqslant \kappa^\theta$.*

Proof. We may suppose that in fact it is the set κ which has been decomposed. So suppose $\kappa = \bigcup \mathscr{A}$ where $|\mathscr{A}| = \lambda$ and $\delta(\mathscr{A}) \leqslant \theta$. For each A in $\mathscr{A}$ with $|A| \geqslant \theta$, choose A^* in $[A]^\theta$. The condition $\delta(\mathscr{A}) \leqslant \theta$ implies that the mapping

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which sends A to A^* is one-to-one. Thus $\{A \in \mathscr{A}; |A| \geqslant \theta\}$ has cardinality at most that of $[\kappa]^\theta$, and so $|\mathscr{A}| \leqslant |[\kappa]^{\leqslant\theta}| = \kappa^\theta$.

The existence of large almost disjoint decompositions will follow from the next theorem. The method of proof is essentially that of the original Sierpinski paper.

Theorem 1.1.3. *Let κ be infinite, suppose $\eta \geqslant 2$ and let θ be the least cardinal for which $\kappa < \eta^\theta$. Let λ be a cardinal with $\lambda \leqslant \theta$ and $\lambda' = \theta'$. Then every set of power κ can be decomposed into a family of η^θ almost disjoint sets each of power λ.*

Proof. We show that such a decomposition is possible for a particular set of power κ; it then follows that this can be done for any set of power κ.

Since θ is least for which $\kappa < \eta^\theta$ we have $\theta \leqslant \kappa$ and consequently $\lambda \leqslant \kappa$. Thus $\kappa \cdot \lambda = \kappa$. The condition $\lambda \leqslant \theta$ and $\lambda' = \theta'$ means that there is a sequence $\langle \alpha(\nu); \nu < \lambda \rangle$ of ordinals $\alpha(\nu)$ with $\alpha(\nu) < \theta$ which is cofinal in θ. Put $S = \bigcup\{{}^{\alpha(\nu)}\eta; \nu < \lambda\}$. The choice of θ ensures that $|{}^{\alpha(\nu)}\eta| \leqslant \kappa$ for each ν, so $|S| \leqslant \Sigma\{|{}^{\alpha(\nu)}\kappa|; \nu < \lambda\} \leqslant \kappa \cdot \lambda = \kappa$. We shall define a decomposition of S. For each function g in ${}^\theta\eta$, put

$$S_g = \{g \upharpoonright \alpha(\nu); \nu < \lambda\},$$

so $|S_g| = \lambda$. And for different f, g from ${}^\theta\eta$, let β be the least ordinal such that $f(\beta) \neq g(\beta)$. Then whenever $\alpha(\nu) > \beta$, for any h in S with $\mathrm{dom}(h) \geqslant \alpha(\nu)$ it follows that $h \notin S_f \cap S_g$. Thus $|S_f \cap S_g| \leqslant |\beta| < \lambda$. This means that if $\mathcal{S} = \{S_f; f \in {}^\theta\eta\}$, then $\mathcal{S}$ is a family of η^θ almost disjoint subsets of S each of power λ.

If $|S| = \kappa$, we have finished. If $|S| < \kappa$ we must add to S to build it up to power κ. Choose any set S_1 of power κ disjoint from S, and divide S_1 into a family $\mathcal{S}_1$ of pairwise disjoint sets each of size λ (which is possible since $\kappa \cdot \lambda = \kappa$). Then the family $\mathcal{S} \cup \mathcal{S}_1$ decomposes $S \cup S_1$ in the required manner.

As a first corollary to this theorem, we deduce the Sierpinski result mentioned above. For given the infinite cardinal κ, let θ be least such that $\kappa < 2^\theta$, so necessarily θ is also infinite. Then by Theorem 1.1.3. with $\eta = 2$ and $\lambda = \theta$, any set of power κ can be decomposed into a family of 2^θ almost disjoint subsets each of power θ, that is, into a family of more than κ almost disjoint infinite sets.

If the Generalized Continuum Hypithesis is assumed, the theorem leads to the following result. Given infinite cardinals κ and λ with $\lambda \leqslant \kappa$ and $\lambda' = \kappa'$, then every set of power κ can be decomposed into a family of κ^+ almost disjoint subsets each of power λ. This follows from Theorem 1.1.3 by putting $\eta = 2$, and noting that by the GCH, $\theta = \kappa$ is the least value of θ for which $\kappa < 2^\theta$. However, this result can be deduced without making use of the CGH, by adopting a different method of approach. When $\lambda = \kappa$, this was known by Erdös in 1934 (see Erdös, Gillman and Henriksen [34, Lemma 4.1]). A different proof is given by Sierpinski [87].

Theorem 1.1.4. *Let κ and λ be infinite cardinals with $\lambda \leqslant \kappa$ and $\lambda' = \kappa'$. Then every set of power κ can be decomposed into a family of more than κ almost disjoint subsets each of power λ.*

Proof. The argument splits into two cases, depending on whether κ is regular or not.

Case 1: κ regular. Here $\kappa = \kappa' = \lambda' \leqslant \lambda \leqslant \kappa$, so $\kappa = \lambda$. Let K be any set with $|K| = \kappa$, and let $\mathcal{K}$ be a decomposition of K into κ disjoint sets each of power κ. We shall show that no family $\mathcal{B}$ with $|\mathcal{B}| < \kappa^+$ is maximal (under the subset relation) in the class $\mathfrak{F}$ of all families extending $\mathcal{K}$ consisting of almost disjoint subsets of K each of power κ. Zorn's Lemma applies to $\mathfrak{F}$, and so there is a maximal family $\mathcal{A}$ in $\mathfrak{F}$. But then we must have $|\mathcal{A}| \geqslant \kappa^+$, and the theorem holds in this case.

So let $\mathcal{B}$ be a member of $\mathfrak{F}$ with $|\mathcal{B}| < \kappa^+$. Since $\mathcal{K} \subseteq \mathcal{B}$, in fact $|\mathcal{B}| = \kappa$. Let $\langle B_\mu; \mu < \kappa\rangle$ be an enumeration (always without repetitions) of $\mathcal{B}$. For each μ with $\mu < \kappa$, note that $|\bigcup \{B_\mu \cap B_\nu; \nu < \mu\}| < \kappa$ since $\delta(\mathcal{B}) \leqslant \kappa$ and κ is regular. Hence we may choose inductively elements b_μ when $\mu < \kappa$ so that

$$b_\mu \in B_\mu - \bigcup \{B_\nu; \nu < \mu\} .$$

Then $b_\mu \neq b_\nu$ whenever $\nu < \mu$. Put $B = \{b_\mu; \mu < \kappa\}$ so $|B| = \kappa$. For each μ with $\mu < \kappa$ we have $B_\mu \cap B \subseteq \{b_\nu; \nu \leqslant \mu\}$ and so $|B_\mu \cap B| < \kappa$. Hence the family $\mathcal{B} \cup \{B\}$ is in the class $\mathfrak{F}$, and so $\mathcal{B}$ is not maximal.

Case 2: κ singular. Choose cardinals κ_σ for σ with $\sigma < \kappa'$ such that $\kappa' < \kappa_\sigma < \kappa_1 < \ldots < \kappa$ and $\Sigma(\kappa_\sigma; \sigma < \kappa') = \kappa$. Since $\lambda' = \kappa'$ we can choose cardinals λ_0 for σ with $\sigma < \kappa'$ such that always $\lambda_\sigma < \lambda$ and $\Sigma(\lambda_\sigma; \sigma < \kappa') = \lambda$. Let K be any set with $|K| = \kappa$, and let $\{K_\sigma; \sigma < \kappa'\}$ be a decomposition of K into κ' disjoint sets each of power κ. This time we show that no family $\mathcal{B}$ with $|\mathcal{B}| < \kappa^+$ is maximal in the class $\mathfrak{G}$ of all families $\mathcal{A}$ of almost disjoint subsets of K with always $|A \cap K_\sigma| = \lambda_\sigma$ for each A in $\mathcal{A}$. Here Zorn's Lemma applies to $\mathfrak{G}$; but any maximal member of $\mathfrak{G}$ must have power greater than κ, as desired.

So let $\mathcal{B}$ be a member of $\mathfrak{G}$ with $|\mathcal{B}| < \kappa^+$. Let $\langle B_\mu; \mu < \theta\rangle$ be an enumeration of $\mathcal{B}$ with $\theta \leqslant \kappa$. For each σ with $\sigma < \kappa'$, choose X_σ with $|X_\sigma| = \lambda_\sigma$ and

$X_\sigma \subseteq K_\sigma - \bigcup \{B_\mu; \mu < \kappa_\sigma\}$. Since

$$|\bigcup \{K_\sigma \cap B_\mu; \mu < \kappa_\sigma\}| \leqslant \lambda_\sigma \cdot \kappa_\sigma < \kappa = |K_\sigma| ,$$

this is always possible. Put $B = \bigcup \{X_\sigma; \sigma < \kappa'\}$, so $|B| = \Sigma(\lambda_\sigma; \sigma < \kappa') = \lambda$. Note that given μ with $\mu < \kappa$, if τ is least for which $\mu < \kappa_\tau$ then $X_\sigma \cap B_\mu = \emptyset$ whenever $\sigma \geqslant \tau$, and so $B \cap B_\mu \subseteq \bigcup \{X_\sigma; \sigma < \tau\}$. Hence $|B \cap B_\mu| \leqslant \Sigma(\lambda_\sigma; \sigma < \tau) < \lambda$. It follows that the family $\mathcal{B} \cup \{B\}$ is in $\mathfrak{G}$, and so $\mathcal{B}$ is not maximal.

If we assume the Generalized Continuum Hypothesis, then we can show that the condition $\lambda \leqslant \kappa$ and $\lambda' = \kappa'$ of Theorem 1.1.4 is also necessary for the existence of such a decomposition. This result goes back to Tarski [94]. Unfortunately there appears no way to avoid the GCH for this proof. We need first the following easy lemma.

Lemma 1.1.5. *Let λ be a cardinal, let β be an ordinal with λ not cofinal in β. For each set A with $A \in [\beta]^{\geqslant\lambda}$ there is an ordinal α with $\alpha < \beta$ such that $|\alpha \cap A| \geqslant \lambda$.*

Proof. Given A with $A \in [\beta]^{\geqslant\lambda}$, take a subset A_1 of A with order type λ. Since λ is not cofinal in β, there is α with $\alpha < \beta$ such that $A_1 \subseteq \alpha$. Then $A_1 \subseteq \alpha \cap A$ so that $|\alpha \cap A| \geqslant |A_1| = \lambda$.

Theorem 1.1.6 (GCH). *Let κ and λ be cardinals with $\lambda' \neq \kappa'$. Then any decomposition $\mathcal{A}$ of a set of power κ into subsets each of power at least λ with $\delta(\mathcal{A}) \leqslant \lambda$, has power at most κ.*

Proof. We may suppose that $\mathcal{A}$ is a decomposition of the set κ, so suppose $\mathcal{A} \subseteq [\kappa]^{\geqslant\lambda}$ with $\delta(\mathcal{A}) \leqslant \lambda$, and we need only consider the case $\lambda \leqslant \kappa$. So in fact $\lambda < \kappa$, because $\lambda' \neq \kappa'$. Since $\lambda' \neq \kappa'$ we know λ is not cofinal in κ. Applying Lemma 1.1.5, for each A in $\mathcal{A}$ there is an ordinal α with $\alpha < \kappa$ and $|\alpha \cap A| \geqslant \lambda$. For each α put $\mathcal{A}_\alpha = \{A \in \mathcal{A}; \alpha \text{ is least for which } |\alpha \cap A| \geqslant \lambda\}$. Then $\mathcal{A} = \bigcup \{\mathcal{A}_\alpha; \alpha < \kappa\}$. Thus $|\mathcal{A}| \leqslant \Sigma(|\mathcal{A}_\alpha|; \alpha < \kappa)$.

We seek an estimate of $|\mathcal{A}_\alpha|$. Since $\delta(\mathcal{A}) \leqslant \lambda$, for distinct A_1, A_2 from $\mathcal{A}_\alpha$ we have

$$|(\alpha \cap A_1) \cap (\alpha \cap A_2)| \leqslant |A_1 \cap A_2| < \lambda ,$$

and so $\alpha \cap A_1 \neq \alpha \cap A_2$. Thus if $\mathcal{B}_\alpha = \{\alpha \cap A; A \in \mathcal{A}_\alpha\}$ then $|\mathcal{B}_\alpha| = |\mathcal{A}_\alpha|$. Now $\delta(\mathcal{B}_\alpha) \leqslant \lambda$ and $\mathcal{B}_\alpha$ is a decomposition of the set α, so by Theorem 1.1.2 we have $|\mathcal{B}_\alpha| \leqslant |\alpha|^\lambda$. However $\lambda < \kappa$ and $|\alpha| < \kappa$ so by GCH, $|\mathcal{B}_\alpha| \leqslant \kappa$. Thus $|\mathcal{A}_\alpha| \leqslant \kappa$. Consequently $|\mathcal{A}| \leqslant \Sigma(|\mathcal{A}_\alpha|; \alpha < \kappa) \leqslant \kappa \cdot \kappa = \kappa$, and the theorem is proved.

Corollary 1.1.7 (GCH). *No set of power κ can be decomposed into a family $\mathcal{A}$ of more than κ sets each of power greater than λ such that $\delta(\mathcal{A}) \leqslant \lambda$.*

Proof. Such a decomposition is certainly impossible when $\lambda' \neq \kappa'$, by Theorem 1.1.6. So suppose $\lambda' = \kappa'$. Since $\lambda' \leqslant \lambda$ and $(\lambda^+)' = \lambda^+$, in fact $\kappa' \neq \lambda^+$. But now Theorem 1.1.6, with λ replaced by λ^+, yields the result.

It is worth remarking that if λ is finite (but κ still infinite) then Theorem 1.1.6 and Corollary 1.1.7 remain true, with the same proofs, and in fact GCH is not needed.

§2. Almost disjoint functions

In this section we shall discuss a refinement of the question considered in section 1. Rather than seeking families of almost disjoint subsets of an arbitrary set, we shall look at families of functions from a fixed set L to a fixed set K, where $|L| = \lambda$ and $|K| = \kappa$. We shall in fact from the start identify L with λ and K with κ, and so shall consider subfamilies of ${}^\lambda\kappa$, the set of all functions mapping λ into κ.

Let $\mathcal{F}$ be a subset of ${}^\lambda\kappa$. Thinking of the members of $\mathcal{F}$ as sets of ordered pairs, we can define $\delta(\mathcal{F})$, the degree of disjunction, as in Definition 1.1.1. So in fact $\delta(\mathcal{F})$ is the least cardinal η such that $|\{\alpha < \lambda; f(\alpha) = g(\alpha)\}| < \eta$ for all f, g in $\mathcal{F}$. For each value of the cardinal θ, we shall seek the largest possible cardinality of a family $\mathcal{F}$ where $\mathcal{F} \subseteq {}^\lambda\kappa$ with $\delta(\mathcal{F}) \leqslant \theta$. Clearly we need only consider the case when $\theta \leqslant \lambda$, since otherwise the condition $\delta(\mathcal{F})$ is no restriction on $\mathcal{F}$.

There is an equivalent formulation of this problem. We make the following definition.

Definition 1.2.1. A set T is said to be a *transversal* of the family $\mathcal{A}$ if $|T \cap A| = 1$ for each A in $\mathcal{A}$.

Given a family $\mathcal{A} = \{A_\nu; \nu < \lambda\}$ of λ pairwise disjoint sets each of power κ, by identifying each A_ν with κ, any transversal of $\mathcal{A}$ can be identified with a function in ${}^\lambda\kappa$. In this way, a family $\mathcal{F}$ with $\mathcal{F} \subseteq {}^\lambda\kappa$ and $\delta(\mathcal{F}) \leqslant \theta$ corresponds to a class of transversals of the family $\mathcal{A}$, any two of which meet in less than θ points. Thus the original problem is equivalent to finding the maximum size of a class $\mathcal{T}$ of transversals of $\mathcal{A}$ with $\delta(\mathcal{T}) \leqslant \theta$.

Clearly one can always find κ pairwise disjoint transversals, so we shall wish to know for which values of λ and θ is it possible that $|\mathcal{T}| > \kappa$.

We shall need to assume the Generalized Continuum Hypothesis almost throughout this section.

We start with several easy remakrs.

Lemma 1.2.2 (GCH). *Suppose either $\theta < \lambda$ and $\kappa^+ < \lambda$ or else $\theta = \lambda$ and $\kappa^+ < \lambda'$. If $\mathcal{F} \subseteq {}^\lambda\kappa$ and $\delta(\mathcal{F}) \leqslant \theta$ then $|\mathcal{F}| \leqslant \kappa$.*

Proof. For a contradiction, suppose that under these conditions there is $\mathcal{F}$ with $\mathcal{F} \subseteq {}^\lambda\kappa$ such that $\delta(\mathcal{F}) \leqslant \theta$ and $|\mathcal{F}| = \kappa^+$. For distinct f, g in $\mathcal{F}$, put $E(f, g) = \{\alpha < \lambda; f(\alpha) = g(\alpha)\}$, so $|E(f, g)| < \theta$. Put $E = \bigcup\{E(f, g); \{f, g\} \in [\mathcal{F}]^2\}$, so $|E| < \lambda$. If α is in $\lambda - E$ then $f(\alpha) \neq g(\alpha)$ for distinct f, g in $\mathcal{F}$, and so $|\mathcal{F}| \leqslant \kappa$, contradicting that $|\mathcal{F}| = \kappa^+$.

Lemma 1.2.3 (GCH). *Let $\lambda' \leqslant \kappa$. If $\mathcal{F} \subseteq {}^\lambda\kappa$ with $\delta(\mathcal{F}) \leqslant \lambda$ then $|\mathcal{F}| \leqslant \kappa^+$.*

Proof. If $\lambda \leqslant \kappa$ clearly $|\mathcal{F}| \leqslant \kappa^+$, so suppose $\kappa < \lambda$ and consequently λ must be singular. Take cardinals λ_τ where $\tau < \lambda'$ such that $\kappa^+ < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda$ and $\Sigma(\lambda_\tau; \tau < \lambda') = \lambda$. For a contradiction, suppose in fact that $|\mathcal{F}| \geqslant \kappa^{++}$. Since $\delta(\mathcal{F}) \leqslant \lambda$ we have a decomposition $[\mathcal{F}]^2 = \bigcup_{\tau<\lambda'} \{\{f, g\}; |f \cap g| < \lambda_\tau\}$. At this stage, we shall appeal to the relation $\kappa^{++} \to (\kappa^+)^2_\kappa$ which follows from Theorem 2.2.4. This symbol means that whenever the class of unordered pairs from a set S of power κ^{++} is decomposed into at most κ parts there is H with $H \in [S]^{\kappa^+}$ such that all the pairs from H fall in the same part of the decomposition. In the situation here, this ensures that there is a family $\mathcal{G}$ with $\mathcal{G} \in [\mathcal{F}]^{\kappa^+}$ such that for some fixed τ with $\tau < \lambda'$ always $|f \cap g| < \lambda_\tau$ for all f, g in $\mathcal{G}$. But now if $\mathcal{H} = \{g \upharpoonright \lambda_\tau^+; g \in \mathcal{G}\}$, it follows that $\mathcal{H}$ is a family of κ^+ functions mapping from λ_τ^+ into κ, with $\delta(\mathcal{H}) \leqslant \lambda_\tau$. This is not allowed by Lemma 1.2.2. Consequently $|\mathcal{F}| \leqslant \kappa^+$.

Lemma 1.2.4. *Let $\kappa' = \lambda'$. Then there is $\mathcal{F}$ with $\mathcal{F} \subseteq {}^\lambda\kappa$, $|\mathcal{F}| = \kappa^+$ and $\delta(\mathcal{F}) \leqslant \lambda$.*

Proof. There are ordinals γ_σ where $\sigma < \kappa'$ and δ_τ where $\tau < \lambda'$ such that $\gamma_0 < \gamma_1 < \ldots < \kappa = \sup\{\gamma_\sigma; \sigma < \kappa'\}$ and $\delta_0 < \delta_1 < \ldots < \lambda = \sup\{\delta_\tau; \tau < \lambda'\}$. We shall show that no family $\mathcal{G}$ with $\mathcal{G} \in [{}^\lambda\kappa]^{\leqslant\kappa}$ is maximal in the class of all families of almost disjoint functions in ${}^\lambda\kappa$; as in the proof of Theorem 1.1.4 this will give the result. So suppose $\mathcal{G} \in [{}^\lambda\kappa]^{\leqslant\kappa}$, and write $\mathcal{G} = \{g_\nu; \nu < \kappa\}$. Define g with $g \in {}^\lambda\kappa$ by choosing $g(\alpha)$ so that if τ is least for which $\alpha < \delta_\tau$ then $g(\alpha) \in \kappa - \{g_\nu(\alpha); \nu < \gamma_\tau\}$. Given μ with $\mu < \kappa$, if σ is least such that $\mu < \gamma_\sigma$ then $g \cap g_\mu \subseteq \{\langle\alpha, g(\alpha)\rangle; \alpha \leqslant \delta_\sigma\}$ and so $|g \cap g_\mu| \leqslant |\delta_\sigma \dotplus 1| < \lambda$. Thus g can be added to $\mathcal{G}$ and still gives an almost disjoint family.

The following lemma is a theorem of Erdös, Hajnal and Milner [36].

Lemma 1.2.5. *Let $\kappa^+ = \lambda$. Then there is $\mathcal{F}$ with $\mathcal{F} \subseteq {}^\lambda\kappa$ such that $|\mathcal{F}| = \kappa^+$ and $\delta(\mathcal{F}) < \kappa$.*

Proof. The plan for the proof is as follows. By thinking of the ordinals less than κ^+ arranged according to order type, it is clear how to define pairwise disjoint functions g_ν for ν with $\nu < \kappa^+$ mapping into κ^+, with the domain of g_ν all ordinals β with $\nu < \beta < \kappa^+$ – simply define $g_\nu(\alpha) = \alpha$.

For each α with $\alpha < \kappa^+$ we have $|\alpha| \leqslant \kappa$, and so we may take a well ordering $\langle \xi_{\alpha\nu}; \nu < \alpha \rangle$ of (a subset of) κ of order type α, and use the above construction on these well orderings to provide pairwise disjoint tail ends of functions $f_\nu : \kappa^+ \to \kappa$ where $\nu < \kappa^+$. One has only to arrange the front ends so that any two of the functions meet in a set of size less than κ, and this gives a family $\mathcal{F}$ as required.

Formally, then, we construct functions $f_\nu : \kappa^+ \to \kappa$ where $\nu < \kappa^+$ by induction as follows. Let ν with $\nu < \kappa^+$ be given, and suppose that the functions f_μ when $\mu < \nu$ have already been suitably defined. If $\alpha > \nu$, put $f_\nu(\alpha) = \xi_{\alpha\nu}$. Then when $\mu < \nu$ certainly $f_\nu(\alpha) \neq f_\mu(\alpha)$. To define $f_\nu(\alpha)$ for values of α where $\alpha \leqslant \nu$, let $\{f_\mu; \mu < \nu\} = \{f_{\nu\beta}; \beta < \min(\nu, \kappa)\}$ and choose $f_\nu(\alpha)$ in $\kappa - \{f_{\nu\beta}(\alpha); \alpha < \beta\}$. Then if $\mu < \nu$, for some β with $\beta < \kappa$ it follows that $f_\mu = f_{\nu\beta}$ and consequently $f_\mu(\alpha) \neq f_\nu(\alpha)$ whenever $\beta \leqslant \alpha \leqslant \nu$. Thus $|f_\mu \cap f_\nu| \leqslant |\beta| < \kappa$. Then if $\mathcal{F} = \{f_\nu; \nu < \kappa^+\}$, we have $|\mathcal{F}| = \kappa^+$ and $\delta(\mathcal{F}) < \kappa$ as desired.

We can now turn to the problem stated at the beginning of this section; to find the maximum cardinality of a family $\mathcal{F}$ where $\mathcal{F} \subseteq {}^\lambda\kappa$ with $\delta(\mathcal{F}) \leqslant \theta$. The case $\theta < \lambda$ is much easier than when $\theta = \lambda$, so we dismiss this case first.

Theorem 1.2.6 (GCH). *Suppose $\theta < \lambda$. Let $\mathfrak{m}$ be the maximum of the cardinalities of families $\mathcal{F}$ where $\mathcal{F} \subseteq {}^\lambda\kappa$ with $\delta(\mathcal{F}) \leqslant \theta$. Then $\mathfrak{m} = \kappa$ unless $\lambda = \kappa^+$, in which case $\mathfrak{m} = \kappa^+$.*

Proof. Suppose $\mathcal{F} \subseteq {}^\lambda\kappa$ with $\delta(\mathcal{F}) \leqslant \theta$.

If $\kappa < \theta$ then $\kappa^+ < \lambda$, and Lemma 1.2.2 shows $|\mathcal{F}| \leqslant \kappa$. If $\kappa > \theta$, consider $\mathcal{F} \restriction \theta^+ = \{f \restriction \theta^+; f \in \mathcal{F}\}$ and note that since $\delta(\mathcal{F}) \leqslant \theta$ the map which sends f to $f \restriction \theta^+$ is one-to-one. Now $\mathcal{F} \restriction \theta^+$ is a decomposition of $\theta^+ \times \kappa$, a set of power κ, into sets of size greater than θ with $\delta(\mathcal{F} \restriction \theta^+) \leqslant \theta$. So by Corollary 1.1.7, $|\mathcal{F} \restriction \theta^+| \leqslant \kappa$ and hence $|\mathcal{F}| \leqslant \kappa$. If $\kappa = \theta$ and $\kappa^+ < \lambda$, again Lemma 1.2.2 gives $|\mathcal{F}| \leqslant \kappa$.

Only the case $\kappa = \theta$ and $\lambda \leqslant \kappa^+$ remains. Since $\theta < \lambda$ we must have $\lambda = \kappa^+$.

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