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Preface

This volumes contains the proceedings of the International Symposium held in Kazimierz Dolny in Spetember 1991. Like the previous seminars organized by Technical University of Radom every two years from 1979, it was devoted to discussions of the results and problems in such fields as: several complex variables, Riemannian and Hermitian geometry, spectral theory in Hilbert space, probability and applications in the mathematical physics. Particular consideration was given to the interrelation of ideas from different areas.
The present volume is the third one in the series. The previous two "Prace Matematyczno-Fizyczne" 1982 and 1988 were published by RTU.

Tomasz Mazur

CONTENTS

On factorization of noncommutative substitutionary matrices with not prime order on complex plane

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ABSTRACT

The main purpose of this paper is the research of connections between the vector boundary Riemann problems with the substitutionary matrices on complex plane and on algebraic surface which covers it. The necessary and sufficient conditions of constructive factorization for noncommutative substitutionary matrix on complex plane are received.

1. INTRODUCTION. During the last years some works 1] - 5], which were dedicated to factorization of substitutionary matrices, have appeared. From the very beginning we must note that the substitutionary matrix Ω of order m is such object which is raised by corresponding substitution $\omega_m = \begin{pmatrix} 1 & \dots & m \\ j_1 & \dots & j_m \end{pmatrix} = (j_1 \dots j_m)$. In matrix Ω all elements in line k are zeroes except the one which is situated in column j_k and is equal to one, ($j_k = \overline{1, m}$, $k = \overline{1, m}$).

The matrices from works 1] - 5] were noncommutative only in case of order $m < 5$.

Here we must add that in this paper the noncommutative matrix means such one that has no linear conversion which is constant everywhere on given curve and transforms this matrix to the diagonal form 6].

So, after formulation above mentioned definition, it becomes quite obvious that the constructive factorization of noncommutative matrix Ω on complex plane $\mathbb{C}$ has no principal difficulties, because it is equivalent to solution of m corresponding scalar boundary Riemann problems. The methods of solution of scalar boundary Riemann problem, either on complex plane $\mathbb{C}$ or on compact Riemann surface, are well known 7], 2], 8].

Hence, the most interesting problem in matrices' factorization is the constructive factorization of noncommutative matrix. The methods of above mentioned works 1] – 5] in case of noncommutative substitutionary matrix consist of scalar boundary Riemann problem's solution on surface, which covers $\mathbb{C}$ and which is constructed according to the initial substitutionary matrix in special way 3], 4]. The constructive solution for scalar problems of this type is rather difficult in ideological and in calculating meanings as well. These difficulties appear already in case of three- and four-leafed covering of $\mathbb{C}$ 3], 4]. According to mentioned method in case of "more"-leafed covering of $\mathbb{C}$, all problems and even the way of solution for constructive factorization of noncommutative substitutionary matrix are unobserved at all.

In this paper such a new approach is offered: the constructive factorization of noncommutative substitutionary matrix with order m on $\mathbb{C}$ becomes attainable because of constructive factorization of corresponding commutative substitutionary matrix with order $n < m$ on compact Riemann surface R which covers $\mathbb{C}$.

At the end of this paragraph we must note that the interest to substitutionary matrices' constructive factorization is connected not only with the researches of new types of vector boundary Riemann problems, but with numerous questions in algebraic functions' theory and in theoretical physics as well, i.e.: the construction of algebraic equation for Riemann surface, the building of algebraic functions' field, etc. 9] - 11]. Such factorization is also the basis for constructive factorization of substitutionary matrix-function (m.-f.). Substitutionary m.-f. means usual substitutionary matrix where instead of all units the functions from Helder class are situated.

2. PRELIMINARIES. By open curve on compact Riemann surface $\mathcal{R}$ we imply open smooth Jordan curve without intersections on $\mathcal{R}$ and which is homeomorphal to the interval (0,1) of the numerical axis.

Then the class $h_0(\mathcal{L};\mathcal{R})$ must be defined. It is the class of vector-functions which are analytical everywhere on surface $\mathcal{R}$ except the open curve $\mathcal{L}$. These functions are H-continued from left and right on $\mathcal{L}$ and are restricted on the edges of $\mathcal{L}$. Also they assume the finite order on infinity.

We say that function $\varphi(\xi), \xi \in \mathcal{R}$, is divisible by divisor $\mathcal{D}$ if $(\varphi):\mathcal{D}$ is the integer divisor. Here (φ) is the divisor which consists of zeroes and infinities of function φ. The notation of this definition looks like: $\mathcal{D} \mid (\varphi)$.

Now the canonical solution matrix (c.s.m.) on compact Riemann surface $\mathcal{R}$ must be defined. This definition is analogous to well known definition of c.s.m. on complex plane $\mathbb{C}$ 12]. So, we say that matrix $X(\xi)$, $\xi \in \mathcal{R}$, is the c.s.m. of boundary vector Riemann problem with matrix-coefficient's dimension m on surface $\mathcal{R}$, if: a) $X(\xi)$ satisfies the given boundary condition; b) $\det X(\xi)$ has no poles anywhere in finite part of $\mathcal{R}$ and may be equal to zero only on the edges of curve $\mathcal{L}$. c) The orders r_j, $(j=\overline{1,m})$, of columns in matrix $X(\xi)$ on infinity can't be lessened.

The new property c') follows from property c), and it sounds as: matrix $\mathcal{X}(\xi)$ satisfies the above mentioned conditions a) and b). Then $\mathcal{X}(\xi)$ is c.s.m. of given boundary vector Riemann problem on $\mathcal{R}$, if $\operatorname*{ord}_{\infty} \det \mathcal{X}(\xi) = \sum_{j=1}^{m} r_j$.

Matrix $\mathcal{X}(\xi)$, $\xi \in \mathcal{R}$, is normal solution matrix (n.s.m.) of boundary vector Riemann problem on $\mathcal{R}$, if $\mathcal{X}(\xi)$ satisfies conditions a), b) and doesn't satisfy the condition c), i.e. $\operatorname*{ord}_{\infty} \det \mathcal{X}(\xi) < \sum_{j=1}^{m} r_j$.

Hence, the following proposition exists: the general solution of vector boundary Riemann problem on compact Riemann surface $\mathcal{R}$ may be expressed with the help of c.s.m. by formulas which are analogous to corresponding formulas on complex plane.

The partial indexes $\varkappa_j$, $(j=\overline{1,m})$, of given vector boundary Riemann problem are defined in such a way: $\varkappa_j = -r_j$, $(j=\overline{1,m})$, and they are invariant concerning the method of c.s.m.'s construction.

Here we must underline that the effective solution of homogeneous vector boundary Riemann problem with matrix-coefficient Ω on open curve $\mathcal{L} \in \mathcal{R}$ is equivalent to constructive factorization of matrix Ω, i.e. matrix-coefficient Ω must be represented in such a way:

$\Omega(\eta) = \mathcal{X}^{+}(\eta)\left(\mathcal{X}^{-}(\eta)\right)^{-1}$. Here $\mathcal{X}^{\pm}(\eta)$, $\eta \in \mathcal{L}$, are limited values of m.-f. $\mathcal{X}(\xi)$, $\xi \in \mathcal{R}$, from left and right on $\mathcal{L}$ accordingly. $\mathcal{X}(\xi)$ is analytical everywhere on $\mathcal{R}$ except the curve $\mathcal{L}$ and its evident representation is known. The dimension of $\mathcal{X}(\xi)$ is equal to m. It must be added that the effective solution of homogeneous vector boundary Riemann problem means the constructive building of its c.s.m. and calculation of its partial indexes.

So, when we say about constructive factorization of matrix Ω, we imply the effective solution of corresponding vector boundary homogeneous Riemann problem with matrix-coefficient Ω , and vice versa.

The last agreement in this paragraph concerns the definition of monodromy group. When we mention it, we imply the substitutions which are fixed on corresponding edges of given open curves 13].

3. PROBLEM'S FORMULATION. Let R be compact Riemann surface with genus $\rho>0$, and its algebraic equation looks like

$$F(z,u)=u^n+\hat{\tau}_1(z)u^{n-1}+\ldots+\hat{\tau}_{n-1}(z)u+\hat{\tau}_n(z)=0, \tag{1}$$

where $\hat{\tau}_j(z),(j=\overline{1,n})$, are polynomials over the field of complex numbers, and (z,u) is considered as point on R. The open curve on R is fixed:

$$L=\bigcup_{k=1}^{M}L_k,\quad L_i\cap L_j=\emptyset \text{ when } i\neq j, \tag{2}$$

where $L_k=(\alpha_k, v(\alpha_k), \beta_k, v(\beta_k))$,$(k=\overline{1,M})$, is open curve on R , and $F(t,v)=0$. The vector-function

$$\Phi(z,u)=\{\Phi_1(z,u),\ldots,\Phi_n(z,u)\}\in h_0(L;R),(z,u)\in R,$$

which satisfies the following boundary condition

$$\Phi^+(t,v)=\Omega(t,v)\Phi^-(t,v),(t,v)\in L,\ D^{-1}|(\Phi), \tag{3}$$

is sought.

Here: $D=\{(\infty,\bigcup_{j=1}^{n}\infty_j)^{\sigma},\ldots,(\infty,\bigcup_{j=1}^{n}\infty_j)^{\sigma}\}$

is m-dimensional vector-divisor of infinities on R ;$\sigma\geq 0$ is the integer number; $\Omega(t,v)$is m-dimensional substitutionary matrix which represents such monodromy group on R

$$\omega_k=(i_1^{(k)}\ldots i_m^{(k)}),\ L_k,\ (k=\overline{1,M}). \tag{4}$$

The structure of matrix-coefficient $\Omega(t,v)$looks like

$$\Omega(t,v)=\sum_{k=1}^{M}\Omega_k\,\delta(t,v;L_k),\quad (t,v)\in L_k, \tag{5}$$
$$\delta(t,v;L_k)=\begin{cases}1,(t,v)\in L_k\\ 0,(t,v)\in L\setminus L_k\end{cases},(k=\overline{1,M}),$$

where Ω_K is matrix-representation of substitution ω_K, $(K=\overline{1,M})$, from system (4). Besides, the monodromy group (4) is considered as the commutative one. Hence, its matrix-representation $\Omega(t,v)$ is the commutative substitutionary matrix too.

Then, let S be the linear conversion which is constant everywhere on L and which transforms the matrix-coefficient (5) to diagonal form T^*. Applying conversion S to boundary condition (3), we receive m scalar boundary Riemann problems on R which are solved effectively 7]. For their constructive solution we must find the normal basis of surface R.

It is quite possible. From algebraic equation (1) follows that R is n -leafed covering of $\mathbb{C}$, its monodromy group is defined by equation (1), and vice versa. Then, let the monodromy group of R has such structure

$$g_\ell=(j_1^{(\ell)}\dots j_n^{(\ell)}),\ \Gamma_\ell,\ (\ell=\overline{1,N}), \tag{6}$$

where: g_ℓ, $(\ell=\overline{1,N})$, is the system of commutative substitutions;

$$\Gamma=\bigcup_{\ell=1}^{N}\Gamma_\ell,\ \Gamma_j\cap\Gamma_i=\emptyset \text{ when } i\neq j, \tag{7}$$

and $\Gamma_\ell=(a_\ell,b_\ell)$, $(\ell=\overline{1,N})$, is open curve on $\mathbb{C}$.

The constructive building of R's normal basis is realized either because of application of Newton's diagram 13] to algebraic equation (1), or because the corresponding vector boundary Riemann problem is solved effectively. Rather often the Newton's diagram works uneffectively, i.e. instead of normal basis we receive the fundamental one (look the examples from 13]).

So, let's consider the corresponding vector boundary Riemann problem in class $h_0(\Gamma,\mathbb{C})$ with such boundary condition

$$\Psi^+(t)=G(t)\Psi^-(t),\ t\in\Gamma. \tag{8}$$

Here: $G(t)$ is n -dimensional substitutionary matrix with such structure

$$G(t)=\sum_{l=1}^{N} G_l \delta(t;\Gamma_l),\ t\in\Gamma_l;\ \delta(t;\Gamma_l)=\begin{cases}1, & t\in\Gamma_l\\ 0, & t\in\Gamma\setminus\Gamma_l\end{cases},\quad (l=\overline{1,N}); \tag{9}$$

G_l, $(l=\overline{1,N})$, is matrix-representation of substitution g_l, $(l=\overline{1,N})$, from monodromy group (6) - (7);

$\Psi(z)=\{\Psi_1(z),\dots,\Psi_n(z)\}\in h_0(\Gamma;\mathbb{C})$, $z\in\mathbb{C}$, is unknown n -dimensional vector-function.

The commutativity of substitutions (6) raises the commutativity of matrices G_l, $(l=\overline{1,N})$. So, let Δ be the linear conversion which is constant everywhere on Γ and which transforms the coefficient (9) to diagonal form Λ. The solutions of n corresponding scalar boundary Riemann problems with piece-constant coefficients on $\mathbb{C}$ 8]

$$\{u_1(z),\dots,u_{n-1}(z),1\} \tag{10}$$

form the sought for normal basis of R, and functions $u_j=u_j(z)$, $(j=\overline{1,n-1})$, are n -leafed over $\mathbb{C}$ 13]. After inverse conversion of Δ is applied to c.s.m. of above mentioned boundary value problem with diagonal coefficient Λ, the c.s.m. of problem (6) - (9) appears

$$X(z)=\begin{bmatrix} u_1^{(q_1)} & \dots & u_{n-1}^{(q_1)} & 1\\ \vdots & & \vdots & \vdots\\ u_1^{(q_n)} & \dots & u_{n-1}^{(q_n)} & 1\end{bmatrix}, \tag{11}$$

where $u_j^{(q_c)}$, $(j=\overline{1,n-1};\ q_c=\overline{1,n};\ c=\overline{1,n})$, is leaf q_c of function u_j which is n -leafed over $\mathbb{C}$. The columns in (11) are situated in such a way that their orders on infinity don't increase from left to right. We must note, that the leaves' distribution of functions (10) in c.s.m. (11)

is realized automatically, because of application of direct and inverse λ -conversion (look, for example, 2]).

Now, let the functions

$$\left\{ \Phi_{11}(z,u), \dots, \Phi_{1,m-1}(z,u), 1 \right\} \tag{12}$$

be the solutions of m corresponding scalar boundary Riemann problems which were received earlier from diagonal matrix-coefficient T^*. These scalar Riemann problems have piece-constant coefficients on surface R 7], and they were received after linear conversion S was applied to boundary condition (3). Functions (12) are m -leafed over R and mn -leafed over $\mathbb{C}$.

After inverse conversion of S was applied to c.s.m. of mentioned earlier boundary value problem with diagonal coefficient T^*, the c.s.m. of problem (1) - (5) on surface R appears

$$X_1(z,u) = \begin{bmatrix} \Phi_{11}^{(p_1)}(z,u) & \dots & \Phi_{1,m-1}^{(p_1)}(z,u) & 1 \\ \vdots & & \vdots & \vdots \\ \Phi_{11}^{(p_m)}(z,u) & \dots & \Phi_{1,m-1}^{(p_m)}(z,u) & 1 \end{bmatrix}, \tag{13}$$

where $\Phi_{1i}^{(p_d)}$, $(i=\overline{1,m-1};\ p_d=\overline{1,m};\ d=\overline{1,m})$, is leaf p_d of function $\Phi_{1i}=\Phi_{1i}(z,u)$ from system (12). The columns in (13) are situated in such a way that their orders on infinity don't increase from left to right. The leaves' distribution of functions (12) in c.s.m. (13) is realized automatically because of application of direct and inverse S -conversion (look, for example, 2]).

Here we must note, that in process of (1) - (5) problem's solution, simultaneously with c.s.m. (13), $n-1$ n.s.m. of problem (1) - (5), were received:

$$X_j(z,u) = \begin{bmatrix} \Phi_{j1}^{(p_1)}(z,u) & \dots & \Phi_{j,m-1}^{(p_1)}(z,u) & 1 \\ \vdots & & \vdots & \vdots \\ \Phi_{j1}^{(p_m)}(z,u) & \dots & \Phi_{j,m-1}^{(p_m)}(z,u) & 1 \end{bmatrix}, (j=\overline{2,n}). \tag{14}$$

Here functions $\Phi_{ji} = \Phi_{ji}(z,u), (j=\overline{2,n}; i=\overline{1,m-1})$, have the same properties as functions (12), except the one: their orders on infinity are bigger than the orders on infinity of corresponding functions from (12), $-\ \underset{\infty}{ord}\,\Phi_{ji} > \underset{\infty}{ord}\,\Phi_{1i}$, $(i=\overline{1,m-1}; j=\overline{2,n})$.

4. MAIN RESULTS. Now we must find out, what the matrix boundary Riemann problem with noncommutative substitutive coefficient on $\mathbb{C}$ was raised by matrix Riemann problem on R (1) - (5).

It's well known 13] that the pasting together of m copies of surface R according to substitutions (4) reduces to oriented closed Riemann surface $\widetilde{\mathcal{R}}$ which is m-leafed covering of R. This surface $\widetilde{\mathcal{R}}$ may be considered also as mn-leafed covering $\mathcal{R}^*$ of complex plane $\mathbb{C}$, because R is n-leafed over $\mathbb{C}$ (1).

Hence, naturally the following problem appears: knowing the monodromy group of surface $\widetilde{\mathcal{R}}$, to build the monodromy group of surface $\mathcal{R}^*$. In this case the monodromy group of $\widetilde{\mathcal{R}}$ may be considered as substitutions (4) which are fixed on the corresponding edges of curves (2). The edges of curves (2) are the $\widetilde{\mathcal{R}}$'s points of ramification. About the unknown monodromy group of surface $\mathcal{R}^*$ we may say beforehand only that substitutions' order is equal to mn, corresponding open curves are situated on $\mathbb{C}$ and their edges are the R's points of ramification.

Looking at this problem from aspect of monodromy group's matrix-representation, we may assert that the effective solution for this problem means the following process: building the surface $\widetilde{\mathcal{R}}$, simultaneously we made the constructive factorization of commutative matrix (5), i.e. the c.s.m. (13) of problem (1) - (5) was constructed. Then we considered the set of vectors which realized this factorization, and some special vectors from this set were chosen. These special vectors accomplish the factorization of noncommutative matrix-coefficient which represents the monodromy group of surface $\mathcal{R}^*$.

This problem is solved because the behaviour of (1) - (5) problem's solution $\Phi(z,u)$ was investigated in neighbourhood of critical points $z=a_\ell$, $z=b_\ell$, $(\ell=\overline{1,N})$, and $z=\alpha_k$, $z=\beta_k$, $(k=\overline{1,M})$, (look (6) - (7) and (2), (4) correspondingly). Taking into consideration the structures of monodromy groups (4), (6) - (7), results and methods of works 13] - 15], we receive the unknown substitutions:

$z=a_\ell$, $z=b_\ell$, $(\ell=\overline{1,N})$:

$$(j_1^{(\ell)} \dots j_n^{(\ell)})(j_1^{(\ell)}+n \dots j_n^{(\ell)}+n)\dots(j_1^{(\ell)}+(m-1)n \dots j_n^{(\ell)}+(m-1)n), \tag{15}$$

$$\Gamma^* = \bigcup_{\ell=1}^{N} \Gamma_\ell^*, \quad \Gamma_\ell^* = (a_\ell, b_\ell),$$

$z=\alpha_k$, $z=\beta_k$, $(k=\overline{1,M})$:

$$(i_1^{(k)}n-\tau_1^{(k)}\; i_2^{(k)}n-\tau_2^{(k)} \dots i_m^{(k)}n-\tau_m^{(k)})(1)\dots(i_1^{(k)}n-\tau_1^{(k)}-1)(i_1^{(k)}n-\tau_1^{(k)}+1)\dots(i_m^{(k)}n-\tau_m^{(k)}-1)(i_m^{(k)}n-\tau_m^{(k)}+1)\dots(mn), \tag{16}$$

$$L^* = \bigcup_{k=1}^{M} L_k^*, \quad L_k^* = (\alpha_k, \beta_k),$$

$$(\tau_\nu^{(k)} = \overline{0, n-1}, \; \nu = \overline{1,m}).$$

Non-commutativity of substitutions (15), (16) is checked up directly.

The surfaces $\widetilde{\mathcal{R}}$ and $\mathcal{R}^*$ are topologically equivalent, they have the same genus

$$\rho^* = m(\rho-1) + 1 + \frac{1}{2}V, \tag{17}$$

where V is index of ramification (branching) 13] , 16] of surface $\widetilde{\mathcal{R}}$. Formula (17) is received according to 16].

So, we have received such

Theorem 1. Let the compact Riemann surface R with ge-