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Edited by A. Dold and B. Eckmann

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Probability in Banach Spaces III

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INTRODUCTION

With each passing biennium, the subject of probability in vector spaces makes more and more impressive gains. Only twenty-five years ago, there was almost nothing in print in the subject, and what did exist was mostly an observation that the methods used in finite-dimensional spaces (which were essentially the methods for one-dimensional spaces) would extend to infinite-dimensional ones with a little coaxing. But there was then no program of study, and no clear reason to investigate the subject except for the Everest Principle: it was there. Even as recently as ten years ago, the subject was considered highly esoteric. There were already strong indications of the essential bonds between measure theory and geometry, indicating the essential role of each in producing theorems, but while the structure had extent, it was without very much substance. It was a mere skeleton on which really important theorems needed to be hung to create a viable body.

It was only five years ago that the accomplishments of a new and gifted generation of mathematicians had accumulated to the point that the subject was ripe for its first international conference (Oberwolfach 1975). By 1978, at the second conference, the volume of work done in the intervening three years exceeded all that had gone before, and now again, we have a new flood of results in only two years. As the 1978 conference had established that no study of Probability could any longer be considered adequate without basic grounding in infinite-dimensional theory, so we now see the infinite-dimensional theory reaching past the finite into the traditional applications of probability to Physics and Statistics.

I would be remiss if I did not at this time make grateful acknowledgement of the contributions of Tufts University and especially of Prof. Marjorie Hahn in making this conference possible, and also note the generous contribution of the National Science Foundation toward some of the expenses. As the Mathematische Forschungsinstitut Oberwolfach supported and fostered the previous two conferences, so these institutions supported this one, and Prof. Hahn's volunteering of many hours of her time was the catalyst which make it all hang together.

Our subject is very healthy and growing at a very substantial pace. We expect to see it recognized as central to mathematical analysis within this decade. This volume exhibits the latest findings, and it is with great pride that I put it forth to the mathematical community.

Anatole Beck,
Editor

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STATISTICS ON BANACH SPACE VALUED GAUSSIAN RANDOM VARIABLES

by A. ANTONIADIS

1. Introduction.

Whereas the classical theory of statistical inference in finite dimensional gaussian models is almost completely developed, in the infinite dimensional case many problems are unsolved. It is the purpose of this paper to provide some methods of statistical inference on gaussian infinite dimensional models. Our approach is based on recent developments of the theory of infinite dimensional statistical spaces and on the use of techniques from the theory of gaussian measures on Banach spaces.

Let us now be more specific and introduce the basic notations and conventions in order to summarize the results.

Let $x = (x(t))_{t\in T}$ be an observation of a real random function $(X(t))_{t\in T}$ such that

$$X = \{ X(t) = m(t) + X_o(t) \ ; \ t \in T \}$$

where T is a compact metric space and $(X_o(t))_{t\in T}$ is a real gaussian function with zero mean and known covariance K on $T\times T$. We shall assume that the mean function m belongs to the reproducing kernel Hilbert space $\mathcal{H}(K)$ of K. The statistical space corresponding to this model is

$$\left(\mathbb{R}^T, \bigotimes_{t\in T} \mathcal{B}(\mathbb{R}) \ , \left\{ \mathbb{N}_{\mathbb{R}^T}(m,K) \ ; \ m \in \mathcal{H}(K) \right\} \right) \qquad (1.1)$$

where $N_{\mathbb{R}^T}(m,K)$ denotes the gaussian measure on $\mathbb{R}^T$ with mean m and covariance K. A statistical space is a triplet $(E,\mathcal{E},\mathcal{P})$, that is, the mathematical model associated with a statistical experiment where E is a real vector space representing the set of all the possible observations x, $\mathcal{E}$ is the σ-field of subsets of E generated by the observable events and $\mathcal{P}$ is a family of hypotheses concerning the probability distribution of the observed random variable X in $(E,\mathcal{E})$. Assuming the model (1.1), statistics for estimating the mean function m are given and tests of hypotheses of the form "$m \in V$" , where V is a finite dimensional subset of $\mathcal{H}(K)$, are performed.

An often adopted model for V is to regard it as the linear span of a family of known functions on T . The tests performed are optimal when the dimension of V is known. Thus, the next problem we are dealing with, is to define an estimation procedure for the dimension of V .

The representation of our model in terms of gaussian measures on Banach spaces is discussed in section 2.

In section 3, we survey some results of [1] which we shall need, concerning the estimation of the mean and the tests. Section 4 is devoted to the estimation of the dimension, which appears as an application of a result valid for models more general than model (1.1).

2. Notations. Preliminaries.

This section covers the basic definitions and notations necessary to the following work. All random variables considered from now on will be defined on a complete probability space $(\Omega, \mathcal{A}, P_r)$.

If B is a real separable Banach space with norm $\| \ \|$, B^* denotes the topological dual of B and the symbol $(\cdot,\cdot)$ denotes the duality between B and B^* . As usual $\mathcal{B}$ denotes the Borel σ-field of B .

Let μ be a centered probability measure on $(B, \mathcal{B})$ such that $\int_B \|x\|^2 d\mu(x) < +\infty$ and let K denote the covariance function of μ defined by

$$K(f,g) = \int_B (f,x)\,(g,x)\,d\mu(x) \qquad (f,g \in B^*)$$

Then according to lemma 2.1 of [3], the reproducing kernel hilbert space $\mathcal{H}(K)$ of K can be realized as a subset of B and the natural inclusion map of $\mathcal{H}(K)$ into B , say J , is linear, one to one and continuous. The same properties are true for the adjoint map J^* from B^* into $\mathcal{H}(K)$ if we identify $\mathcal{H}^*(K)$ and $\mathcal{H}(K)$ in the usual way. Furthermore, $J^*(B^*)$ is dense in

$\mathcal{H}(K)$. Hence, there is a subset $\{e_j^* ; j \geq 1\}$ of B^* such that $\{J^*(e_j^*) = e_j ; j \geq 1\}$ is a C.O.N.S in $\mathcal{H}(K)$. Further, the linear operators

$$\pi_N(x) = \sum_{k=1}^{N} (e_k^*, x)\, e_k \quad \text{and} \quad Q_N(x) = x - \pi_N(x) \quad , \quad N \geq 1$$

are continuous from B to B .

For every $N \geq 1$, let H_N denote the linear span of $\{e_j ; 1 \leq j \leq n\}$ which is also the range of π_N ; π_N and Q_N when restricted to $\mathcal{H}(K)$ are orthogonal projections onto their range.

Finally, if P_o denotes the gaussian measure on $(B, \mathcal{B})$ with zero mean and covariance K , then $\int_B \|x\|^2 \, dP_o(x) < +\infty$ and the above holds. For every m in $\mathcal{H}(K)$, let P_m be the image measure of P_o by the map $x \longmapsto x + m$, $x \in B$.

3. Estimation and quadratic tests.

For applications, it is natural to suppose that the C.O.N.S $\{e_j ; j \geq 1\}$ of $\mathcal{H}(K)$ is given. With the notations of the above section let Σ be the gaussian statistical space

$$\Sigma = (B, \mathcal{B}, \{P_m ; m \in H_N \subset \mathcal{H}(K)\})$$

Here, we give the estimation procedure of the unknown mean m and some quadratic tests of the hypothesis "$m \in H_N$" against "$m \notin H_N$" . The proof of these results will appear elsewhere [1] .

Proposition 3.1. Let $\Sigma = (B, \mathcal{B}, \{P_m ; m \in H_N \subset \mathcal{H}(K)\})$ be a gaussian statistical space. The statistic

$$x \longrightarrow \pi_N(x)$$

is sufficient for m , of maximum likelihood and defines an unbiased linear estimation of minimum variance.

Remark. In the statistical space $\Sigma_\infty = (B, \mathcal{B}, \{P_m ; m \in J^*(B^*)\})$

by using the extension of the Cramer-Rao inequality in Banach spaces as it is stated in [2], it is easy to check that the identity is an efficient estimator of m. By efficiency of an unbiased estimator $\hat{m}$ of m we mean that for any unbiased estimator $\bar{m}$ of m and any f in B^* we have

$$E_{P_m}\left((f, \hat{m}-m)^2\right) \leq E_{P_m}\left((f, \bar{m}-m)^2\right).$$

It is also known that in the gaussian case, $\pi_N(x)$ converges to x P_m a.s. It follows that the sequence $\{\pi_N(x) ; N \geq 1\}$ is asymptoticaly efficient.

Next, when we assume B to be the space $C(T)$ of continuous functions on a compact metric space T, we have

Proposition 3.2. Let $(X(t))_{t \in T}$ be a gaussian random function with a.s. continuous sample paths on T, with covariance function K continuous on $T \times T$ and mean function m in $\mathcal{H}(K)$. Then, for every α, $0 \leq \alpha \leq 1$, there exists an unbiased quadratic test of size α for testing "$m \in H_N$" against "$m \notin H_N$". Such a test is of the form

$$x \in C(T), \int_T (Q_N(x)(t))^2 \, d\nu(t) > \ell_\alpha \Longrightarrow m \in H_N$$

for some positive Borel measure ν on T and some positive real number ℓ_α.

4. Estimation of the dimension.

The estimation procedure presented in section 3, requires the knowledge of the dimension N and it is then optimal. Thus, we are faced with the problem of choosing the appropriate dimension that will fit a given set of observations. A typical example of this problem is the choice of degree for a polynomial regression. In this section we present an estimation of the dimension of the model. Before stating the main result we shall establish some terminology.

Let us consider an n-sample in the statistical space $\Sigma_\infty = (B, \mathcal{B}, \{P_m ; m \in J^*(B^*)\})$. Since we can consider the parameter space $J^*(B^*)$ partitioned in the following way

$$J^*(B^*) = \bigcup_{N=1}^{\infty} M_N$$

where $M_N = H_N \setminus H_{N-1}$ and $M_\infty = J^*(B^*) \setminus \bigcup_{1 \le N < +\infty} H_N$, we shall call dimension of our model the subscript N of M_N such that m belongs to M_N in Σ_∞.

For each x in B and φ strictly positive, set

$$r(x,\varphi) = \min \{ j \, ; \, \| Q_j(x) \| < \varphi \} \qquad (4.1)$$

where $Q_j = I_d - \pi_j$, π_j being the orthogonal projector onto H_j.
The estimation procedure can be formulated in the following corollary whose proof will follow easily from the general convergence result obtained in theorem 4.1.

Corollary 4.1. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d random variables in Σ_∞. There is a positive constant M such that, for any $\epsilon > 0$, $r(\overline{X}_n, \varphi_n(\epsilon, M))$ defines an estimator of the dimension of the model, P_r a.s. convergent, where r is defined in (4.1), $\varphi_n(\epsilon, M) = \left(\frac{2 \log \log n}{n} \right)^{1/2} (\epsilon + M)$ and $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Before stating our result let us recall a definition which is used in theorem 4.1.

Definition 4.1. Let μ a probability measure on $(B, \mathcal{B})$, with mean zero, covariance K and such that $E_\mu(\|X\|^2) < +\infty$. The measure μ is said to satisfy the law of the iterated logarithm if for any sequence of i.i.d B-valued random variables on $(\Omega, \mathcal{Q}, P_r)$ such that $\mathcal{L}(X_1) = \mu$, we have

$$P_r \left(\left\{ \omega \in \Omega \, ; \left\{ \frac{S_n(\omega)}{\sqrt{2n \log \log n}} \, ; \, n \ge 1 \right\} \text{ is conditionally compact in } B \right\} \right) = 1$$

where $S_n = \sum_{i=1}^{n} X_i$.

If μ satisfies the LIL then, according to [3],

$$P_r(\lim d \, ((2n \log \log n)^{-1/2} S_n, S) = 0) = 1 \qquad (4.2.)$$

where S denotes the unit ball of $\mathcal{H}(K)$.

Theorem 4.1. Let $(X_n)_{n\in\mathbb{N}}$ a sequence of i.i.d B-valued random variables such that $\mathcal{L}(X_n - m) = \mu$ where μ is as in the definition 4.1 and m belongs to $J^*(B^*)$. Let d be the dimension of m. There exists a constant M such that, for any $\epsilon > 0$, we have

$$P_r\left(\{\omega\in\Omega\ ;\ \lim_{n\to+\infty} (r(\overline{X}_n(\omega), \varphi_n(\epsilon,M)) = d\}\right) = 1$$

with $r(\overline{X}_n, \varphi_n(\epsilon, M))$ as given in corollary 4.1.

Proof. It is well known that the unit ball S of $\mathcal{H}(K)$ is compact in B and that $Q_j(S) \subseteq S$ for all $j \geq 1$. Thus,

$$M = \sup\{\|Q_N(x)\|\ ;\ N\geq 1, x\in S\} < +\infty .$$

Let d be the dimension of m. First we will establish the proof for d finite. In this case we have by definition of d, $m \in H_d \setminus H_{d-1}$ and therefore $\pi_d(m) = m$.

Let us consider

$$\|Q_d(\overline{X}_n)\| = \|Q_d(\overline{X}_n - m)\| = \left(\frac{2\log\log n}{n}\right)^{1/2} \left\|Q_d\left((2n\log\log n)^{-1/2} S_n\right)\right\| \quad (4.3)$$

where $S_n = \sum_{i=1}^{n} (X_i - m)$. Since $\mathcal{L}(X_i - m) = \mu$ and μ satisfies the LIL, (4.2) holds. On the other hand, Q_d maps B into B continuously. Hence (4.2) implies that

$$P_r\left(\lim d(Q_d(\xi_n), Q_d(S)) = 0\right) = 1 \quad (4.4)$$

where $\xi_n = (2n\log\log n)^{-1/2} S_n$. Therefore, there exists $A \in \mathcal{Q}$ with $P_r(A) = 1$ such that for any ω in A and $\epsilon > 0$, there is an integer $n(\epsilon,\omega)$ such that

$$\text{for any } n \geq n(\epsilon,\omega) \quad d(Q_d(\xi_n(\omega)), Q_d(S)) < \epsilon \quad (4.5)$$

It follows from (4.5) that for $n \geq n(\epsilon,\omega)$ we have

$$\|Q_d(\xi_n(\omega)\| < (\epsilon + M) \quad (4.6)$$

set

$$\varphi_n(\epsilon, M) = \left(\frac{2 \log \log n}{n}\right)^{1/2} (\epsilon + M)$$

It follows from (4.3), (4.6) and (4.1) that for any $n \geq n(\epsilon, \omega)$

$$r(\overline{X}_n(\omega), \varphi_n(\epsilon, M)) \leq d$$

and consequently

$$P_r(\limsup_n r(\overline{X}_n, \varphi_n) \leq d) = 1 \quad (4.7).$$

Set $F = \{\omega \in \Omega ; \liminf r(\overline{X}_n(\omega), \varphi_n) < d\}$ and let ω be an element of F. Thus, for any integer n there exists $k(\omega) \geq n$ such that

$$r(\overline{X}_k(\omega), \varphi_k) < d$$

and since $H_1 \subset H_2 \subset \ldots\ldots \subset H_d$ it follows that

$$d(\overline{X}_k(\omega), H_{d-1}) < \varphi_k \quad (4.8).$$

But $d(\overline{X}_k(\omega), H_{d-1}) = d(\overline{X}_k(\omega) - m, H_{d-1} - m)$ and therefore

$$d(m, H_{d-1}) \leq d(\overline{X}_k(\omega), m) + \varphi_k$$

or equivalently

$$\left(\frac{k}{2 \log \log k}\right)^{1/2} d(m, H_{d-1}) \leq \| \xi_k(\omega) \| + (\epsilon + M) \quad (4.9).$$

Since the dimension of m is d, we have $d(m, H_{d-1}) > 0$. Hence, for any ω in F we have $\limsup_n \| \xi_n(\omega) \| = +\infty$. Combining this with (4.2) we then have $P_r(F) = 0$. Now (4.7) and $P_r(F) = 0$ implies the assertion of the theorem 4.1.

To finish the proof, let us consider the case of an infinite d, that is $m \in J^*(B^*) \setminus \bigcup_{\infty > N \geq 1} H_N$. We will prove here that

$$P_r(\liminf r(\overline{X}_n, \varphi_n) = +\infty) = 1 \quad (4.10).$$

Set $A = \{ \omega \in A \ ; \ \liminf r(\overline{X}_n, \varphi_n) < +\infty \}$ and let ω be an element of F. There exists an integer $K(\omega)$ such that for any n, there is a $k \geq n$ which satisfies

$$r(\overline{X}_k, \varphi_k) < K(\omega) \ .$$

Thus inequality (4.8) holds with $d-1 = K(\omega) - 1$. Hence we have

$$\left(\frac{k}{2 \log \log k} \right)^{1/2} d\left(m, H_{K(\omega)-1}\right) - (\epsilon + M) \leq \| \xi_k(\omega) \| \ . \tag{4.11}$$

But $d = +\infty$ and therefore $d\left(m, H_{K(\omega)-1}\right) > 0$. Now (4.10) follows from (4.11) and (4.2), so the proof is complete.

The proof of corollary 4.1 is immediate since the law P_o satisfies the LIL. However we stated it because it is the most useful in applications.

Acknowledgement

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MARTINGALES, AMARTS AND RELATED STOPPING TIME TECHNIQUES

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The purpose of this paper is to describe the state of affairs in martingale and amart theory from the point of view of convergence, maximal inequalities and generalized Fatou inequalities (which imply convergence), the key tool in these considerations being the notion of simple stopping time.

There are of course many other important areas of research in Martingale Theory, such as: martingale differences and martingale transforms (see the important work of Pisier [55] giving a proof of Enflo's theorem that every super-reflexive space admits an equivalent uniformly convex norm, via martingale inequalities; see also the beautiful recent work of D. L. Burkholder [15]), strong laws, H spaces (see Garsia's monograph [32] and Maurey's recent paper [45] establishing the Banach space isomorphism of the classical Hardy H_1 space and the H_1-space of martingales), stochastic integration (now a vast field in itself), etc. These aspects of the theory will not be discussed in the present paper.

The paper is divided as follows:

§1. Convergence in the presence of RNP.

§2. Convergence in the absence of RNP.

§3. The case of directed index sets.

§1. Convergence in the presence of RNP.

Let $(\Omega,\mathcal{F},P)$ be a probability space and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ a filtration, i.e., an increasing sequence of sub-σ-fields of $\mathcal{F}$. Denote by $T = T((\mathcal{F}_n)_{n\in\mathbb{N}})$ the set of all simple (i.e., taking finitely many values) stopping times relative to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Then T is ordered for the natural pointwise order relation $\sigma \leq \tau$ and T is filtering to the right.

We first consider the real-valued case: $E = R$.

We recall that:

Definition 1.1. An integrable process $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ is an amart if the net $(E(X_\tau))_{\tau\in T}$ converges to a finite limit.

Note that

$(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ is a martingale <=> $E(X_\tau)$ = constant, for $\tau \in T$;

$(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ is a submartingale <=> $(E(X_\tau))_{\tau\in T}$ is an increasing net.

[1]Research supported in part by the National Science Foundation.

It was soon noticed that Doob's downcrossing proof can be adapted from the martingale to the amart. This led to the amart convergence theorem:

Theorem 1.1 ([2];[17]). Every L^1-bounded amart converges a.s.

Remarks. 1) The idea of using simple stopping times to derive a.s. convergence from the convergence of the expectation of the stopped random variables, first appears as such in [46] and then [3] (for the case of uniformly bounded $(X_n)_{n\in\mathbb{N}}$). The general L^1-bounded case of the amart convergence theorem was first explicitly stated and proved in [2]; see also [17]. The amart convergence theorem is in fact contained in the earlier paper [44]; Lamb's result, however, is not stated in the language of stopping times. The amart convergence theorem (in the uniformly bounded, or dominated case) also follows from Sudderth's paper [59]. The latter paper studies the commutativity of E (expectation) and $\bar{L}$ (the operation of taking upper limits), that is: when does $\bar{L}E(X_\tau) = E(\bar{L}X_\tau)$ hold, τ running through the set of arbitrary stopping times; this problem was apparently inspired by the notion of "utility of a measurable strategy" introduced earlier by Dubins and Savage ([23] or [24]). See also [58].

2) It should be stressed that the essential feature of the amart is that it is defined in terms of simple stopping times. The first systematic study of amart theory was given in [26]; it was also Edgar and Sucheston who introduced the term amart (which now replaces the former term of asymptotic martingale). In addition to the proofs of the amart convergence theorem mentioned above, various other proofs are available (for instance [4],[10], Dvoretzky's proof in [6], p. 282; see also [25], [20]).

The class of L^1-bounded amarts has good stability properties. It is a vector lattice; it has the optional sampling property. Also the Riesz decomposition holds; in fact, every amart $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ can be written in the form

$$X_n = Y_n + Z_n \quad \text{where} \begin{cases} (Y_n,\mathcal{F}_n)_{n\in\mathbb{N}} & \text{is a martingale} \\ E(|Z_\tau|) \xrightarrow[\tau\in T]{} 0 & \end{cases}$$

(see [26],[6]).

In what follows we shall use the notation

$$T(\sigma) = \{\tau \in T \mid \tau \geq \sigma\}, \quad \text{for } \sigma \in T.$$

Before proceeding further let us note (see [9]):

Lemma 1.1. Let $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ be an integrable process. Then for each $n \in \mathbb{N}$ we have

$$(1) \qquad \sup_{\substack{(\sigma,\tau) \\ \sigma,\tau\in T(n)}} |E(X_\tau) - E(X_\sigma)| = \sup_{\substack{(\sigma,\tau) \\ \sigma\leq\tau \\ \sigma,\tau\in T(n)}} \|E^{\mathcal{F}_\sigma}[X_\tau] - X_\sigma\|_1$$

and hence

(2) $$\limsup_{\substack{(\sigma,\tau)\\ \sigma,\tau \in T}} |E(X_\tau) - E(X_\sigma)| = \limsup_{\substack{(\sigma,\tau)\\ \sigma \leq \tau \\ \sigma,\tau \in T}} \|E^{\mathcal{F}_\sigma}[X_\tau] - X_\sigma\|_1 .$$

In particular, in the real-valued case we have:

$$(X_n,\mathcal{F}_n)_{n\in\mathbb{N}} \text{ is an amart} \Leftrightarrow \lim_{\substack{(\sigma,\tau)\\ \sigma \leq \tau \\ \sigma,\tau \in T}} \|E^{\mathcal{F}_\sigma}[X_\tau] - X_\sigma\|_1 = 0.$$

We now consider the case when the random variables X_n take values in a Banach space E (always assumed to be separable).

Definition 1.2. An integrable process $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ is said to be of class (B) if

$$\sup_{\sigma \in T} \|X_\sigma\|_1 < \infty,$$

that is, if the set $\{X_\sigma | \sigma \in T\}$ is L^1-bounded.

The definition of amart extends to Banach spaces without difficulty:

Definition 1.3. An integrable process $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ is an amart (= a strong amart) if $\lim_{\tau\in T} E(X_\tau)$ exists in E.

The extension of the convergence theorem to strong amarts was obtained by Chacon and Sucheston:

Theorem 1.2 ([18]). Assume that E has RNP and that E' is separable. Let $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ be an amart of class (B). Then there is $Y \in L^1_E$ such that $(X_n)_{n\in\mathbb{N}}$ converges to Y weakly a.s.

The fact that only weak convergence obtains may be rather disappointing but this is the best one can get ([5]). In fact, using the Dvoretzky - Rogers lemma or the Dvoretzky theorem that ℓ^2 is "finitely representable" in every infinite-dimensional B-space, one can show (see [5]; see also [28]) that:

Every amart of class (B) converges strongly in E a.s. $\Leftrightarrow$ E is finite-dimensional.

The reason for this strange behavior in B-space is that if we replace R by E, the equality sign in (2) of Lemma 1.1 no longer holds: in general we have strict inequality.

We are thus led to introduce the notion of uniform amart ([7]):

Definition 1.4. An integrable process $(X_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ is a uniform amart if

$$\lim_{\substack{(\sigma,\tau)\\ \sigma \leq \tau \\ \sigma,\tau \in T}} \|E^{\mathcal{F}_\sigma}[X_\tau] - X_\sigma\|_1 = 0.$$