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Paul B. Garrett

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University of Minnesota

Wadsworth & Brooks/Cole Advanced Books & Software
Pacific Grove, California

Brooks/Cole Publishing Company
A Division of Wadsworth, Inc.

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging in Publication Data
Garrett, Paul B.
Holomorphic Hilbert modular forms.

Bibliography: p.
Includes index.
1. Hilbert modular surfaces. 2. Holomorphic mappings.
I. Title.
QA573.G37 1989 516.3'5 88-33753
ISBN 0-534-10344-8

Sponsoring Editor: *John Kimmel*
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Interior and Cover Design: *Vernon T. Boes*
Typesetting: *Asco Trade Typesetting Ltd.*
Cover Printing: *Phoenix Color Corporation*
Printing and Binding: *Arcata Graphics-Fairfield*

Introduction

This book is intended as a systematic, self-contained, and straightforward introduction to a substantial part of the theory of holomorphic Hilbert modular forms, associated L-functions, and especially their arithmetic. As such, it is an introduction to the theory of automorphic forms in general, especially the arithmetic of holomorphic automorphic forms. Beginning from the most standard algebraic number theory and function theory, one can develop matters far enough to recover some of Shimura's results on special values of L-functions attached to Hilbert modular forms. Also within reach are the theorem on special values of L-functions of totally real number fields (after Klingen), and examples of results on special values of certain grossencharacter L-functions (due to Damerell over $\mathbb{Q}$ and to Shimura in general).

One of many reasons for the study of Hilbert modular forms is for applications to Dirichlet series. One emphasis of this book is the construction of Dirichlet series

$$D(s) = \sum_{n \geq 1} c_n/n^s$$

which have analytic continuations and functional equations. Many such are obtained as (Mellin) integral transforms of holomorphic Hilbert modular forms: the analytic continuations and functional equations follow almost immediately from the properties of such modular forms. Further, the existence of an Euler product factorization

$$D(s) = \prod_{p \text{ prime}} 1/P_p(p^{-s})$$

(where P_p is a polynomial) of a Dirichlet series arising from a modular form f is essentially equivalent to properties of f: f should be an eigenfunction for certain operators (the Hecke operators) on automorphic forms.

The first three chapters bear upon such matters. A considerably subtler issue is the determination of the nature of the values of such Dirichlet series $D(s)$ at special integers s. Some samples of results in this direction are given in sections 6.2–6.5. The discussion of the arithmetic of the special values of these Dirichlet series relies profoundly upon arithmetic properties of modular forms themselves.

It is possible to give a reasonably full account of these special-values results by virtue of new direct proofs (chapter 7) of a theorem (stated in 6.1) of which all these special values results are corollaries. One half of this theorem is part of a result of Shimura concerning the arithmetic properties of Fourier coefficients of Hilbert modular forms. The other half is an apparently new comparison of inner products of cuspforms, that allows an omission of some delicate matters concerning Hecke operators.

In the first five chapters I have attempted to collect in one place fundamental ideas and methods that arose during the period 1904–1961. The diffuseness of the literature and the desirability of illustrating the efficacy of "modern methods" I have taken as justification of this reprise. Chapters 6 and 7 bear upon more recent (and more arithmetic) developments. All but the method of 7.9–7.10 and the choice of presentation must be attributed to many other authors—notably, Blumenthal, Hecke, Siegel, Maass, Rankin, Klingen, Shimura, and Selberg.

There is no presumption of familiarity with the theory of elliptic functions and elliptic modular forms, (although some experience might prove helpful). Indeed, the beautiful but very special ideas and calculations arising from the theory of elliptic functions may not be the best paradigm for general expectations: there is no convenient generalization of the Weierstrass *P*-function. There are even more mundane coincidences that are implicitly exploited in the classical theory of elliptic modular forms; for example, that the rational integers are a Euclidean (hence principal) domain. But, as it turns out there is no genuine need to depend on such fortuitous coincidences, and because the general situation is important in its own right, I have tried to give a portrayal of Hilbert modular forms that is essentially blind to special features of the ground field. It is possible to do so coherently and palatably by considering modular forms on adele groups.

Currently, it may seem that the theory of elliptic modular forms (that is, Hilbert modular forms with ground-field $\mathbb{Q}$) is most immediately relevant to other parts of arithmetic. However, Klingen's determination of the arithmetic nature of the special values of *L*-functions of totally real number fields (see 6.2) is a pointed example of the necessity and utility of more general considerations. Also, the treatment here (6.3) of Damerell's and Shimura's results on grossencharacter *L*-functions makes essential use of the arithmetic properties of Fourier coefficients of Hilbert modular forms (due to Shimura). (See 6.1.) Further, the arguments of 6.3–6.5 rely upon comparisons of inner products of automorphic forms; 6.1 gives a new result in this direction.

The new proof (Chapter 7), which yields a part of Shimura's theorem on the arithmetic properties of the Fourier coefficients of Hilbert modular forms and gives the comparison of inner products, suggests another broadening of perspective and methods: the discernible arithmetic of certain (Siegel's) Eisenstein series on a larger group (a symplectic group of rank 2) is the starting point of the proof. Thus, Siegel's already striking idea (in [Si2]) for understanding the arithmetic nature of the Fourier coefficients of more general Eisenstein series becomes more significant. Shimura's original proof (in [Sh3]) of the theorem on Fourier coefficients depends on a special case of his profound results (in [Sh1]) on canonical models of arithmetic quotients, that in turn depend on quite serious results from algebraic geometry.

While the method of 7.9 and 7.10 seems incapable of recovering Shimura's theorems on generation of class fields, it is profitable to see that the arithmetic nature of the Fourier coefficients can be understood directly. Previous methods of comparison of inner products of cuspforms relied essentially on the delicate notion of "newform" (as in [M], [A], [C]); the idea here is less technical and is relevant in more general situations as well.

Another point hopefully illustrated here is the central role of Eisenstein series—in Rankin's integral representation of L-functions (4.10), in Shimura's applications (6.3–6.5), and in the present proof of the theorem of 6.1 (Chapter 7).

The first chapter studies Hilbert modular forms from a "classical" viewpoint—that is, as it would have been done prior to 1960. Siegel's book [Si1] also gives an introduction to this material. Most of this material merely imitates the theory of elliptic modular forms. An exception is the method of proof of the finite-dimensionality of spaces of cuspforms in 1.7; this method is due to Siegel and Maass. This method, applicable as well to elliptic modular forms, supplants the use in that theory of either residues or the Riemann-Roch theorem, neither of which is as helpful in general. Precise attribution of all historic sources is beyond my ability; the results here are due to many authors, going back as far as Blumenthal's pursuit of Hilbert's suggestion in 1904. Some of the more important sources are [B1], [B2], [H1], [H2], [H3], [KL1], [M], [P], and [Se]. With regard to some matters (Hecke operators and L-functions) the treatment in this chapter is only introductory, as these topics are best treated later in a different way. For the classical versions of these topics—that is, where the base field is $\mathbb{Q}$, one might consult [Sh2] or [Gu].

The second chapter introduces a general notion of automorphic form on $GL(2)$ over the adeles. The influential books [GGP] and [JL] make the usefulness of this viewpoint clear. This viewpoint makes any ground field tractable by uniform methods. For example, a classical treatment of Hecke operators for congruence subgroups over rings of integers not of narrow class number one is excruciatingly technical and not very helpful; an adelic treatment supplants this approach by a discussion even simpler than consideration of Hecke operators for $SL(2, \mathbb{Z})$. Further, in this setting the Euler product expansion of Dirichlet series obtained as Mellin transforms of modular forms arises as inexorably as in Tate's thesis's treatment of L-functions of number fields (as in [La] or [CF]). In 2.7 some references to recent developments regarding analytic continuation of higher L-functions are given. Appendices A.1, A.2, and A.4 give some relevant background on integration theory on homogeneous spaces, harmonic analysis on the adeles, and invariant differential operators on $SL(2, \mathbb{R})$.

Both for completeness and for clarity, the third chapter compares the classical and adelic versions of these things. An essential ingredient is a Strong Approximation Theorem for $SL(2)$; this theorem is proven in Appendix A.3, which specializes and simplifies a proof found in [Kn]. Although it is not hard to give a quite elementary proof over $\mathbb{Q}$ (for example, in Chapter 3 of [Sh2]), it is not easy to do so over arbitrary number fields; the original references for the ideas involved in even more general strong-approximation theorems are the papers [E1] and [E2]. I do not go far in discussing Hecke operators at bad primes; for such a discussion, see [A], [M], and [C].

Chapter 4 is concerned with Eisenstein series. Consideration of analytic and arithmetic properties of Eisenstein series goes back almost 200 years and is still central to number theory and automorphic forms. Most of this chapter is a study of the formal properties of Eisenstein series, which become even more transparent in an adelic formulation. The method of proof of analytic continuation and functional equation given in 4.2 reformulates an idea I first saw in Godement's exceptional article [Go]; perhaps the idea should be attributed to Maass and Selberg in some form. This avoids direct consideration of confluent hypergeometric functions and their p-adic analogues, although the latter have virtues of their own. Section 4.10 is a mildly adelic version of a part of Rankin's papers, [R1] and [R2]. (See also [P].) Although Rankin's first paper on these matters appeared in the 1930's, the potential of the general idea is not yet exhausted. Sections 6.3–6.5 pursue the arithmetic side of the idea (decisively illustrated in Shimura's papers [Sh4], [Sh5], and [Sh6]).

In Chapter 5 the treatment is again somewhat classical, for a non-trivial reason: theta series that are of half-integral weight cannot be considered as automorphic forms on an adelic $GL(2)$; indeed, the study of half-integral-weight automorphic forms is a subject in its own right and is not pursued here. The main point of this chapter is the proof that theta series are modular forms; the proof uses an adelic version of the Poisson summation formula. Theta series have been studied for a long time; some references are [B2], [KL2], and [Sc]. Theta series of integral weight reappear in 6.3 in connection with grossencharacter-L-functions.

Chapter 6 discusses some developments since 1961. Section 6.1 states part of a fundamental theorem of Shimura's (from [Sh3]); the theorem concerns the arithmetic properties of Fourier coefficients of Hilbert modular forms. I cannot believe that this result was not known until 1975, considering that versions of this result over $\mathbb{Q}$ were known before 1900 via the theory of elliptic functions. (Certainly the other parts of the results of [Sh3], those regarding generation of class fields, could not have been known earlier.) Also, the theorem stated in 6.1 contains an equally important fact concerning comparison of inner products of cuspforms; this fact is indispensable for applications. Section 6.2 recapitulates Klingen's papers [K1] and [K2]; that is, it combines the result of 6.1 over $\mathbb{Q}$ with 4.8's calculation of Fourier coefficients of Eisenstein series to determine the arithmetic nature of special values of certain L-functions of totally real number fields. Sections 6.3–6.5 give a substantial class of examples of Shimura's results on special values of L-functions attached to Hilbert modular cuspforms ([Sh4], [Sh5], and [Sh6]). The results of 6.5 for ground-field $\mathbb{Q}$ were obtained in a different way in [D]. All these special-value results fit into the general conjectural pattern enunciated in [De].

The last chapter is devoted to a new proof of a part of Shimura's theorem (6.1) on Fourier coefficients and to a proof of the comparison results concerning inner products. Sections 7.1–7.8 follow Siegel's idea (in [Si2]) to prove the rationality of Fourier coefficients of some specific Eisenstein series on a symplectic group of rank 2. Such calculations can be done in much greater generality, as in [Sh7], for example, Section 7.9, following the idea of [G1], determines the restriction of this Eisenstein series to an imbedded copy of $SL(2) \times SL(2)$, embodied in the "Main Formula." Section 7.10 combines these two items to obtain the theorem of 6.1.

As complementary and supplementary reading, one might consult the books [Sh2], [Si1], [GGP], [JL], [Ge], and [Gu]. I have omitted many topics and applications, most of which would have the subject matter of this volume as prerequisite: algebraic geometry of Hilbert modular varieties, compactifications of arithmetic quotients, moduli of abelian varieties, generation of class fields, base change, non-holomorphic modular forms, half-integral-weight forms, converse theorems, representation theory, trace formulas, Kronecker limit formulas, congruence properties of modular forms, application to Hasse-Weil zeta functions, Siegel-Weil formulas, theta correspondences, triple-product L-functions, ...

I find it interesting to trace the arguments used here to obtain the rationality results. The arithmetic of Dirichlet L-functions over $\mathbb{Q}$ is quite old; in appendix A.5 the results are proven by residues and by the classical Poisson summation formula. From this result, following Siegel in 7.1–7.8 and using the method of 7.9–7.10, Shimura's theorem of 6.1 over $\mathbb{Q}$ is obtained, as is the comparison of inner products. Then from this result, following Klingen in 6.2, the special-values result for L-functions over totally real number fields is obtained. From this result in turn, 7.1–7.10 yield the theorem of 6.1 over arbitrary totally real number fields. Finally from this result, the results of 6.3–6.5 follow by Shimura's arithmetic application of Rankin's method. I take the viewpoint that all these arguments have the effect of reducing rationality questions to questions about Dirichlet L-functions over $\mathbb{Q}$: no new numbers are approached from first principles; rather, every step is a reduction. Diagramatically:

Dirichlet L-functions over $\mathbb{Q}$ (A.5) Siegel's method (7.1–7.8)
⇘ ⇓
rationality of Fourier coefficients of Siegel's Eisenstein series over $\mathbb{Q}$
⇓
the Main Formula (7.9), methods of 7.10 ⇓
⇘ ⇓
rationality properties for $GL(2, \mathbb{Q})$ (6.1 over $\mathbb{Q}$)
⇓
Eisenstein series Fourier expansion (4.8) ⇓
⇘ ⇓
Klingen's method (6.2) ⇘ ⇓
⇘ ⇘ ⇓
special values of Dirichlet L-functions over totally real number fields (6.2)
⇓
Siegel's method (7.1–7.8) ⇓
⇘ ⇓
rationality of Fourier coefficients of Siegel's Eisenstein series over totally real fields
⇓
the Main Formula (7.9), methods of 7.10 ⇓
⇘ ⇓
rationality properties for $GL(2)$ over totally real fields (6.1)

$$\Downarrow$$

Rankin-Shimura method (4.10, 6.3–6.5) $\quad \searrow \quad \Downarrow$

special values of grossencharacter L-functions over totally imaginary quadratic extensions of totally real number fields,
special values of standard L-functions and convolution L-functions for $GL(2)$

If one views the theory of Hilbert modular forms as but the second example (after elliptic modular forms) of much more general phenomena, then distilling the facts and finding the most economical viewpoint becomes important: a long voyage with heavy baggage is unpleasant. Partly for this reason, I have attempted to be conservative in choosing what to include: almost everything in this book is indispensable for getting to the results of Chapter 6. The new methods of 7.9–7.10 allow all these matters to fit into one volume, most of which is prerequisite for a reasonable understanding of further developments in the subject.

Given the modest prerequisites, this book ought to be accessible to graduate students and non-specialists. I also hope that it will be useful as a handbook of some standard methods in the subject.

I was introduced to this subject through Shimura's lectures at Princeton University from 1975 to 1977. The text is a rewriting of parts of courses and lectures I have given at the University of Minnesota from 1982 to 1988. While almost all these results can be found somewhere in the literature, the precise form of this text is the result of my own reflections upon the subject; therefore, I take responsibility for any imbalances, unwarranted omissions, or misattributions. The bibliography is not aimed at exhaustiveness, either in historical references or in references to current literature; such an attempt would serve no purpose. In any case, my aim has been to make the present treatment sufficiently self-contained so that it can be read fruitfully without too many auxiliary references.

The author would like to thank Professor V. Kumar Murty, University of Toronto, for his helpful suggestions in reviewing the manuscript.

This work was partially supported by a grant from the National Science Foundation.

Paul Garrett

Contents

1 Classical Theory of Hilbert Modular Forms

This first chapter develops most of the significant classical features of holomorphic Hilbert modular forms. Some aspects are different from the theory of elliptic modular forms. For example, Koecher's principle 1.4 holds only for fields other than $\mathbb{Q}$. On the other hand, some results from the theory of Riemann surfaces (for example, the Riemann-Hurwitz formula and the Riemann-Roch theorem) are no longer available in as usable a form, so other methods are required.

The first two sections are mainly definitions, with a few very easy lemmas. The third section is the first illustration of a novelty: The number of cusps "at level one" is the class number of the ring of integers of the field involved. Section 1.4 shows another novelty: Over fields larger than $\mathbb{Q}$, there is no need to impose growth conditions or Fourier coefficient conditions on holomorphic Hilbert modular forms, unlike the case of elliptic modular forms. Section 1.5 returns to some examples which very much resemble the elliptic modular case: holomorphic Eisenstein series of level one.

The next section, on Siegel sets, is a simple example of a very general phenomenon: It is not practical to determine a precise fundamental domain for the action of Hilbert modular groups on products of upper half-planes, but it is possible to determine in qualitatively simple terms a superset of a fundamental domain. Indeed, one may easily find examples of exact fundamental domains which are polyhedral but require a very large number of faces. Section 1.7 then provides an illustration of the fact that this approximation of a fundamental domain suffices for many purposes; the purpose at hand is to give a function-theoretic (rather than algebro-geometric) proof of the finite dimensionality of spaces of cuspforms (due to Siegel and Maass). Along the way we obtain some useful estimates on the asymptotic behavior of cuspforms and their Fourier coefficients. The next section shows that the space spanned by Eisenstein series is a complement (in the space of all Hilbert modular forms) to the space of cuspforms.

The two sections on Dirichlet series and Hecke operators associated to cuspforms are actually only first approximations; in this classical setting there are various complications arising from class numbers and units. These complications may best be eliminated by adopting an adelic viewpoint, which we do later.

To integrate functions on a quotient space, without having an explicit fundamental domain, requires some preparatory consideration of integration theory. This in hand, we compute the volume of the quotients (assuming class number one) in a quite amusing manner (after Siegel). Then the Petersson inner product on spaces of cuspforms can be defined, and properties of the Poincare series considered. In this setting we also have a reproducing kernel for cuspforms of a fixed level and weight (studied by Petersson and Selberg).

1.1 The Hilbert Modular Group

Here we define the Hilbert modular groups and certain subgroups, and examine the elementary aspects of the action of these groups on products of complex upper half-planes.

Let $GL^+(2, \mathbb{R})$ be the collection of elements of $GL(2, \mathbb{R})$ with positive determinant, where $GL(2, \mathbb{R})$ is the group of invertible two-by-two real matrices. The group $GL^+(2, \mathbb{R})$ acts on the upper complex half-plane $\mathfrak{H}$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = (az + b)/(cz + d).$$

The center of $GL^+(2, \mathbb{R})$ acts trivially. It is important to note that

$$\operatorname{Im}(gz) = (\det g)\operatorname{Im}(z)/|cz + d|^2.$$

The isotropy group in

$$SL(2, \mathbb{R}) = \{g \in GL(2, \mathbb{R}): \det g = 1\}$$

of the point $i \in \mathfrak{H}$ is the special orthogonal group

$$SO(2) = \{g \in SL(2, \mathbb{R}): g^T g = 1_2\},$$

and the action of $SL(2, \mathbb{R})$ on $\mathfrak{H}$ is transitive. Letting Z be the center of $GL^+(2, \mathbb{R}$ $SO(2)Z$ is the isotropy group of $i \in \mathfrak{H}$ in $GL^+(2, \mathbb{R})$.

Let F be a totally real number field of degree m over $\mathbb{Q}$, with ring of integers $\mathfrak{o}$. Let $\sigma_1, \ldots, \sigma_m$ be the real imbeddings of F; fix an ordering of them. Via the σ_i we have m imbeddings of $GL(2, F)$ into $GL(2, \mathbb{R})$, componentwise; together, we obtain an imbedding of $GL(2, F)$ into $GL(2, \mathbb{R})^m$. The image of $GL(2, \mathfrak{o})$ in $GL(2, \mathbb{R})^m$ is discrete, since the image of $\mathfrak{o}$ in $\mathbb{R}^m$ (via $\sigma_1, \ldots, \sigma_m$) is discrete, and $GL(2, \mathfrak{o}) \subset M(2, \mathfrak{o}) \approx \mathfrak{o}^4$ is discrete in $M(2, \mathbb{R})^m \approx (\mathbb{R}^m)^4$. Let $GL^+(2, F)$ (respectively, $GL^+(2, \mathfrak{o})$) be the collection of elements of $GL(2, F)$ (respectively, $GL(2, \mathfrak{o})$) with totally positive determinant. We have an action $z \to gz$ of elements g of $GL^+(2, \mathbb{R})^m$ on $z \in \mathfrak{H}^m$ componentwise, and an action of $GL^+(2, \mathfrak{o})$ on $\mathfrak{H}^m$ by our choice of ordering of the real imbeddings of F. This group $GL^+(2, \mathfrak{o})$ is the *full Hilbert modular group* (attached to the field F).

Definition For a non-zero ideal $\mathfrak{n}$ of $\mathfrak{o}$, let

$$\Gamma(\mathfrak{n}) = \{\gamma \in GL^+(2, \mathfrak{o}) : \gamma \equiv 1_2 \text{ modulo } \mathfrak{n}\}.$$

This is the *principal congruence subgroup of level* $\mathfrak{n}$. Let $Z(\mathfrak{o})$ be the center of $GL^+(2, \mathfrak{o})$. Any subgroup Γ of $GL^+(2, F)$ so that $Z(\mathfrak{o})\Gamma$ contains some $\Gamma(\mathfrak{n})$ with *finite index* is a *congruence subgroup* (of $GL^+(2, F)$).

PROPOSITION Let Γ and Γ_2 be congruence subgroups.

i. Then $\Gamma \bigcap \Gamma_2$ is also a congruence subgroup and is of finite index in both Γ and Γ_2.
ii. For $g \in GL(2, F)$, $g\Gamma g^{-1}$ is a congruence subgroup.
iii. $SL(2, \mathfrak{o})$ is a congruence subgroup.
iv. There is a congruence subgroup $\Gamma(\mathfrak{n})$ which is a normal subgroup of $Z(\mathfrak{o})\Gamma$.

Proof All these assertions are easy exercises. ∎

Remarks In the literature, there is an ambiguity about the definition of the principal congruence subgroups and some other special types of congruence subgroups. As we have defined them, they are subgroups of $GL^+(2, \mathfrak{o})$, satisfying some congruence properties. Sometimes they are defined as subgroups of $SL(2, \mathfrak{o})$, satisfying congruence properties. (Since there is no distinction for $F = \mathbb{Q}$, there is no ambiguity for elliptic modular forms). This ambiguity is essentially harmless, except for engendering commensurate ambiguities in certain subsequent definitions.

1.2 Hilbert Modular Forms

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$ and $z \in \mathfrak{h}$, put

$$\mu(g, z) = \det g^{-1/2}(cz + d).$$

For $g = (g_1, \ldots, g_m) \in GL^+(2, \mathbb{R})^m$ and $z = (z_1, \ldots, z_m) \in \mathfrak{h}^m$, and for $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$, we use a standard multi-index notation and write

$$\mu(g, z)^k = \prod_{j=1,\ldots,m} \mu(g_j, z_j)^{k_j}.$$

Definition Let f be a function on $\mathfrak{H}^m$, let $g \in GL^+(2, \mathbb{R})^m$, let $z \in \mathfrak{H}^m$, and let $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$. Then define

$$(f|_k g)(z) = f(g(z))\mu(g, z)^{-k}.$$

Definition Let Γ be a congruence subgroup, $k \in \mathbb{Z}^m$. Let $\chi : \Gamma \to \mathbb{C}^\times$ be a group homomorphism so that $\chi(\Gamma)$ is a finite group. The space of *weak Hilbert modular forms of weight k for* Γ *with character* χ is

$$\mathrm{Wfm}(\Gamma, k, \chi) = \{f \text{ holomorphic function on } \mathfrak{H}^m \colon f|_k\gamma = \chi(\gamma)f \text{ for all } \gamma \in \Gamma\}$$

With χ the trivial character, we write

$$\mathrm{Wfm}(\Gamma, k) = \{f \text{ holomorphic function on } \mathfrak{H}^m \colon f|_k\gamma = f \text{ for all } \gamma \in \Gamma\}.$$

Let

$$\mathrm{Wfm}(k) = \bigcup_{\text{congruence subgroups } \Gamma} \mathrm{Wfm}(\Gamma, k)$$

denote the space of all weak holomorphic Hilbert modular forms of weight k with trivial character.

Remarks Very often, one is most interested in the case that the kernel of a character χ as above is a congruence subgroup Γ'. In that event, one sometimes considers $\mathrm{Wfm}(\Gamma', k)$ and neglects the character χ. However, this is not always appropriate and not always possible.

PROPOSITION Let Γ be a congruence subgroup, and let $\Lambda = \{u \in F \colon \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma\}$. Then any $f \in \mathrm{Wfm}(\Gamma, k)$ has a *Fourier expansion*

$$f(z) = \sum_{\xi \in \Lambda^*} c_\xi \exp(2\pi i \operatorname{Tr}(\xi z)),$$

where Tr is the $\mathbb{C}$-linear extension to $\mathbb{C}^m \to \mathbb{C}$ of the Galois trace $F \to \mathbb{Q}$, and where

$$\Lambda^* = \{u \in F \colon \operatorname{Tr}(u\Lambda) \subset \mathfrak{o}\}$$

is the dual $\mathbb{Z}$-module to Λ. This Fourier series is absolutely convergent, and uniformly so for z in compact subsets of $\mathfrak{H}^m$.

Proof First, note that since Γ contains some $\Gamma(\mathfrak{n})$, Λ contains some non-zero ideal $\mathfrak{n}$. Writing $z = x + iy$ with $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, $f(x + iy)$ is certainly as smooth as one could desire, as a function of $x \in \mathbb{R}^m$ Therefore, as a function of x, f has a Fourier expansion

$$f(x + iy) = \sum_{\xi \in \Lambda^*} c_\xi(y) \exp(2\pi i \operatorname{Tr}(\xi x))$$

which is absolutely convergent, and uniformly so for x in compact subsets of $\mathfrak{H}^m$. It is easy to see that the only exponentials which enter are those given by $\xi \in \Lambda^*$. For f to be holomorphic in $z = (z_1, \ldots, z_m) \in \mathfrak{H}^m$, the Cauchy-Riemann equation

$$i\partial f/\partial x_j = \partial f/\partial y_j$$

must be satisfied for $j = 1, \ldots, m$. That is, for each j, we must have

$$i \sum_{\xi \in \Lambda^*} c_\xi(y)(2\pi i \xi_j) \exp(2\pi i \operatorname{Tr}(\xi x)) = \sum_{\xi \in \Lambda^*} \partial c_\xi(y)/\partial y_j \exp(2\pi i \operatorname{Tr}(\xi x)),$$

where $\xi_j = \sigma_j(\xi)$. By uniqueness of Fourier expansions (in x) we must have

$$-2\pi \xi_j c_\xi(y) = \partial c_\xi(y)/\partial y_j.$$

Therefore, solving the indicated system of differential equations, there is a constant c_ξ so that

$$c_\xi(y) = c_\xi \exp(-2\pi \operatorname{Tr}(\xi y)).$$

Finally, to see that the Fourier series is absolutely convergent and uniformly so in compacta in $\mathfrak{H}^m$, we observe that as any component y_j increases, $\exp(-2\pi \operatorname{Tr}(\xi y))$ does not increase. As we have already ascertained the uniform absolute convergence in x, this gives the result. ∎

Definition Let $f \in \mathrm{Wfm}(\Gamma, k)$. Then say that f is a *holomorphic Hilbert modular form of weight k with respect to* Γ if, for every $g \in GL^+(2, F)$, the Fourier expansion

$$(f|_k g)(z) = \sum_\xi c_\xi(g) \exp(2\pi i \operatorname{Tr}(\xi z))$$

has $c_\xi(g) = 0$ unless $\xi = 0$ or ξ is totally positive. Let $\mathrm{Mfm}(\Gamma, k)$ be the $\mathbb{C}$-vectorspace of holomorphic Hilbert modular forms of weight k with respect to Γ; let

$$\mathrm{Mfm}(k) = \bigcup_{\text{congruence subgroups } \Gamma} \mathrm{Mfm}(\Gamma, k)$$

denote the space of all *holomorphic Hilbert modular forms of weight k*.

Definition A function $f \in \mathrm{Mfm}(\Gamma, k)$ is a holomorphic Hilbert modular *cuspform* of weight k with respect to Γ if, for every $g \in GL^+(2, F)$, the Fourier expansion

$$(f|_k g)(z) = \sum_\xi c_\xi(g) \exp(2\pi i \operatorname{Tr}(\xi z))$$

has $c_\xi(g) = 0$ unless ξ is totally positive. Let $\mathrm{Cfm}(\Gamma, k)$ be the $\mathbb{C}$-vectorspace of holomorphic Hilbert modular cuspforms of weight k with respect to Γ; let

$$\mathrm{Cfm}(k) = \bigcup_{\text{congruence subgroups } \Gamma} \mathrm{Cfm}(\Gamma, k)$$

denote the space of all *holomorphic Hilbert modular cuspforms of weight k*.

PROPOSITION Let Γ be a congruence subgroup, and let $g \in GL^+(2, F)$. Take f in $\mathrm{Wfm}(\Gamma, k)$ (respectively, in $\mathrm{Mfm}(\Gamma, k)$, $\mathrm{Cfm}(\Gamma, k)$). Then $f|_k g$ is in $\mathrm{Wfm}(g^{-1}\Gamma g, k)$ (re-