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Function Spaces and Applications

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Foreword

These are the proceedings of a seminar held in Lund in June 1986, devoted to function space methods in analysis. The organizers felt that recent developments in Interpolation Theory have important implications in various areas of analysis, and that further work in Interpolation would benefit from the input of new problems and applications. Thus the makeup of the seminar was considerably broader than its two predecessors (Lund 1982 and 1983). It is particularly appropriate that these seminars have taken place in Lund, where Interpolation Theory was born some 60 years ago. To further emphasize the historical origin of the subject, the collection is preceded by a historical lecture on the life of Marcel Riesz.

The organizers wish to thank Naturvetenskapliga Forskningsrådet and the N.S.F. for supporting the conference. The two non-Swedish organizers feel that they express the sentiments of all participants in thanking Jaak Peetre and Hans Wallin as well as the members of the mathematics department at Lund for their warm hospitality.

The Editors

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MARCEL RIESZ IN LUND

Jaak Peetre

(Public lecture delivered on June 18, 1986 at 7 PM.)

Ladies and gentlemen,

Marcel Riesz was born on Nov. 16, 1886 in Györ, Hungary and died on Sept. 4, 1969 in Lund, Sweden. Thus this conference has also given us a convenient opportunity to celebrate the 100th anniversary of his birth.

The organizers had originally planned this lecture to be delivered by Prof. Lars Gårding. Unfortunately, Prof. Gårding is currently abroad and so, as we have not been able to materialize his spirit, I shall try to take over his role. So imagine now a much more charismatic personality. At least, I hope you will not end up by throwing tomatoes at me.

We all wish to believe that we are forerunners of our science, but is also important not to lose the contact with its glorious past. Here is an appropriate quotation: "We may say that its (mathematical history) first use to us is to put or to keep before our eyes 'illustrious examples' of first-rate mathematicians" ([We], p. 229). Therefore this talk certainly has a place within the framework of this conference.

Sources. There are two orbituaries by Lars Gårding [G1], [G2]; [G2] is an expanded Swedish version of [G1], with more personal recollections and a few anecdotes.

I myself once wrote a bibliography of Marcel Riesz for an Italian encyclopedia [P1]. It turned out to be a double catastrophe: 1st it was translated into Italian and 2nd it was cut by a factor 2 (in the middle). So don't read it. But it was this preoccupation with Marcel Riesz some 15 years ago that makes that I possess some of my present knowledge.

John Horvath, one of Riesz's last associates and a personal friend, has written a very detailed account of his early work [H1].

The very interesting correspondence between Marcel Riesz and the British mathematicians (read: Hardy), is analyzed at length by Dame Mary Lucy Cartwright [C].

It should also be mentioned that Gårding is now writing a book on Swedish mathematics. Below I draw heavily on his expert knowledge.

Pre-history of Lund mathematics. Before I start my subject proper, let me say a few words about the status of mathematics in Lund prior to Marcel Riesz's times.

You have already encountered the name Tycho Brahe (b. 1546 at Knutstorp mansion (50 km NW of Lund), d. 1601 in Prague), but he has no bearing upon the University of Lund, founded only in 1668, after the Swedish conquest in 1660. It has been maintained that the university was created mainly for geo-political reasons, to prevent the sons of the local clergy to attending Danish schools.

There are at least three names of some note in the history of mathematics connected with Lund. Characteristically, none of these were professional mathematicians.

18^{th} century. Erlang Samuel Bring (1736 - 1798) was a professor of history. His name is attached to the Bring-Jerrard theorem, stating that an algebraic equation of the 5^{th} degree can be brought to the normal form $x^5 + ax + b = 0$, using a suitable Tschirnhaus transform. Some people have seen in Bring a precursor of Abel's but, according to Gårding, this is a gross overestimation.

19^{th} century. Albert Victor Bäcklund (1845-1922) was a professor of physics. His name again is connected with the Bäcklund transform, now very fashionable in non-linear p.d.e. (sine-Gordon and all that).

Turn of the century. Finally, Carl Ludwig Wilhelm Charlier (1862 - 1934) was an astronomer. Being an pioneer in stellar statistics, it is natural that he became involved in questions of probability. The Charlier polynomials form a system of "discrete" orthogonal polynomials corresponding to the Poisson distribution.

The modern era of Lund mathematics started - if we exclude a short interlude with the Dane Niels Norlund, famous for several monographs on topics such as difference equations and special functions, who was a professor here for a year or so - with Marcel Riesz being appointed a full professor in 1926. After his retirement in 1952 he spent some ten years at various places in the United States - Bloomington, Chicago and finally College Park, Maryland. When he became ill, he returned to Lund and died here.

His brother. Marcel Riesz was one out of three brothers. An elder brother was the famous F. Riesz (1880 - 1956). The third brother devoted himself to practising of law.

The general concensus among mathematicians seems to have been that F. was by far the more powerful and influential among the two. But such judgements change with time and perhaps we are now going to witnes a phase when the importance of Marcel Riesz's work is being reevaluated. Gårding tells us ([G2], p. 72) that during his last encounter with him - Riesz was then already hospitalized - his thoughts wandered to this very subject. "Remember", Riesz told Gårding, "that I am just a brightly colored copy of my brother".

Stockholm period 1910 - 1925. Marcel Riesz came to Sweden in 1911 upon the invitation of G. Mittag-Leffler. The two had met, apparently, in 1908 on the occasion of the ICM in Rome and, as Riesz's first mathematical subject was summation methods, Mittag-Leffler took a liking for the young man. Remember that Marcel Riesz was only in his 20's at the time. Riesz had begun his studies in Budapest. After a shorter stay at Göttingen, he had spent the year 1910 - 1911 in Paris.

One of his first assignments in this country was to compile an index for the first 35 volumes of Acta Mathematica, as you know, a journal founded by Mittag-Leffler himself. As a unique specimen in the mathematical literature, it contains a complete collection of portraits of all the authors!

As Mittag-Leffler was already an aged man, Marcel Riesz soon became the leader of the young generation of Stockholm mathematicians. His students from that period include names such as Harald Cramér, known for his achievements in probability, who also later became Chancellor of the Swedish universities (universitetskansler) - he died only recently aged 92 - and Einar Hille, a Swedish-American, born in New York, who had returned to his country of origin to study mathematics; Hille, apparently, was underestimated in Sweden and ultimately went back to the U.S. and ended up as a professor at Yale.

Addresses. Personaliae. Marcel Riesz address in Stockholm - Döbelnsgatan 5 - is one of the famous ones in the history of mathematics - it is here that he proves his celebrated conjugate function theorem. In Lund he soon moves to Kävlingevägen 1, to live in on the top floor of what was then and still is one of the most fashionable apartment buildings in town. I have been told that the flat was originally intended for Nils Zeilon, who became a professor in the same year as Riesz. But Zeilon had a large family so ultimately the flat turned out to be too expensive for him. One of Riesz's daughter Birgit Riesz-Larsson has after her own retirement moved back there. The other (twin) daughter Margit Riesz-Pleijel is married to my thesis advisor Åke Pleijel.

Competition. As you may know, in Sweden one has to compete for a professorship. In 1925? when Riesz applied for a professorship at Stockholm his competitors were Torsten Carleman and Harald Cramér. Riesz lost to Carleman and had to content himself with a chair at Lund. To become a professor at Lund, at the time, was considered, or at least Riesz himself considered it as an exile. Is it still?

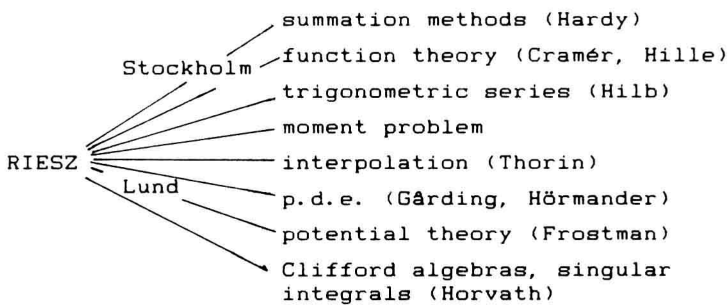
Scientific profile. Students. For Marcel Riesz his arrival in Lund means, scientifically speaking, a watershed. Presumably through his contacts with Nils Zeilon, Riesz turned away from classical analysis and became interested in p.d.e. and became thereby the founder of the Swedish or Lund school in p.d.e. (Gårding, Hörmander etc.). Zeilon is a name which is little known in the mathematical community but Gårding has tried to reevaluate him on several occasions (see e.g. his address to the 1970 ICM [G3]). In the 1910's Zeilon wrote several penetrating studies on the singularities of hyperbolic equations but, being published in an obscure Swedish periodical, their influence has been negligible. In particular, he reached in this way an understanding of the physically important problem of double refraction in crystals, a question where Sonja Kowalewsky, incidentally another one of Mittag-Leffler's "protégés", had grossly erred (actually by misusing the Cauchy--Kowalewsky theorem!; this issue has recently again aroused some controversy; see several articles in the Intelligencer). Zeilon was also an amateur painter and painting seems to have been one of his major activities in later professorial years. In contrast to Riesz, he never had students.

As I have mentioned already, Riesz's first mathematical field had been summation methods, something that was very fashionable in the first decades of the century - remember that L. Fejér was a Hungarian and in 1901 had proved his famous theorem on the Cesaro summability of Fourier series. (It may well be that there will soon be a revival of interest in summation; this is connected with advances on the numerical (computer) side;

see e.g. [Wi] for a recent book on the subject.) But he also worked in classical function theory - the F. and M. Riesz theorem is surely well-known to everybody - and on trigonometric series, where he collaborated with Hilb on an Encyklopädie-article.

Then he invested some effort (ca. 1920) on another problem fashionable for the day, the moment problem. In this connection he did some work in which he in a way anticipated the Hahn-Banach theorem. However, by and large his achievements here fall into the shadow of men such as Carleman and Nevanlinna, generally considered to be mathematicians of greater stature.

Riesz's contribution to mathematics from the Lund (or post-Lund) period is intimately tied up with the work of his many talented students and associates. Before proceeding it will be convenient to draw a chart, depicting some general trends.



Among less known names, let me mention Carl Hylthén-Cavallius and Lennart Sandgren. The former published some original work on positive trigonometric sums, while the latter wrote on convexity and also a thesis on the Steklov eigenvalue problem; later he made a career as a civil servant, ending up as "landshövding" in "Kristianstads län".

Riemann-Liouville integral. If one looks for a common denominator in Riesz's work, there is one thing par excellence that comes to ones mind and this is the Riemann-Liouville integral of fractional integration:

$$\boxed{I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy} \qquad (\alpha > 0),$$

which he came to investigate from various angles. He was, presumably, led to this through his previous preoccupations with summation, where he had invented what he himself termed the typical means and other people have begun to call the Riesz means. In terms of integrals, rather than series, the basic fact (on the Abelian side) is that if f is a function on the real line such that $f(x) \to A$ (as $x \to \infty$) then, for any $\alpha > 0$, also

$$\frac{\alpha}{x} \int_0^x \left(1 - \frac{y}{x}\right)^{\alpha-1} f(y)\, dy \to A.$$

The Riesz method of summation has several advantages compared to the older Césaro method. Riesz's early work in this area is summarized in the Hardy-Riesz Cambridge tract [HR] from 1915, which, however, due to war circumstances was mostly written by Hardy alone; the two authors happened to be in opposite camps. During the war (World War I), Riesz, still a citizen of Austria-Hungary, served at his country's embassy in Stockholm.

Returning to the Riemann-Liouville integral itself, Riesz observed that the operators I possess the semigroup property:

$$I^{\alpha}I^{\beta} = I^{\alpha+\beta}, \qquad I^{0} = \text{identity}.$$

Using analytic continuation (taking, at an intermediate stage, α to be complex, with Re α > 0) he could define I^{α} also for negative α. Especially, one has I^{-1} = derivation, whereas I^{1} = (indefinite) integration. It is curious that Gel'fand in the preface of the book [GS] maintains that Riesz in this way became a precursor of distributions or, as Gel'fand himself says, generalized functions; the name of L. Schwartz hardly occurs in [GS].

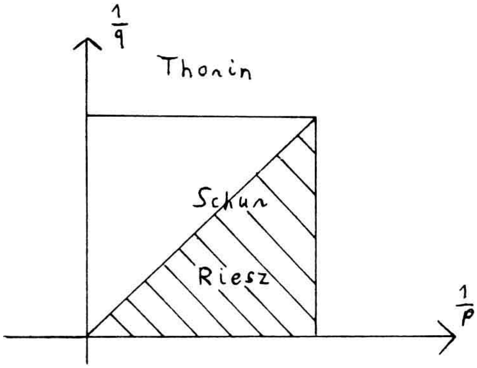
Riesz then goes on to generalizations in several variables. In particular, he defines the generalized or Riesz potentials

$$I^{\alpha}\mu(x) = c_{\alpha,n} \int |x - y|^{\alpha-n} \, d\mu(y) \qquad (0 < \alpha < n),$$

where $c_{\alpha,n}$ is a suitable "gamma factor", for an arbitrary "mass distribution" μ in R^{n} (equipped with the Euclidean metric); if $\alpha = 2$ (and $n \neq 2$) one has the Newton potential. Now formally $I^{-2} = \Delta$ = Laplacean. The subject was then followed up by Otto Frostman in his famous 1935 thesis [F], which still may serve as a most readable introduction to "modern" potential theory. Frostman was a secondary school teacher for many years; from my own boyhood I remember his characteristic profile on the yard of the Lund "Cathedral School" (Katedralskolan; incidentally one of the oldest institutions of higher learning in Scandinavia). When at last in 1952 (the very year of Riesz's retirement) he obtained a nomination as a university professor at Stockholm, his energy seemed to have been spent, and he published little after that.

In another direction, one can define operators I^{α} generalizing the Riemann-Liouville integral also in R^{n+1} equipped with the Minkowski metric. Then $I^{-2} = \Box$ = d'Alembertian. Riesz could exploit this to give a treatment of the Cauchy problem for the wave equation. Thereby he got a most elegant substitute for Hadamard's "partie finie", used by the French mathematician in his studies of general second order hyperbolic equations. Riesz's investigations in this area, dating from the 30's, were published only much later in a truly monumental Acta paper [R1].

The Riemann-Liouville integral recurs, for instance, also in a little known paper by Gårding [G4], where the author - prior to his later work on general higher order hyperbolic equations with constant coefficients - investigates certain hyperbolic equations related to Siegel's generalized upper halfplanes. This is perhaps a matter into which one ought to dig deeper, as Siegel halfplanes connect to such now popular issues as Jordan



triple systems (JTS). It strikes me now also that a few years ago I pointed out a connection between Gårding's theory and the then freshly solved van der Waerden's conjecture [P2].

Interpolation (for "aficionados" only). I come now to the topic which really has served as the chief raison d'ètre of this meeting. The first result about interpolation in the literature is however the one by Schur (1911), which says, roughly speaking, that T: $L^1 \to L^1$, T: $L^\infty \to L^\infty$ entails T: $L^p \to L^p$ $(1 < p < \infty)$, that is, what we now understand as the "diagonal" case of the Riesz-Thorin theorem. For some reason, this is a fact that never has established itself as a genuine "theorem"; it is being "rediscovered" over and over again. Riesz is of course fully aware of his predecessor, whom he duly quotes.

The road by which Riesz reaches his "convexity" theorem [R2] is curiously enough connected with the conjugate function theorem, another of Riesz's lasting contributions to Analysis (discovered in 1923 but published only in 1927 [R3]). For the real line, H denoting the Hilbert transform, it simply states that H: $L^p \to L^p$, $1 < p < \infty$. Riesz's proof of this result, reproduced many times before, goes, rougly speaking, as follows:

1) He proves it for p = 2, 4, 8,...

2) Then he "interpolates" to get it for $2 < p < \infty$ (p not an even integer).

3) Finally, he applies duality to incorporate the case $1 < p < 2$ as well.

It is thus the middle step when generalized, which is the genesis of the whole discipline of (abstract) interpolation.

As for step 1, the case p = 2 is trivial. If, for instance, p = 4 the argument is the following. By Cauchy's theorem $\int_R f^4\,dx = 0$ where $f = u + iv$ is an analytic function in the upper halfplane with $v = Hu$ on the boundary R. Expanding and taking real parts, one gets the identity

$$\int u^4 - 6\int u^2v^2 + \int v^4 = 0.$$

From this, using again Hölder's inequality, to take care of the middle term, one then readily obtains $\|v\|_4 \le c\|u\|_4$. For p a general even integer, the proof is analogous.

Riesz tells himself how he hit on this argument. He intended to give the (trivial) case p = 2 as a problem to a student (not one of the famous ones!) in an examination and, as he suspected that the poor man did not know Plancherel's theorem, he began to think if there was not an alternative route... Whence Hardy's laconic comment ([C], p. 502): "Your student's life is not entirely without value (though I suppose he will never understand why)."

Nowadays the conjugate function theorem is subsumed in the Calderón-Zygmund theory of singular integral, thus real variable theory. But in questions of obtaining sharp estimates of the constant, the function theoretic approach, thus having its origin in Riesz's work, still is of importance; see e.g. [E1], [E2] for a recent contribution.

In [P3] I gave a "contemporary" version of Riesz's original proof of his interpolation theorem, interpreting it in terms of a suitable interpolation method, the "Riesz method". Can other known proofs of Riesz's theorem, e.g. the ones by Paley or the one by Thorin (see [Th2], appendix and [Th3]), be given a similar reinterpretation?

The extension to the whole square was open for some time and Riesz himself seems to have been of the opinion that this was not possible (probably because of an example in [R2]). Therefore, Thorin's proof [Th1] in 1939 of the full Riesz-Thorin theorem which also forms the basis for Calderón's complex method of interpolation, came as a surprise to everybody. Littlewood alludes to Thorin's achievement as "the most impudent idea in mathematics" ([L], p. 1). Olov Thorin, who is an utterly modest man, has himself described in detail the process of discovery [Th3]. Apparently, it was a cursory remark of Frostman's that triggered him off. In his 1948 thesis [Th2] Thorin independently proves also Sobolev's theorem about the L^p-continuity of the Riesz potentials (the n-dimensional version of a theorem by Hardy-Littlewood). Most of his life he worked as an actuary - Riesz too served as a consultant for Swedish insurance companies - but upon his retirement he is reported to have said: "Now at last I am free to do mathematics!" His name is well-known to probabilists and he has done important work on i.d.r.v. (cf. [BP]).

Clifford algebras. Singular integrals. In his later life Marcel Riesz worked on somewhat more esoteric subjects. I shall bypass here his preoccupations with number theory and "pure" algebra, concentrating instead on "geometry". Riesz, apparently, was always fascinated by the internal beauty of geometry, in particular, the interplay between geometry and "physics" (relativity, quantum mechanics). In particular, he tried to promote the use of Clifford numbers in Analysis, notably in connection with the Dirac equation. At the time, people did not think very highly of this. At least, his Maryland notes [R4] got a bad review [Ti]. But, as we have seen before, customs change even in such a conservative science as ours.

An 2^n-dimensional Clifford algebra C is an algebra with generators $e_1,\dots,e_n$ subject to relations $e_i^2 = \pm 1$, $e_i e_k + e_k e_i = 0$ $(i \neq k)$. Then, formally, a solution ϕ (with values in a suitable C-module) of the equation

$$\sum_{i=1}^{n} e_i \frac{\partial \phi}{\partial x_i} = 0$$

will have the property that each component is a nullsolution of the "wave" operator $\Sigma \pm \partial \phi^2/\partial x_i^2 = 0$. Whence the connection with Dirac's theory. If we add one more variable x_0, taking $e_0 = 1$, we get a generalization of the Cauchy-Riemann equations, which leads to a generalized function theory based on the notion of "monogenic" function (cf. e.g. the monograph [BDS]). Monogenic functions have recently atracted a lot of interest. For instance, they have been used in the context of singular integrals (see e.g. [McI]). The famous Riesz transforms (generalizing the Hilbert transform in the case of one variable) presumably also arose in this context (see [H2]). Later they came to be the substratum of the Fefferman--Stein-Weiss theory of generalized H^p spaces [FS], [SW].

At the '83 Lund conference Svante Janson and I suggested [JP], in an attempt to generalize Calderón's method, to use, instead of analytic functions in a strip (or a disk) in the complex plane, harmonic vector fields in an analogous domain in R^{n+1}, and similar considerations can be made more generally with Clifford numbers. The Thorin construction, in the original L^p context, basically uses only the properties of one special function, the exponential. If one could find a suitable Clifford analogue of the exponential, perhaps one could do something... I wonder what Marcel Riesz himself would have said had somebody in his lifetime pointed out to him this connection between two apparently quite separate aspects of his work.

Conclusion. Compared with other great mathematicians of the past, Marcel Riesz published relatively little and, as we have seen, often with a great delay. This has in part to do with his personality, about which I have said very little. Above all, he was a perfectionist. Let me conclude with a piece of happy news. After many years of delay, his collected works will now be published by Springer-Verlag, with Lars Hörmander as editor. The sponsors are N.F.Riesz - the same as for this conference - and the Swedish Actuarial Society, this as a tribute to his toils in actuarial mathematics.

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N.B. - A complete list of Marcel Riesz's scientific publications can be found in [G1]; here we have listed only those few items explicitly mentioned in the text ([HR], [R1-4]).

INTERPOLATION OF TENT SPACES AND APPLICATIONS

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In a series of papers [CMS] , [CMS1] , Coifman, Meyer and Stein have studied a new family of function spaces, the so-called "tent spaces." These spaces have proved to be useful for the study of a variety of problems in harmonic analysis. Moreover, their study has led to interesting simplifications and refinements of some basic techniques of real analysis. In particular we mention the results of [CMS1] concerning the Cauchy integral on Lipchitz curves, the natural and simple approach to the atomic decomposition and interpolation theory of H^p spaces (cf. [CMS1]) and the connection with the theory of Carleson measures (cf. [CMS1] , [AM] , [BJ]).

Tent spaces on product domains (cf. [AM]) have been used in [GM] to settle the problem of describing the complex interpolation spaces between H^p and BMO on product domains. We also mention the authors' work on vector valued tent spaces and their duality and interpolation properties (cf. [AM1]). An interesting consequence of this development is a generalization of the Lions-Peetre interpolation theorem of vector valued L^p spaces to the setting of vector valued H^p spaces.

These papers also provide applications to the study of maximal operators, square functions, balayages, weighted norm inequalities....

In this note we survey results concerning the interpolation theory of tent spaces and spaces of Carleson measures. The plan of the paper is as follows: the first three sections give the definition and basic properties of tent spaces and spaces of Carleson measures, §4 treats the interpolation theory of spaces of Carleson measures, §5 and §6 provide the real and complex interpolation of tent spaces. In particular §6 includes a Wolff type theorem for Quasi Banach lattices (cf. [GM1]), §7 provides an application to the interpolation theory of H^p spaces, while §8 contains a brief synopsis of tent spaces on product domains, vector valued tent spaces and further applications.

1. The T_q^p spaces

We shall work in $\mathbf{R}_+^{n+1}$ but most of the results can be stated in the more general context of homogeneous spaces. Points in $\mathbf{R}_+^{n+1}$ shall be denoted by (x,t) or (y,t) , $x,y \in \mathbf{R}^n$, $t > 0$. Given $x \in \mathbf{R}^n$, let $\Gamma(x)$ denote the cone $\{(y,t)/|x-y| < t\}$. Let Ω be an open set, $\Omega \subsetneq \mathbf{R}^n$, we let $T(\Omega)$ ("the tent over Ω ") be the subset of $\mathbf{R}_+^{n+1}$ defined by $T(\Omega) = \{(x,t)/B(x,t) \subset \Omega\}$, where $B(x,t)$ denotes the ball with centre x and radius t .

The tent spaces T_q^p , $0 < p \leq \infty$, $0 < q \leq \infty$, are defined using two families of functionals. The A_q functionals defined by

$$A_q(f)(x) = \{\int_{\Gamma(x)} |f(y,t)|^q \frac{dydt}{t^{n+1}}\}^{1/q} \qquad (1.1)$$

$$A_\infty(f)(x) = \sup_{\Gamma(x)} |f(y,t)| \qquad (1.1)'$$

The C_q functionals given by

$$C_q(f)(x) = \sup\{\frac{1}{|B|}\int_{T(B)} |f(y,t)|^q \frac{dydt}{t}\}^{1/q} \qquad (1.2)$$

where the sup is taken over all balls containing x .

The tent spaces T_q^p , $0 < p$, $q < \infty$ are defined by $T_q^p = \{f/A_q(f) \in L^p(\mathbf{R}^n)\}$, and we let $||f||_{T_q^p} = ||A_q(f)||_p$. The case $q = \infty$ and $p < \infty$ requires a natural modification: T_∞^p will denote the class of continuous functions in $\mathbf{R}_+^{n+1}$ such that $A_\infty(f) \in L^p(\mathbf{R}^n)$ and such that $\lim_{\varepsilon \to 0^+} ||f_\varepsilon - f||_{T_\infty^p} = 0$, where $f_\varepsilon(x,t) = f(x + \varepsilon, t)$, and $||f||_{T_\infty^p} = ||A_\infty(f)||_p$. Finally the T_q^∞ spaces $(q < \infty)$, are defined by the condition $C_q(f) \in L^\infty$ and $||f||_{T_q^\infty} = ||C_q(f)||_\infty$.

It is readily verified that the T_q^p spaces are complete and that for $1 \leq p$, $q \leq \infty$, the T_q^p are Banach spaces.

2. Spaces of Carleson measures

For measures on $\mathbf{R}_+^{n+1}$ we can also consider variants of the A_1 and C_1 functionals. For a measure w on $\mathbf{R}_+^{n+1}$ we define

$$A_1(w)(x) = \int_{\Gamma(x)} t^{-n} d|w|(y,t) \qquad (2.1)$$

$$C_1(w)(x) = \sup_{B \ni x} \frac{1}{|B|}\int_{T(B)} d|w|(y,t) \qquad (2.2)$$

For $0 < \Theta < 1$ the spaces V^Θ of Carleson measures of order Θ (cf. [AB]) consist of those measures w such that for some $C > 0$

$$|w|(T(\Omega)) < C|\Omega|^\Theta \qquad \forall\ \Omega \text{ open in } \mathbf{R}^n \qquad (2.3)$$

We let

$$||w||_{V^\Theta} \inf\{C/\ (2.3) \text{ holds } \forall\ \Omega \text{ open, } \Omega \subset \mathbf{R}^n\}$$

It is also convenient to define $V^0 = \{w|w$ finite measure on $\mathbf{R}_+^{n+1}\}$.

The spaces V^Θ can be also characterized in terms of the A_1 functionals (cf. [AB]):

$$V^\Theta = \{w|A_1(w) \in L(\tfrac{1}{1-\Theta},\infty)\} \qquad (2.4)$$

It is shown in [CMS1] that

$$(T_\infty^1)^* = V^1 \qquad (2.5)$$

The duality result (2.5) is a consequence of the basic inequality valid for Carle-