A FIRST COURSE IN MATHEMATICAL MODELING

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To my mother, Margaret McCarthy Giordano
and in loving memory of my father, Samuel Rudolf Giordano

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PREFACE: TO THE INSTRUCTOR

The Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM) recommended in 1981 that "Students should have an opportunity to undertake 'real world' mathematical modeling projects, either as term projects in an operations research course, as independent study, or as an internship in industry."* That report goes on to add that a *modeling experience* should be included within the *common core* of all mathematical sciences majors. Further, this experience in modeling should begin early: ". . . to begin the modeling experience as early as possible in the student's career and reinforce modeling over the entire period of study."† We would like to describe how our book is designed for a *first course* in modeling.

Goals and Orientation

While this text can be used in a variety of ways, its main purpose is to provide the textual material supporting a modeling course that can be taught as soon as possible after the introductory engineering or business calculus sequence. The course is a bridge between calculus and the applications of mathematics to various fields, and it is a transition to the significant modeling experiences recommended by CUPM. The course affords the student an early opportunity to see how the pieces of an applied problem fit together. By using fundamental calculus concepts in a modeling framework, the student investigates meaningful and practical problems chosen from common

* Mathematical Association of America, Committee on the Undergraduate Program in Mathematics, *Recommendations for a General Mathematical Sciences Program* (Mathematical Association of America: Washington, D.C., 1981), p. 13.
† Ibid., p. 77.

experiences encompassing many academic disciplines, including the mathematical sciences, operations research, engineering, and the management and life sciences.

This text provides an introduction to the entire modeling process. The student will have occasions to practice the following facets of modeling:

1. *Creative and Empirical Model Construction*: Given a real-world scenario, the student learns to identify a problem, make assumptions and collect data, propose a model, test the assumptions, refine the model as necessary, fit the model to data if appropriate, and analyze the underlying mathematical structure of the model in order to appraise the sensitivity of the conclusions.

2. *Model Analysis*: Given a model, the student learns to work backward to uncover the implicit underlying assumptions, assess critically how well those assumptions fit the scenario at hand, and estimate the sensitivity of the conclusions when the assumptions are not precisely met.

3. *Model Research*: The student investigates a specific area to gain a deeper understanding of some behavior and learns to use what already has been created or discovered.

It is our perception that many mathematics students lack real problem solving capability, and we have designed our modeling course to help rectify that deficiency. For purposes of discussion we identify the following steps of the problem solving process:

1. Problem identification
2. Model construction or selection
3. Identification and collection of data
4. Model validation
5. Calculation of solutions to the model
6. Model implementation and maintenance

In many instances the undergraduate mathematical experience consists almost entirely of doing step 5: calculating solutions to models that are given. There is relatively little experience with "word problems," and that is spent with problems that are short (in order to accommodate a full syllabus) and often contrived. Such problems require the student to apply the mathematical technique currently being studied, from which a *unique* solution to the model is calculated with great *precision*. For lack of experience, consequently, students often feel anxious when given a scenario for which the model is *not* given or for which there is no unique solution, and then are told to identify a problem and construct a model addressing the problem "reasonably well."

With this in mind, we feel that in an introductory modeling course students should spend a significant amount of time on the first several steps of the process described above—learning how to identify problems, construct or select models, and figure out what data needs to be collected—progressing from relatively easy scenarios to more difficult ones. It is probably unreasonable to expect an average student to excel in a semester-long project on the first attempt. It takes time and experience to develop skill and confidence in the modeling process. We have found that involving students in the mathematical modeling process as early as possible, beginning with short projects, facilitates their progressive development and confidence in mathematics and modeling.

Many modeling texts present "type models" such as various inventory models for determining optimal inventory strategies. Students then learn to select an appropriate model for a particular situation. This approach has merit, and model selection is a valid step in the problem-solving process. However, undergraduate students often do not comprehend the assumptions behind a model, and only rarely do they take into consideration the appropriateness and sensitivity of those assumptions. Therefore, we emphasize *model construction* in order to promote student creativity and to demonstrate the artistic nature of model building, including the ideas of experimentation and simulation. Although we do discuss fitting data to chosen model types, our concentration is still on the entire model-building process, leaving the study of type models for more advanced courses.

Student Background and Course Content

Since our desire is to initiate the modeling experience as early as possible in a student's program, the only prerequisite for this text is a basic understanding of single-variable differential and integral calculus. (Occasionally we use partial derivatives, but other than a quick explanation by the instructor demonstrating how to compute such derivatives, the student does not need multivariable calculus.) While some unfamiliar mathematical ideas are taught as part of the modeling process, the emphasis is on using mathematics already known by the student after completing calculus. The modeling course will then motivate students to study the more advanced courses such as linear algebra, differential equations, optimization and linear programming, numerical analysis, probability, and statistics. The power and utility of these subjects are intimated throughout the text.

Although there are strong arguments to include such courses as advanced calculus, linear algebra, differential equations, linear programming, probability, and statistics as prerequisites to an introductory modeling course, this requirement necessitates postponing the course until the junior or senior undergraduate year, delaying the student's exposure to real-world applications. It also cuts off a number of student beneficiaries (namely, those nonmathematics majors who cannot satisfy all the prerequisites). Though our philosophy differs somewhat, this text still serves the more advanced student who has taken more mathematics courses. Certain sections of the text can be covered more rapidly by the advanced student, allowing more time for deeper extensions of the material as suggested by the projects for each chapter. The advanced student might also solve some of the models—such as a linear programming model—that would be beyond the capabilities of a student knowing only calculus.

While the text is designed so that it can be studied early in a student's program, readers at all levels should find the scenarios and problems both interesting and challenging. They are not designed for the application of a particular mathematical technique. Instead, they demand thoughtful ingenuity in using fundamental concepts to find reasonable solutions to "open-ended" problems. Certain mathematical techniques (such as dimensional analysis, curve fitting, and Monte Carlo simulation) are presented because they often are not formally covered at the undergraduate level. As an instructor, you should find great flexibility in adapting the text to meet the particular needs of your students through the problem assignments and student projects. We have used this material to teach courses to both undergraduate and graduate students, and even as a basis for faculty seminars.

Organization of the Text

The organization of the text is best understood with the aid of Figure 1. Part One consists of the first three chapters and is directed toward creative model construction and to an overview of the entire modeling process. We begin with the construction of graphical models, which provides us with some concrete models to support our discussion of the modeling process in Chapter 2. This approach also naturally extends the student's calculus experience, providing a transition into model construction by first involving the student in *model analysis*. Next we classify models and analyze the modeling process. At this point students can really begin to analyze scenarios, identify problems, and determine the underlying assumptions and principal variables of interest in a problem. This work is preliminary to the models they will create later in the course. (The order of Chapters 1 and 2 may be reversed, although we have found the current order best for capturing student interest and reducing student anxiety.) In their first modeling experience, students are quite anxious about their "creative" abilities and how they are going to be evaluated. For these reasons we have found it advantageous to start them out on familiar ground by appealing to their understanding of graphs of functions and having them learn model analysis. The book blends mathematical modeling techniques with the more creative aspects of modeling for variety and confidence building, and gradually the

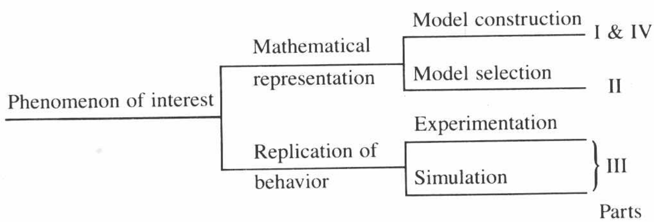

FIGURE 1. The organization of the text follows the above classification of the various models.

transition is made to the more difficult creative aspects. Students will find Parts Three and Four more challenging than the first parts of the text.

In Chapter 3 we present the concepts of proportionality and geometric similarity and use them to construct mathematical models for some of the previously identified scenarios. The student formulates tentative models or submodels and begins to learn how to test the appropriateness of the assumptions.

After Chapter 3 the book provides a number of models that can be related to curve fitting and optimization. These models motivate the study in Part Two. In Chapter 4 model fitting is discussed, and in the process several optimization models are developed. These optimization models are then analyzed in Chapter 5. The area of optimization is so rich in practical applications for modeling that it is tempting to teach optimization solution techniques (such as linear programming) as part of the course. However, such an approach detracts from the time allowable for model formulation and construction. Thus we have chosen to give students the opportunity to practice model construction while addressing a wide variety of scenarios. In a subsequent section the students are asked to solve those optimization problems requiring only calculus. Although students may not be able to solve some of the optimization models they will formulate here, nevertheless they do obtain needed additional practice in model construction, through which they gain confidence in their modeling skills. Moreover, the motivation is provided for studying linear optimization later in a full course. For those instructors who do wish to cover linear programming or other optimization topics as part of their course, we have suggested a sequence of UMAP (Undergraduate Mathematics and Its Applications Project) modules that provide excellent introductory material.* The modules include both graphical and analytical treatments of the Simplex method, for instance. While Part Two can be viewed more as model "solving," nevertheless, once completed, it enables a student to fit constructed or selected models to a set of data.

Part Three of the book consists of Chapters 6–8 and is dedicated to empirical model construction. It begins with fitting simple one-term models to collected sets of data and progresses to more sophisticated interpolating models, including polynomial smoothing models and cubic splines. The next topic is dimensional analysis and it presents a means of significantly reducing the experimental effort required when constructing models based on data collection. We also include a brief introduction to similitude. Finally, simulation models are discussed. An empirical model is fit to some collected data, and then Monte Carlo simulation is used to duplicate the behavior being investigated. The presentation motivates the eventual study of probability and statistics.

* UMAP modules are developed and distributed through COMAP, Inc./UMAP, 271 Lincoln St., Lexington, MA 02173.

In Part Four dynamic (time varying) scenarios are treated. Modeling based on differential equations in lieu of difference equations is motivated by our desire to emphasize the use of mathematics that students already know, namely the calculus. We begin by modeling initial value problems in Chapter 9 and progress to interactive systems in Chapter 10, with the student performing a graphical stability analysis. Students with a good background in differential equations can pursue analytical and numerical stability analyses as well, or they can investigate the use of difference equations or numerical solutions to differential equations by completing the projects.

The text is arranged in the order we prefer in teaching our modeling course. However, once the material in Chapters 1–4 is covered, the order of presentation may be varied to fit the needs of a particular instructor or group of students. Figure 2 shows how the various chapters are interdependent or independent, allowing progression through the chapters without loss of continuity.

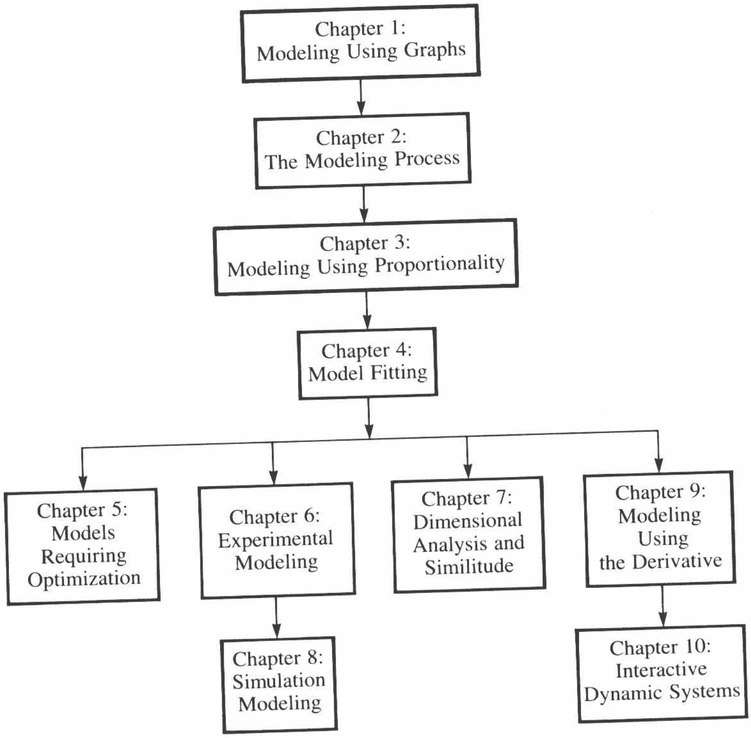

FIGURE 2. Chapter organization and progression.

Student Projects

Student projects are an essential part of any modeling course. This text includes projects in creative and empirical model construction, model analysis, and model research. Thus we recommend a course consisting of a mixture of projects in all these three facets of modeling. These projects are most instructive if they address scenarios that have no unique solution. Some projects should include *real* data that the student is either given or can *readily* collect. A combination of individual and group projects can also be valuable. Individual projects are appropriate in those parts of the course where the instructor wishes to emphasize the development of individual modeling skills. However, the inclusion of a group project early in the course gives students the exhilaration of a "brainstorming" session. A variety of projects is suggested in the text, such as constructing models for various scenarios, completing UMAP modules, or researching a model presented as an example in the text or class. It is valuable for each student to receive a mixture of projects requiring either model construction, model analysis, or model research for variety and confidence building throughout the course. Students might also choose to develop a model in a scenario of particular interest, or analyze a model presented in another course. We recommend six to eight short projects in a typical modeling course. Detailed suggestions on how the student projects can be assigned and used are included in the Instructor's Manual written for this text.

In terms of the number of scenarios covered throughout the course, as well as the number of homework problems and projects assigned, we have found it better to pursue a few that are developed very carefully and completely. Two or three good problems are about the maximum that an average student can handle in one week. We have provided many more problems and projects than can reasonably be assigned in order to allow for a wide selection covering many different application areas.

The Role of Computation

Although computing capability is *not* a requirement in using this book, computation does play a role of increasing importance after Part One. We have found a combination of programmable calculators, microcomputers, and mainframe computers to be advantageous throughout the course. Students who have programming experience can write computer code as part of a project, or software can be provided by the instructor as needed. We include some programs in the Instructor's Manual for this text. Typical applications for which students will find computers useful are in graphical displays of data, transforming data, least-squares curve fitting (and possibly the Simplex method if the Chebyshev criterion is pursued with more advanced students), divided difference tables and cubic splines, programming simulation models, and numerical solutions to differential equations. The use of computers has the added advantage of getting the student to think early about numerical methods and strategies, and it provides insight into how "real-world" problems are actually attacked in business and industry. Students appreciate being provided with or developing software that can be taken with them after completion of the course.

Resource Materials

We have found material provided by the Undergraduate Mathematics and Its Applications Project (UMAP) to be outstanding and particularly well suited to the course we propose. UMAP started under a grant from the National Science Foundation and has as its goal the production of instructional materials to introduce applications of mathematics into the undergraduate curriculum.

The individual modules may be used in a variety of ways. First, they may be used as instructional material to support several lessons. (We have incorporated several modules in the text in precisely this manner.) In this mode a student completes the self-study module by working through its exercises (the detailed solutions provided with the module can be removed conveniently before it is issued). Another option is to put together a block of instruction covering, for example, linear programming or difference equations, using as instructional material one or more UMAP modules suggested in the projects sections of the text. The modules also provide excellent sources for "model research" since they cover a wide variety of applications of mathematics in many fields. In this mode, a student is given an appropriate module to be researched and is asked to complete and report on the module. Finally, the modules are excellent resources for scenarios for which students can practice model construction. In this mode the teacher writes a scenario for a student project based on an application addressed in a particular module and uses the module as background material, perhaps having the student complete the module at a later date. Information about Project UMAP (now COMAP) may be obtained by writing the director, Ross L. Finney.

Several other excellent sources can be used as background material for student projects. The comprehensive four-volume series *Modules in Applied Mathematics*, edited by William F. Lucas and published by Springer-Verlag, provides important and realistic applications of mathematics appropriate for undergraduates. The volumes treat differential equations models, political and related models, discrete and system models, and life science models. Another book, *Case Studies in Mathematical Modelling*, edited by D. J. James and J. J. McDonald and published by Halsted Press, provides case studies explicitly designed to facilitate the development of mathematical models. Finally, scenarios with an industrial flavor are contained in the modular series edited by J. L. Agnew and M. S. Keener of Oklahoma State University. While we feel that this text provides abundant material for an introductory modeling course, any of the preceding references can be used selectively to complement the text and suit the tastes of the individual instructor.

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* Optional Section

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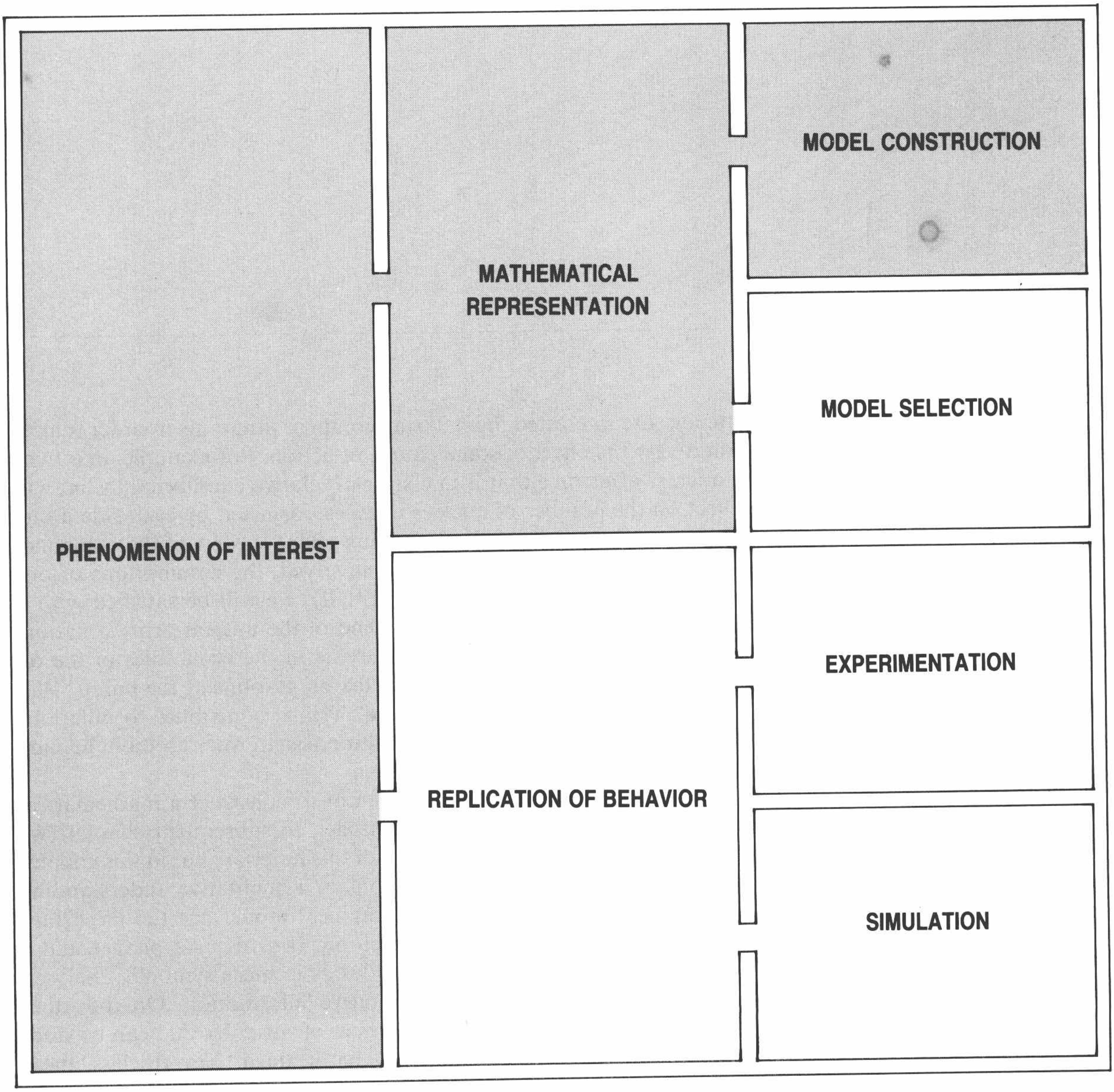
PART ONE

CREATIVE MODEL CONSTRUCTION AND THE MODELING PROCESS



1

GRAPHS OF FUNCTIONS AS MODELS

INTRODUCTION

Quite often we are interested in analyzing complex situations in order to predict qualitatively the effect of some course of action. For example, in a two-country nuclear arms race that is in a state of relative equilibrium, what will be the effect on the number of nuclear missiles possessed by each side if one of the countries introduces mobile launching pads? In view of the many factors affecting the behavior of the parties involved, the assumptions of our model will necessarily be rather crude. Initially, we will be satisfied with a model that simply captures the general trend of the nuclear arms situation. Another example of a complex situation exists in the economics of the oil industry. What is the effect of a surcharge tax on gasoline at the pump? Will the tax actually reduce consumer demand? Will it contribute to inflation? In cases like those just described, graphical models are very useful in helping us understand the situation being studied.

In mathematical modeling we often attempt to construct a mathematical function relating variables to serve as a model. In subsequent chapters we attempt to determine a precise description of this function, but in this chapter the graph of the function is used simply to gain a qualitative understanding of the behavior under investigation. A graphical model has the important advantage of appealing to our visual intuition. It gives us a picture and a "feel" for what is happening that often eludes us in more symbolic analyses. Graphs are very good for gaining qualitative information. On the other hand, graphical analysis does limit the number of variables that can be studied effectively and the precision that can be attained. Nevertheless, these