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MATHEMATICAL AND CONCEPTUAL FOUNDATIONS OF 20TH-CENTURY PHYSICS

Gérard G. EMCH

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PREFACE

This book is primarily intended for Mathematicians with no, or little, background in Physics; much emphasis is laid on the interlocked historical development of mathematical and physical ideas.

For didactic purposes, the book is divided in three parts. Parts II & III can be covered in one semester each, as part of an introductory survey course on Applied Mathematics in the Graduate mathematics curriculum. Part I is elementary enough to be assigned for independent readings. The mathematical level of discourse should present no problem for a beginning graduate student in an American mathematics department. If this student were to consider the book for self-study, (s)he might find useful to have had, or to take concurrently, one-semester introductory courses in Functional Analysis and in Differential Geometry: these are two amongst the main tools of today's Mathematical Physics; they, however, are not prerequisites: all terms appearing in the text, mathematical as well as physical, are defined. What is required from the reader is the curiosity and the breadth necessary to undertake repeated crossings of the bridges which exist between Mathematics and Physics, two disciplines traditionally considered as closely allied fields; (s)he will follow in this journey the steps of such luminaries as Archimedes, Fermat, Newton, Euler, the Bernoullis, Lagrange, Laplace, Fourier, Cauchy, Gauss, Poincaré, Cartan, Hilbert, Weyl and von Neumann, not to mention the names of many Mathematicians or Mathematical Physicists now active.

For the mathematical audience thus defined, this book will present the main ideas and fundamental concepts of 20th-century Physics, with special attention to the concurrent mathematical developments. This century has indeed been marked by two conceptual revolutions from which Mathematics drew considerable impetus: the theory of Relativity and Quantum theory, both owing much to the seminal ideas of Einstein, one of the great geniuses of all times.

As no intellectual revolution can be properly understood without some knowledge of the paradigms prevalent at the time of its inception, Part I provides a survey of Classical Physics, which we divided in three chapters: Mechanics, Thermodynamics and Statistical Mechanics, and Electromagnetism. This study provides opportunities to place in perspective the successive advents of Calculus, of Probability and Statistics, of Differential and Symplectic Geometry, and of classical Functional Analysis.

Relativity is presented in Part II of this book, and Quantum Theory in Part III. The motivation provided by physical problems in the development of mathematical disciplines such as, for instance, pseudo-Riemannian Geometries, Hilbert Spaces and Operator Algebras, are emphasized.

Aside from the primary aim of this book, which is to present a unified mathematical account of the conceptual foundations of 20th-century Physics, under a single cover and in a form suitable for use in a survey course in Applied Mathematics, it is hoped that the book will also serve another function, namely that various parts of the work will be excerpted, and incorporated in separate courses pertaining to the Pure Mathematics curriculum, to provide illustrative examples, further motivations, and testimony to the unity of the Mathematical Sciences.

Finally, the author hopes that this book will help mathematicians broaden their exchanges with physicists, and with philosophers and historians of science.

Acknowledgements

Some long time ago, Leopoldo Nachbin impressed upon me that I should take some "rest", and write this book. I have had the advantage of being able to discuss both text and tune with him over the several years I have tested a patience that, by now, must be proverbial. This one-hundredth volume of the *Notas de Mathemática* is a testimony to his perseverance.

The materials presented here have been taught in courses I gave at the University of Rochester; the Ecole Polytechnique Fédérale à Lausanne; the Universität Tübingen; the Universidade do São Paulo; the Université de Paris VII; and the Virginia Polytechnic Institute and State University. I wish to acknowledge with thanks the stimulating comments I received from my colleagues at these Institutions: William Eberlein, Nicolas Gisin, Christian Günther, Lawrence Helfer, Richard Lavine and Malcolm Savedoff; Philippe Choquard and Philippe Martin; Peter Kramer, Burckhard Kümmerer, Alfred Rieckers and Wolfgang Schröder; Moysés Nussenzveig and Kalyan Sinha; Jacqueline Bertrand, Raymond Jancel and Guy Rideau; George Hagedorn, Gerhard Hegerfeldt and Paul Zweifel.

I would like to put on record the generosity of the leave of absence policy of the University of Rochester which also allowed me to spend sizable stretches of time with colleagues at the Zentrum für interdisziplinäre Forschung der Universität Bielefeld; the Université de Genève; Harvard University; the Mathematical Centre at the University of Warwick; the Centre de Physique théorique du CNRS à Marseille; and the University of Pennsylvania. For arranging the other head of these bridges, I am grateful to Ludwig Streit; Jean-Pierre Eckmann and Constantin Piron; George Mackey and Shlomo Sternberg; Klaus Schmidt; Daniel Kastler and Madeleine Sirugue-Colin; and Dick Kadison.

The always supportive attitude of Antoinette Emch-Dériaz over the twenty-five years of our association took a new turn in the present venture; sometimes at the cost of delays in her own historical researches, she tried to inform me away from the naturally whiggish views scientists tend to have on matters of intellectual history. For one, I appreciate her unrelenting efforts, even if the reader will soon realize that I still have a long way to tread on the road to a proper historiographical method. If anything is right here on that account, it is probably hers.

My students Stephan De Bièvre, Sung-Pyo Hong and Orietta Protti have responded with juvenile enthusiasm to the suggestion that they read first drafts of this book; they have helped clear many ambiguities.

Joan Robinson produced the work on $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX with extraordinary celerity, more than professional conscience, and remarkable resources of ingenuity. Arnie Pizer, the expert TEX tamer on our Faculty, saw to it that the beast would roar in harmony with a true mathematician's taste.

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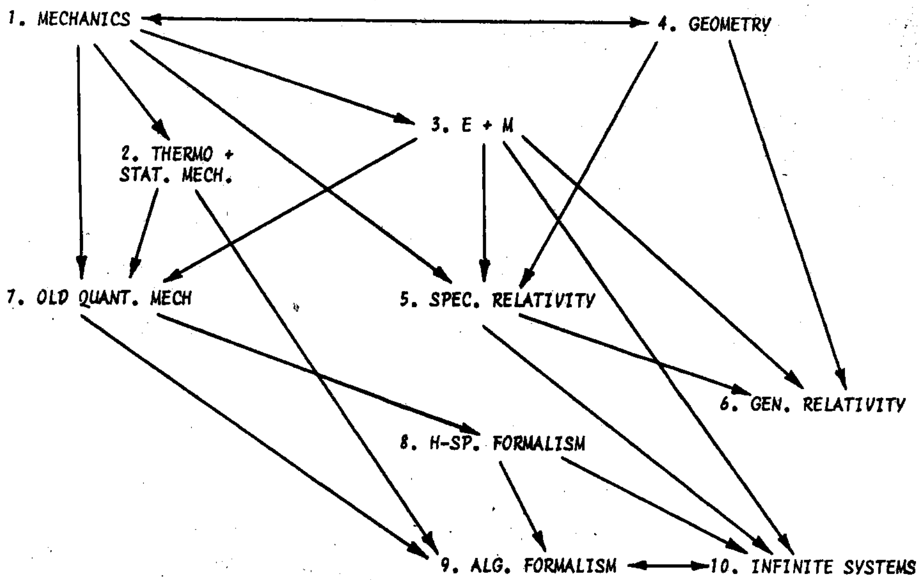
PART III. QUANTUM THEORY

Chapter Interdependence



The arrows indicate logical dependences. As much as possible in two dimensions, the lay-out has been organized in such a manner that, along most paths, the mathematical analytic sophistication increases from left to right, while the physical synthetic conceptualization increases from top to bottom of the diagramme.

CHAPTER 1. MECHANICS

SYNOPSIS

There is a good deal of truth in A. N. Whitehead's caveat: "... in a memoir one's whole trouble is with the first chapter, or even the first page. For it is there, at the very outset, where the author will probably be found to slip in his assumptions. Further, the trouble is not with what the author does say, but with what he does not say. Also, it is not with what he knows he has assumed, but with what he has unconsciously assumed". Still, the author must make the plunge somewhere, swim as he may in whatever murky currents he encounters, reserving all the while his right to come back, stir and plumb for new depths those calm expanses he once thought had been properly charted. We make our plunge in classical mechanics.

By the turn of the Twentieth Century, Hamilton mechanics was the widely accepted form of mechanics, the one to be used when a correspondence was to be found with the new quantum mechanics. For instance, the first formula on the first page of the first chapter—"Principes de la dynamique"—of Poincaré's *Leçons de mécanique céleste* (1905) is the system of first order differential equations of Hamilton, which we write:

$$(1) \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i} \quad ; \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i} \quad i = 1, 2, \ldots, d$$

The system of equations reflects the presence of a mathematically interesting underlying structure, namely that of a 2d-dimensional symplectic manifold, i.e. a differentiable manifold $\mathbf{M}$ endowed with a symplectic form ω which, by Darboux theorem, can be written (at least locally):

$$(2) \qquad \omega = \sum_{i=1}^{d} dp^i \wedge dq^i$$

In many applications $\mathbf{M}$ is T^*M, the cotangent bundle of a manifold M, called the "configuration space"; and ω is the canonical form $\omega = d\theta$ [where θ is the 1-form defined, for every $p \in T^*M$ and every $\xi \in T(T^*M)_p$, by $\theta(\xi) = p(\pi_*\xi)$ with π the projection $\pi : T^*M \to M$]. When $M = \mathbb{R}^{3N}$, $\mathbf{M}$ can simply be identified with the usual "phase space" $\mathbb{R}^{6N} = \{(q^1, \ldots q^{3N}; p^1, \ldots p^{3N}\}$; the general form is nevertheless necessary, as soon as holonomic constraints are considered.

To every smooth (e.g. C^∞ —) function F on M one can then associate the vector field ξ_F defined by

(3) $$\xi_F \rfloor \omega = -dF$$

from which one defines in turn the Poisson bracket

(4a) $$\{F, G\} = -\omega(\xi_F, \xi_G)\,;$$

in local coordinates this becomes

(4b) $$\{F, G\} = \sum_{i=1}^{d} \frac{\partial F}{\partial q^i} \cdot \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \cdot \frac{\partial G}{\partial q^i}$$

Note that one obtains for the coordinate functions q^i and p^j:

(4c) $$\left.\begin{aligned} \{q^i, q^j\} = 0 = \{p^i, p^j\} \\ \{p^i, q^j\} = -\delta^{i,j} 1 \end{aligned}\right\} \quad \forall\, i, j = 1, \ldots, d$$

Note further that the Hamilton equations (1) now appear as the equations for the integral curves of the vector field ξ_H associated to a special function H, the Hamilton function. In the particular case of a system of N particles moving freely in the ordinary one-particle configuration space $\mathbb{R}^3$, except for their mutual (velocity independent) interactions, H takes the form

(5a) $$H = T + V$$

with

(5b) $$T = \sum_{n=1}^{N} \frac{1}{2m_n} |\mathbf{p}_n|^2$$

(5c) $$V = V(\mathbf{q}_1, \ldots, \mathbf{q}_N)$$

In this case, Equations (1) reduce to:

(6a) $$\dot{q}_n^i = \frac{1}{m_n} p_n^i \quad ; \quad \dot{p}_n^i = -\frac{\partial V}{\partial q_n^i}$$

so that

(6b) $$\mathbf{F}_n = m_n \ddot{\mathbf{q}}_n \quad \text{with} \quad F_n{}^i = -\frac{\partial V}{\partial q_n^i}$$

To understand the facts of experience lying behind the interpretation of $\mathbf{F}_n$ as the "force" acting on the n-th particle, m_n as its "mass", $\mathbf{p}_n$ as its "momentum"; of T as the "kinetic energy" of the system, V as its "potential energy", and H as its "total energy", it is helpful to back up in history to what should be called Euler's mechanics. It is traditional and convenient to divide classical mechanics into three parts, the first being associated to Newton's *Philosophiae Naturalis Principia Mathematica* (1687), the second to Lagrange's *Mécanique Analytique* (1788), and the third to Hamilton's *General Method in Dynamics* (1834, 1835) and Jacobi's *Vorlesungen über Dynamics* (published by Clebsch in 1866).

Our three sections' headings perforce have to use this discrete nomenclature—Newtonian, Lagrangean and Hamiltonian mechanics—which artificially parcels out a continuous development starting before Galileo and continuing after Poincaré. We will also allude to some of the problems of current interest: however "classical" it may be called, mechanics is still a field of active research where new mathematics and, thus, new physical understandings are in the process of being developed.

Section 1. NEWTONIAN FORMULATION

Mechanics is a science of experience, an experience apprehended by observation, controlled by experiment, and comprehended by theory.

One of the established sources of experience that played an important role in Newton's formulation of rational mechanics and of universal gravitation was observational astronomy, culminating in Kepler's three laws of planetary motion, published in Prague, the first two in his *Astronomia Nova* (1609) and the third in his *De Harmonice Mundi* (1619). These laws are: (1) *each planet moves in an elliptical orbit, with the sun at one focus of the ellipse;* (2) *the focal radius from the sum to a planet sweeps equal areas of space in equal intervals of time;* (3) *the square of the sideral periods of the planets are proportional to the cube of their mean distance to the sun,* a statement which we transcribe as:

$$A^3 = k\,T^2 \tag{1}$$

where T is the period of the planet on its orbit, and A is the semi-major axis of its elliptical orbit, i.e. the average of the distances between the sun and the planet at its aphelion and perihelion; as we shall see below, the proportionality factor k can be computed in term of the gravitational constant of Newton's theory of universal gravitation.

The use of controlled experiments was persuasively advocated by Galileo in his *Discorsi e Dimostrazioni Matematiche intorno à due nuove Scienze attenti alla Mecanica & i Movimenti Locali* (1638). Whether Galileo was faithfully reporting the results of experiments he had himself actually performed is not terribly relevant, although we should mention that doubts were raised on that account, already by some of his contemporaries, e.g. Mersenne in France. The main point we want to make here is that Galileo was eloquently expounding, as a reflection of contemporary concerns, that the primary role of controlled experiments is to discriminate between opposing theories. Hence, theory has to come first, to organize observational experience and to inform the sensible planning of experiments.

The rational mechanics of the 17th century, however, did not happen as a sudden reaction to some hitherto unchallenged misconceptions inherited from antiquity; it was instead slowly brought into focus through the perceptive work of the medieval critics of Greek science. These now obscure scholars started the programme that would eventually replace the qualitative notions, that emerge from common experience of motion, by qualitative statements on: the geometric description of motion (*kinematics*); the analysis of its causes or manifestations (*dynamics*); the notions of force, torque, pressure, stress and their effects; the concepts of mass and inertia; the derived quantities we now call linear and angular momentum, kinetic and potential energy, work, power and action.

One of the achievements of medieval science was the logical distinction between the computational geometric questions of kinematics and the conceptually more involved problems of dynamics. Between 1328 and 1350, the scholars of Merton College in Oxford succeeded indeed in formulating clear enough ideas of instantaneous velocities and accelerations, allowing them to state a rule to the effect that in a rectilinear uniformly accelerated motion, the space $(x - x_0)$ travelled in an interval of time $(t - t_0)$ is given by:

$$(x - x_0) = \frac{1}{2}(v + v_0)(t - t_0) \tag{2}$$

where v_0 (resp. v) is the velocity at time t_0 (resp. t). Similarly dim, but sound, ideas on the change of rate of changes can be found in *De Uniformitate et Difformitate Intensionum* (1350) and *Tractatus de Latitudinibus Formarum* (n.d.) by Nichole Oresme in France, who can be credited with a geometrical proof of the "*Merton rule*" (2) akin, again in modern language, to the computation of the trapezoidal area under the graph of v as a function of t. The fact that v is here a linear function of t did obviously help since calculus was still a long way ahead. Indeed, it took some three centuries until enough power and confidence were built into the mathematical apparatus to allow the elegance and economy of concepts on which Newton erected his *Principia* (1687). What can be regarded as the first, even if incomplete, axiomatic formulation of mechanics holds indeed in the following three laws, stated in the beginning of the *Principia*: (1) *every body continues in its state of rest, or of uniform motion along a straight line, unless it is compelled to change that state by forces impressed upon it;* (2) *the change of motion is proportional to the motive force impressed, and it takes place in the direction of the straight line in which that force is impressed;* (3) *to every action there is always an* *opposite and equal reaction; or, the mutual actions of two bodies upon each other are always equal and opposed in direction.* Newton complements these three laws with a few "definitions", among which it is necessary to quote here the following ones: (a) "the quantity of matter is the measure of the same, arising from its density and bulk conjunctly ... it is this quantity that I mean ... under the name of body or mass. And the same is known by the weight of each body, for it it proportional to the weight ... "; (b) "the quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly ... "; (c) "an impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly along a straight line. This force consists in the action only; and remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its *vis inertiae* [force of inactivity] only".

These laws and definitions call for some immediate comments.

Firstly, the reader will have noticed that Newton assumed an acquaintance with kinematics. He stated explicitly—in a separate commentary, or scholium—that he does "not define time, space, place [i.e. volume occupied by a body] and motion [in particular velocities, absolute or relative, and acceleration], as being known to all." One might add that, although Newton refrained from using explicitly the notation of the fluxions in the *Principia*, the notions and methods of infinitesimal and integral calculus nonetheless pervade implicitly his exposition throughout. If he tried to hide this, as some authors would have it, the subterfuge would be so transparent to the modern eye, that little (if any) insight would result from trying to hide the obvious in the present survey: our aim is primarily to outline the substratum of concepts on which 20th-century physics was erected.

Secondly, we should point out that the definition of mass in Newton's *Principia* requires some elaboration. Mach (1883a) noticed that Newton's third law gives an empirical mean to determine the ratio of two masses; he started with the empirical definition: "all those bodies are bodies of equal mass, which, mutually acting on each other, produce in each other equal and opposite accelerations"; on that basis, Mach then argued that this empirical determination of the equality of two masses can be extended to an empirical determination of the ratio of two arbitrary masses, and that this relation is transitive. It might be of some incidental interest to notice that Mach (1883b) used a similar reasoning to reach an empirical definition of the quantity of electricity, i.e. electric charge (see Chapter Three). Of more immediate interest here, we must notice that Newton, in his definition of mass, identifies the *inertial mass* (resistance to changes of linear motion) and the *gravitational mass* (as measured through the weight of the body). This identification is by no means trivial; it was to be elevated, by Einstein, to the status of a fundamental *principle of equivalence*, according to which no external static homogeneous gravitational field can be detected in a laboratory in free fall in this field, since both the observer and his measuring apparatus will respond to the field with the same acceleration. Newton himself felt it necessary to state explicitly that he did verify the identification between these two notions of mass by experiments "very accurately made" on pendulums of equal lengths, but different compositions, and that he found no detectable differences in their period. This result was subsequently verified with increasing precision. At the beginning of the 20th century, as a result of delicate balancing experiments conducted by Eötvös, the difference between the ratio of the inertial and gravitational masses of wood and platinum was known to be less than one part in 10^9; the method was further improved by Dicke et al. (1964) to reach an experimental coincidence with 10^{-11} between aluminum and gold. We can, therefore, build with some confidence on a theory that identifies inertial mass and gravitational mass, as Newton did, followed on that account by Einstein.

We now continue our analysis of Newton's theoretical set-up with a third remark, the purpose of which is to emphasize that Newton's originality, at the beginning of the *Principia*, was mostly in his determination to write down basic axioms from which the theory could proceed deductively by mathematical reasoning, and his ability to extract concise axiomatic statements from the mist of ideas that had accumulated on mechanics, ideas some of which were correct, others were irretrievably wrong-headed, but most were vague, or blurred by extraneous circumstances. For instance, Buridan (c. 1300–c. 1360) had discussed a quantity that he called *impetus* which, once imparted to the motion of a body, would continue with the motion until it is destroyed by some external agency, such as the resistance of the air to the motion of a projectile. However, Buridan does not properly distinguish between the nature of air-resistance and of gravity, nor does he seem to recognize the unique role of linear motion. Newton's definition (b) of the *quantity of motion* is crisp; without hesitation, we can write it

$$p = m v \tag{3}$$

and then note that Newton's second law of motion can be now transcribed to read:

$$F = \frac{d}{dt}(m v) \tag{4a}$$

or, upon assuming that the quantity of matter, as measured by the mass m, is independent of the state of motion of the body:

$$F = ma \quad \text{with} \quad a \equiv \frac{dv}{dt} = \frac{d^2x}{dt^2} \equiv \ddot{x} \tag{4b}$$

which, after the work Euler did in the mid-18th century, has become the textbook form of Newton's second law. Anticipating further the post-Newtonian developments into the 18th century, we can already remark here that if it happens, as is the case for gravitational forces, that there exists a function

$V : x \in \mathbb{R}^3 \mapsto V(x) \in \mathbb{R}$ such that $F = -\operatorname{grad} V$, i.e. in modern language, if the force derives from a potential, Newton's second law takes the form

$$\dot{p}_k = -\partial_k V \quad (k = 1, 2, 3) \tag{4c}$$

These formal manipulations of Newton's second law, while legitimate and useful, may nevertheless divert attention from the fact that this second law subsumes a wealth of unconscious assumptions and idealizations. We shall discuss later the question of whether Newton's formulation suffices or not for establishing the equations of motion of bodies that are more complicated than systems of point masses. Our next remark is directed, rather, to the motivation behind the concept of force. Newton's contribution suggests, at least implicitly by the use he makes of it, that *force is a primary concept*. It is nevertheless of interest to understand how it came about that forces should be represented by vectors—or by vector fields, as is the case for gravitation—even before they could be fitted so snugly into Newton's second law. In this context, it is useful to remember that dynamics not only assumes kinematics, but also contains statics as a particular case. *Statics* is circumscribed, in modern language, as the science concerned with the conditions under which an array of forces (and torques), acting on material bodies—e.g. particles, rigid or elastic bodies, fluids, or assemblages thereof—results in an equilibrium state, i.e. a state of rest or, more generally, a state characterized by the absence of accelerated motions. Devoting here a few lines to statics seems justified on account of the following three circumstances: (i) elementary situations come more easily under experimental scrutiny; (ii) the theoretical analysis of these situations does not require, at the start, the full epistemological apparatus necessary to distinguish all the fine threads running through the fabrics of the shared intuitions stored under the name of experience; (iii) the historical development of statics, especially in the 18th century, but also earlier and later well into the 20th century, shows that its purview is broad enough to provide essential clues into the general case of dynamics.

In particular, the mathematical nature and rules of composition of forces as vectors attached to different points of space, came to light in the 16th-century study of statics, although one finds a forerunner in Archimedes' law of levers. We find for instance that in *De Beghinselen der Weeghconst* [The Elements of the Art of Weighing] (1586), Stevin starts with a discussion of various systems of levers, and then begins his theory of the inclined plane with the following result: "Proposition XIX. Given a triangle, whose plane is at right angle to the horizon, with its base parallel thereto, while on each of the other sides there shall be a rolling sphere, of equal weight to one another: as the right side of the triangle is to the left side, so is the apparent weight of the sphere on the left side to the apparent weight of the sphere on the right side." Although Stevin does not seem to care giving a clear-cut definition of what he means by the term "apparent weight", the meaning springs out from the context, namely the "proof" he proposes, and the "corollaries" he derives from his proposition. Indeed his reasoning amounts to showing that if two spheres are of such unequal weights that their apparent weights would maintain them in equilibrium were they to be linked by a (weightless, inextendible) string, then their apparent weight would be what we call today the tension in the string or, more abstractly, the components of the weights of the spheres along a direction parallel to the inclined plane on which they are placed. From the variants of this problem, which Stevin analyses using various combinations of strings attached to hanging weights, to pull first on a sphere on an inclined plane, and then in subsequent examples, to pull on freely hanging rigid bodies of arbitrary shapes, it becomes clear that Stevin had recognized that "apparent weights"—i.e. forces—are characterized by their magnitude, their direction, and the point of the body on which they act. Moreover, the nature of his argument is such that he has been credited for having arrived at the *law of the parallelogramme of forces*, i.e. the law of addition of vectors. It should, nonetheless, be noted that Stevin's proof of his proposition XIX is based on what he presents as a reductio ad absurdum involving the absence of perpetual motion; the reasoning, however, was formalized by Varignon in his *Nouvelle Mécanique ou Statique* (1725) where he shows that the law of the parallelogramme of forces can be obtained as an application of the principle of virtual work. In that treatise moreover, Varignon reproduces a geometric argument, he had presented to the Paris Academy in 1687, to the effect that the law of the parallelogramme of forces implies the *law of levers*. This law states that if n forces F_k $(k = 1, 2, \ldots, n)$ act at n points x_k $(k = 1, 2, \ldots, n)$ of a rigid system whose motions are constrained to be rotations around a fixed point x_0, then the equilibrium condition is:

$$(5) \qquad M = 0 \quad \text{with} \quad M \equiv \sum_{k=1}^{n} (x_k - x_0) \wedge F_k \,.$$

$M_k \equiv (x_k - x_0) \wedge F_k$ is called the *moment* (or torque) *of the force* F_k *with respect to* x_0, and M is the total torque applied to the constrained system. To maintain this constraint, namely that x_0 be fixed, a force

$$(6) \qquad F_0 = -\sum_{k=1}^{n} F_k$$

has to be exerted at the point x_0. Hence the conditions of equilibrium of such a rigid system are

$$(7a) \qquad F = 0 \quad (\text{ with } \quad F \equiv \sum_{k=0}^{n} F_k \,)$$

$$(7b) \qquad M = 0 \quad (\text{ with } \quad M \equiv \sum_{k=0}^{n} x_k \wedge F_k \,)$$

Note that, because of (7a), we can replace x_k in (7b) by $\xi_k = x_k + \xi$ with ξ arbitrary.