

$(F: \leq \partial \int_{-\infty}^{\infty} x + y^2$   
 $\sqrt[3]{\rightarrow} \approx \Delta \delta +$

ANALYTIC GEOMETRY AND THE CALCULUS

*A. W. Goodman*

**A. W. GOODMAN**  
UNIVERSITY OF KENTUCKY

*Analytic Geometry*  
*and the Calculus*

**THE MACMILLAN COMPANY, NEW YORK**  
**COLLIER-MACMILLAN LIMITED, LONDON**

© A. W. GOODMAN 1963

All rights reserved. No part of this book may be reproduced in any form without permission in writing from the publisher, except by a reviewer who wishes to quote brief passages in connection with a review written for inclusion in a magazine or newspaper.

FIRST PRINTING

Library of Congress catalog card number: 63-8395

The Macmillan Company, New York

Collier-Macmillan Canada, Ltd., Galt, Ontario

*Divisions of The Crowell-Collier Publishing Company*

Printed in the United States of America

# **ANALYTIC GEOMETRY AND THE CALCULUS**

## PREFACE FOR THE TEACHER

Recent textbooks on the Calculus have generally three common characteristics: (a) the inclusion of analytic geometry as an integrated part of the course, (b) the use of vectors, and (c) the introduction of more rigor. There are, I believe, good reasons for (a) and (b), and since these are now in style there is no need to repeat here the arguments in their favor. As the title indicates, this book is a unified treatment of Analytic Geometry and the Calculus. Further, we introduce vectors as early as possible and use vectors whenever we can.

But item (c) is quite different. It seems to me that a really rigorous book on the Calculus would begin with the axioms of set theory, derive the Peano axioms, and then reproduce most of Landau's great *Foundations of Analysis*. After this preparation, the class (if there are any students left) would have no trouble following the proofs needed in the Calculus. Since a completely rigorous presentation of the Calculus is obviously neither practical nor desirable, the central question is which theorems should be proved and which should be left to the intuition of the student. It is my hope that this book comes somewhere near the correct answer for the majority of students.

I have attempted to encourage the proper attitude toward rigor by stating definitions and theorems clearly. In this connection, I contend that one should not try to state all of the hypotheses in a theorem, because the statement can become so long as to be incomprehensible to the average student. Certain hypotheses are tacitly understood by both author and reader and hence, for brevity, should be omitted. As an illustration of the problems involved, consider the following:

**THEOREM S.** *If  $y = f(u)$  and  $u = g(x)$ , then the derivative of the composite function  $y = f(g(x))$  is given by*

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Here is a statement that is brief and simple, and the average student has a reasonable chance of understanding it. Now let us look at the same theorem when stated in a rigorous fashion.

**THEOREM R.** *Let  $f$  and  $g$  be two real-valued functions of a real variable and suppose that the range of  $g$  is a subset of the domain of  $f$ . Let  $h = fg$  be the composite function defined over the domain of  $g$  by setting  $h(x) = f(g(x))$  for each  $x$  in the domain of  $g$ . If  $x_0$  is an interior point of an interval contained in the domain of  $g$ , and  $g$  is a differentiable function at  $x_0$ , and if  $f$  is a differentiable function at  $u_0 = g(x_0)$ , then  $h$  is a differentiable function at  $x_0$ , and further the derivative is given by the formula*

$$h'(x_0) = f'(u_0)u'(x_0).$$

There is no doubt that R is the correct statement and S is full of gaps. However, the average student can learn and use S, but when R is presented he will either fall asleep or totally ignore it. It is just too complicated for him to master at this stage of his mathematics study. The presentation of R rather than S does real harm because it serves to repel many students who are originally attracted to mathematics and who might turn out to be capable technicians or teachers (if not creative thinkers) if they are given a reasonable chance to develop.

In this book we always give the short, simple, attractive, and perhaps erroneous statement in preference to the long, complicated, unattractive, but certainly correct version. For the very bright student (say one out of every twenty-five in the average university) who wants a rigorous treatment, it is a simple matter for the teacher to assign additional outside reading, such as Landau's *Foundations of Analysis*, his *Differential and Integral Calculus*, Hardy's *Pure Mathematics*, the two-volume work by Courant and McShane, *Differential and Integral Calculus*, Blachie and Don, Std. London, 1934, or the teacher's favorite book on Advanced Calculus.

The content of a course in Analytic Geometry and the Calculus, is reasonably standard, and a glance at the table of contents of this book will show that all of the essentials have been included, and these items do not require any further discussion. However, this book does have several minor innovations, which are listed here for convenience, together with a word or two of explanation.

1. The appendix contains short chapters on (a) mathematical induction, (b) inequalities, and (c) determinants. A student beginning the Calculus should know this material. It is included for the convenience and use of the poorly prepared student, but it is relegated to the appendix in order to avoid breaking into the natural sequence of ideas in the text proper.

2. Chapter 4 contains a detailed explanation of the summation notation and a list of problems devoted exclusively to this topic.

3. Centroids and moments of inertia have been postponed until Chapter 15. It has been my experience that students always have difficulty with this material, and the trouble may well disappear if we first allow them to develop more maturity. As a bonus for this postponement, we are able to amuse the student with a collection of weighted points that does not possess a center of gravity.

4. Chapter 11 contains some material on the computation of integrals by the method of undetermined coefficients.

5. In Chapter 13, on infinite series, we begin by stressing the computation and use of series, and postpone most of the theorems and proofs till the later part of the chapter. In this way the student can see the need for, and value of, the theorems before he tackles the proofs.

6. Chapter 16 includes a very brief introduction to the descriptive properties of point sets, together with a list of problems.

All topics are important in mathematics, but it must be admitted that some are more important than others. In order to devote enough space to a decent and detailed development of the essential ideas, and still hold this book to a reasonable length, it was necessary to omit certain items. The following topics were selected (with deep regrets and apologies) for exclusion: (a) equations of the angle bisectors for a pair of lines, (b) the radical axis of two circles, (c) average value of a function, (d) Kepler's laws of planetary motion, (e) contour lines as a method of graphing surfaces, and (f) line integrals and work.

Although the physical labor of organizing and writing this book was mine alone, it is obvious that any merit the book may have is ultimately due to the many persons who from every direction extended their helping hands. Just the list of all the mathematicians who have taught, guided, and inspired me is already too long for inclusion here, but the three who have had the deepest influence on my work are Otto Szasz, Paul Erdős, and Hans Rademacher. Unfortunately Otto Szasz is no longer with us, and Paul Erdős has been barred from entering this country for the last eight years, much to the detriment of our own young mathematical scholars. I must also acknowledge my great debt to my parents who understood and encouraged me in my efforts to study mathematics. Finally, special mention is due my wife, who while raising a family found the time and energy to type the entire manuscript of this book, and two other books as well.

*Lexington, Kentucky*

A. W. GOODMAN

## PREFACE FOR THE STUDENT

The Calculus is not an easy subject, and yet every year thousands of students manage to master it. You can do the same. The real difficulty is that the Calculus deals with *variables*. Because it is hard to give a precise definition of a variable this word is at present in ill-favor with some mathematicians. But the concept is easily understood by anyone who keeps his eyes and mind open while observing nature in action. For example, the distance of a car from some fixed reference point is a constant if the car is not moving, but the distance is a variable as soon as the car is in use. In the hands of a steady driver the speed will be constant, at least over short intervals of time, but in the vast majority of cases the speed is a variable. The height of a tree in a windstorm is a variable. The length of a steel bar changes as it is heated or cooled. The pressure of the gas in the cylinder of an automobile varies quite violently. The distance from the moon to Mars changes in a very complicated way.

The Calculus is a branch of mathematics that deals with variable quantities, and it is here that the student has his troubles. In trigonometry the student learns to solve triangles that are fixed (at least while he is solving them), or to prove identities (that are also fixed). In Algebra he learns how to solve a fixed equation for its fixed roots, or to find the value of a certain fixed determinant, etc. In the Calculus, however, we consider a variable quantity and ask such questions as "How fast is the quantity varying?", "What is the maximum value of the quantity?", "When is it increasing, and when is it decreasing?" etc.

As a result of this enlarged viewpoint, we can use the Calculus to solve problems that are very difficult or almost impossible by the means previously at our disposal. Probably the outstanding example is the computation of the area of the region bounded by a given curve. As long as the region is *fixed*, we are powerless to find the area, as all of the mathematicians prior to Newton and Leibniz (with the possible exception of Archimedes) will testify. But once we are willing to let one boundary of

the region be variable, then the computation of the area becomes almost trivial. (See Chapter 4.)

The basic concept of the Calculus is expressed by the mysterious looking collection of symbols

$$\lim_{x \rightarrow a} y = b,$$

(read, the limit of  $y$ , as  $x$  approaches  $a$ , is  $b$ ). Here  $y$  depends on  $x$ , and as  $x$  changes  $y$  also changes. Equation (1) tells us that as  $x$  gets closer to  $a$ , then  $y$  gets closer to  $b$ . How close is  $y$  to  $b$ ? As close as we please. This means that if we select any small number, usually denoted by  $\epsilon$  (the Greek letter epsilon) then we can make  $y$  within  $\epsilon$  of  $b$ , in symbols

$$|y - b| < \epsilon, \tag{1}$$

by taking  $x$  sufficiently close to  $a$ .

To feel at home with (1) and its definition, you must have some knowledge of inequalities, and to assist you a brief treatment of this subject is given in Appendix 1. It is not necessary to master this material before starting the Calculus, but you should have some familiarity with the ideas and the symbols  $<$  and  $>$ . You should also have some feeling for the relative magnitude of quantities. For example, it should be immediately obvious that if  $x$  is very large then  $1/x$  is very small. This concept is expressed by the symbols

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0. \tag{2}$$

Here  $x$  is a variable that is growing without bound ( $x \rightarrow \infty$ ), and (2) merely states that as  $x$  grows without bound  $1/x$  gets closer and closer to zero. More examples and details will be found in §3 and 4 of Chapter 2.

Every author hopes that his book can be read by the student, and this author is no exception. The trouble is that the author, in writing his book, cannot raise his voice or pound on the table for emphasis. You must supply these stage effects for yourself while reading. The best that we can do is to put the important formulas in boxes, and to put the theorems and definitions in italics so that you can spot the essential items. It is a good idea for you to memorize all of the theorems, definitions, and boxed formulas. Memorization is not a substitute for learning, but it frequently happens that once an item is memorized, the subconscious mind will mull it over, and this will hasten eventual understanding. Naturally, learning and understanding are the ultimate goals, but memorization plays a very important role that is not properly recognized today. A child who memorizes Lincoln's Gettysburg address, when the words are meaningless will come to understand its meaning and appreciate its beauty far more quickly than the child who does not memorize it.

Most of the exercise lists contain more problems than can be done in one study session. You should be content to work a representative selection (probably the teacher will make a definite assignment) and reserve the rest to be used in review or in studying for examinations. Some problems are more difficult than others, either involving more complicated concepts or more extensive computation. We have marked such problems with stars(\*) so that you can be on guard. The double star (\*\*) means that the problem is even harder than the ones marked with a single star.

The Calculus was discovered almost simultaneously by Isaac Newton (1642–1727) in England and Gottfried Leibniz (1646–1716) in Germany. Parts of the Calculus were anticipated much earlier by Archimedes (287?–212 B.C.) in Syracuse and by Pierre Fermat (1601–1665) in France. As first presented, the material was difficult to understand, but during the second century of its life it was smoothed, polished, simplified, and extended by a host of geniuses. The leaders in this activity were Leonard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), and Augustin-Louis Cauchy (1789–1857). With such an array of mental giants contributing to the subject we should expect it to be rich, elegant, and beautiful—but not easy.

As you begin your study of the Calculus, I am tempted to say “Good luck” to you. But it is not luck, just good hard work that will see you safely through. The climb is difficult, but when you reach the peak and look back on the material covered, you will see a mathematical design of great beauty in the mountain “Calculus.” There are of course still other mathematical mountains to climb that are even higher and more difficult, and from their tops the scenery is still more beautiful. But the Calculus seems to be the one that separates the men from the boys.

*Lexington, Kentucky*

A. W. GOODMAN

# **ANALYTIC GEOMETRY AND THE CALCULUS**

# INDEX

## INDEX

- Abscissa, 2
- Absolute area, 146
- Absolute convergence, 431-432
- Absolute value, 3
- Acceleration, 87, 334  
normal, 351-355
- Addition, of series, 420  
of vectors, 313-315, 318
- Additivity, 649 (prob. 21)
- Age of the earth, 662 (prob. 4)
- Agnesi, Maria Gaetana, 328 (prob. 17)
- Air resistance, 337, 660-661, 663 (probs. 11-13)
- Algebraic area, 146, 159
- Algebraic function, 281
- Algebraic operations, 281
- Alternating series, 427-431
- Anchor ring, *see* torus
- Angle, between two curves, 214-217, 392 (probs. 7, 8, 9)  
between two lines, 214-217  
between a line and a plane, 488 (prob. 15)  
between two planes, 487 (prob. 14)  
bisector 217 (prob. 9), 473 (probs. 16, 17)  
of inclination 14
- Antidifferentiation, 89
- Applications of differential equations, 659-665
- Approximations, by differentials, 125-127, 556-558  
by infinite series, 437-441  
of integrals 444, 447 (probs. 16-19), 453 (probs. 30-33)
- Arc length, 182-185  
as parameter, 332-334, 348-351, 495  
in polar coordinates, 390-393
- Archimedes (287-212 B.C.), 128, 143 (prob. 17)  
spiral, 380, 392 (prob. 4)
- Area, 134-149, 153-164, 165-169  
axioms for, 135-136  
algebraic, 144, 159  
in polar coordinates, 393-396  
of a circular sector, 229  
of a surface, 603-608  
of revolution, 186, 189, 393 (probs. 13-18)  
parametric equations, 328 (probs. 22-25)
- Arithmetic operations, 280
- Associative law, 472, 475
- Astronaut, 355 (prob. 12)
- Asymptotes, 195-203
- Atmospheric pressure, 663 (probs. 18, 19)
- Auxiliary equation of a differential equation, 646
- Average density, 516-519
- Axes, coordinate, 2-4, 454-456  
of ellipse, 30  
of hyperbola, 32  
rotation of, 221-227  
translation of, 191-195
- Axis, of parabola, 27  
of symmetry, 24, 203
- Barrow, Isaac (1630-1677), 129
- Bell, Eric T., 4
- Bernoulli equation, 644-645 (probs. 9-13)
- Binomial series, 411 (prob. 23), 440 (prob. 4)
- Bisector of angle, 217 (prob. 9), 473 (probs. 16, 17)
- Boundary, 551

- conditions, 652 (probs. 18–21)  
 point, 552  
 Bounded sequence, 416  
 Box, 455  
 Boyle's law, 663 (prob. 18)  
 Branch of a function, 82–85, 239–242, 268  
 for  $\sec^{-1} x$ , 248–249 (probs. 31–33)
- Cardan, G. (1501–1576), 102  
 Cardioid, 377, 390  
 Carslaw, H. S., 162  
 Cartesian coordinate system, 4, 454  
 Cartesian equation, 323  
 Catenary, 519 (prob. 14)  
 Cauchy integral test, 424  
 Cauchy's inequality, 473 (prob. 13)  
 Center, of expansion, 404  
 of gravity, 511  
 invariance of, 513 (probs. 7, 8)  
 of similitude, 385 (prob. 24)  
 of symmetry, 203  
 Centroids, 519, 525  
 of curves and surfaces, 531–533  
 Chain, rule, 72, 559–561  
 sliding from a table, 664 (prob. 22)  
 Change of base in logarithm, 253 (probs.  
 12, 13), 259 (prob. 39)  
 Change of variable in a definite integral,  
 284, 291, 329 (probs. 22–25)  
 Circle, 23–25, 219–220  
 area of a sector, 229  
 parametric equations, 326 (prob. 12)  
 polar coordinate equation, 378, 381  
 (prob. 13), 385 (prob. 12)  
 unit, 356  
 Circle of curvature, 345  
 Circular helix, 494, 495  
 Closed, interval, 34, 93  
 region, 552  
 set, 552  
 Closure, 552  
 Coefficient of friction, 353  
 Commutative law, 469  
 Comparison test, 421  
 generalization, 427 (prob. 16)  
 Component, scalar, 318  
 vector, 314  
 Composite function, 71  
 Concave curve, 108  
 Conditional convergence, 431–432
- Cone, area of frustum, 186  
 equation, 502  
 moment, 529  
 volume, 175 (prob. 13)  
 Conic sections, 26, 193, 208–213, 224, 227,  
*see also ellipse, hyperbola, parabola*  
 in polar coordinates, 385–388  
 Conical helix, 497 (prob. 13)  
 Connected set, 552  
 Continuous function, 52–53, 682–684  
 of two variables, 541  
 Convergence, definition, 41  
 of a sequence, 415–420, 685–687  
 of a series, 398, 418–440  
 properties of, 41–52  
 uniform, 441–445  
 Convergence set of a series, 407, 433, 434  
 Convex curve, 393 (probs. 19, 20)  
 Cooling, Newton's law of, 665 (prob. 30)  
 Coordinate axes, 2–4, 454–456  
 rotation of, 221–227  
 translation of, 191–195  
 Coordinate system, Cartesian, 4  
 cylindrical, 504–505, 620–623  
 polar, 375–377  
 rectangular, 2–4, 454, 455  
 spherical, 505–508, 623–626  
 Cos  $x$ , series, 411 (prob. 3), 440 (prob. 2)  
 Cramer's rule, 691, 705–706  
 Critical point, 61, 582  
 Cross product, 469, 473–477, 478–480  
 Curvature, 339–346, 494, 496  
 circle of, 345  
 invariance of, 346 (probs. 17, 18)  
 radius of, 345  
 Curves, 10, 35  
 concave, 108  
 convex, 393 (prob. 19, 20)  
 families of, 217–221  
 intersection of two, 38–40  
 length of, 182–183  
 of intersection of two surfaces, 577  
 (probs. 19–28)  
 on a surface, 502, 503 (probs. 22–25)  
 parametric equations, 322–329  
 Cusp, 359  
 Cycloid, 324, 325, 337 (probs. 11, 12),  
 351 (prob. 14)  
 Cylinder, 457, 502, 529  
 Cylindrical coordinates, 504–505, 620–623  
 element of volume, 621  
 to rectangular coordinates, 504

- Decimal fractions, 403–404
- Decreasing, function, 94  
sequence, 416
- Definite integral, 140–190  
change of variables in, 284, 291, 329  
(probs. 22–25)  
definition, 140, 144, 161–162  
improper, 370–374
- Degree, of a differential equation, 630  
of a homogeneous function, 637
- Density, 516–519
- Dependent variable, 35
- Derivative, applications, 86–127  
definition, 55  
formulas, see differentiation formulas  
higher order, 78–81  
of a composite function, 71–74  
of an implicit function, 75–78  
of an integral, 274–275, 641  
of a power series, 408, 445–446  
partial, 541–544
- Descartes, René (1596–1650), 1, 4
- Descriptive properties of sets, 552
- Determinant, 104, 476–485, 688–708  
definition, 690, 693–694, 696  
for area of triangle, 477 (prob. 7)  
for cross product, 476  
for equation of a plane, 484  
for triple scalar product, 482 (prob. 3)  
expansion of, 694  
minor of, 702  
order of, 690, 696
- Development of a polynomial, 404–407  
of a function, 407–411
- Differential formulas, 126
- Differential area in polar coordinates, 610
- Differential volume, in cylindrical  
coordinates, 621  
in spherical coordinates, 624
- Differential equations, 133–134, 630–665
- Differentials, 123  
total, 561–563
- Differentiation, of series, 408, 445–446  
of vectors, 488–493
- Differentiation formulas, 126  
exponential function, 260  
inverse trigonometric functions, 243–245  
hyperbolic functions, 269  
logarithm function, 255  
power, 62, 68, 77  
product, 66, 70 (prob. 19, 20)  
quotient, 67  
trigonometric functions, 230–234
- Directed distance, 1
- Direction, angles, 460  
cosines, 461–462  
numbers, 461  
on a curve, 323
- Directional derivative, 566–569
- Directrix, 26, 208
- Discontinuous function, 53
- Discriminant, 224
- Disk method, for volumes, 169–170  
for moment of inertia, 530
- Distance, 4  
between two lines, 481, 482, 587 (prob. 15)  
between two planes, 487 (prob. 6)  
between two points, 7, 460, 462–463  
in  $n$ -dimensional space, 553 (prob. 18)  
from point to line, 480, 487 (prob. 13)  
from point to plane, 480, 481, 485, 486,  
586 (prob. 14)
- Distributive law, for dot product, 471, 472  
(probs. 7, 8)  
for cross product, 475, 479–480
- Divergence, of a sequence, 415  
of a series, 398–399
- Division of power series, 449
- Dot (scalar) product, 469–473
- Double integrals, 595–601  
applications, 601–607  
in polar coordinates, 608–613
- Duhamel's Theorem, 188
- Electrical circuits, 664 (probs. 23–26)
- Eccentricity, 208–212
- Ellipse, area, 329 (prob. 24)  
as a conic section, 26  
axes, 28–29  
definition, 28, 208  
equation, 29, 208–210  
length of arc, 351 (prob. 16)  
polar coordinates, 385–388  
test for, 224  
vertices, 28
- Ellipsoid, 500  
oblate, prolate, 503 (prob. 13)  
volume of, 175 (probs. 15, 16)
- Elements of a cylinder, 457
- Epicycloid, 329 (prob. 26)
- Equation of a graph, 12
- Equation of a line, in polar coordinates,  
385 (prob. 11)

- in space, 464–468
- intercept, 20 (prob. 11)
- point-slope, 16, 17
- slope-intercept, 19
- two-point, 18
- vector, 326 (prob. 11)
- Equation of a plane, 483–485, 497
- Equations, equivalent, 204
- Equiangular spiral, 392 (prob. 5)
- Equivalent equations, 204
- Errors, 556–558
- Escape velocity, 663 (prob. 17)
- Euler, Leonhard (1707–1783), 673
- Euler's formula, 647–648
  - for polyhedrons, 708 (prob. 9)
- Even function, 266
- Exact differential equation, 640–643
- Excluded regions, 206–207
- Expansion, of a function, 407–412
  - of a polynomial, 404–407
- Exponential function, 250–251, 260–263
  - series for, 410, 437, 439, 440
- Exponents, laws of, 250–251
- Extraneous roots, 39
- Extreme values, 96–100, 114–122
  - in several variables, 580–588
  - sufficient condition, 583–586
- Families of curves, 217–221, 631–634
- Fermat, Pierre (1601–1665), 673
- Fluid pressure, 176–178, 181–182, 535–537
- Focus, 26, 208
- Force, 312, 314
- Fractions, 403–404
- Friction, 353
- Frustum of a cone, 186
- Functions, 34–38
  - composite, 71
  - continuous, 36, 52–54, 682–684
  - discontinuous, 36, 53
  - implicit, 75–78, 571–577
  - increasing, decreasing, 93–96
  - inverse, 81–85, 238–249, 251, 266–268
  - notation, 36–37
  - of several variables, 540–545
- Fundamental theorem of the calculus, 163
- Gauss, Carl F., 294
- General solution of a differential equation, 631
- Geometric, area, 146
  - series, 401, 407
- Generalized mean value theorem, 359
- Gradient, 569–571
- Graph, 10–12, 84
- Gravitational attraction, 626–629
- Gravity, 89–91, 337
- Growth and decay, 659, 662
- Gyration, radius of, 526
- Half-life, 662 (probs. 2–4)
- Half-plane, 206
- Hardy, G. H., 101
- Harmonic, function, 544 (probs. 15, 16), 642 (prob. 9)
  - series, 421, 423, 426
- Helix, circular, 494, 495
  - conical, 497 (prob. 13)
- High point, on a curve, 61
  - on a surface, 569 (probs. 4, 5)
- Homogeneity, 649 (prob. 20)
- Homogeneous, differential equation, 638–640, 645
  - function, 637–638
  - material, 516–519
- Hyperbola, as a conic section, 26
  - axis, 32, 34 (prob. 21)
  - definition, 30, 208
  - equation, 31, 210, 211
  - in polar coordinates, 385–388
  - test for, 224
  - unit, 356
  - vertices, 32
- Hyperbolic functions, 263–272
  - definition, 263
  - differentiation, 269
  - graphs, 265
  - geometric interpretation, 355–357
- Hyperbolic paraboloid, 501
- Hyperbolic substitutions, 292–293 (probs. 18–23)
- Hyperboloid, of one sheet, 500–501
  - of two sheets, 503 (prob. 14)
- Hypocycloid, 328, 329 (probs. 18–20), 351 (prob. 13)
- Implicit functions, 75–78, 571–577
- Improper integrals, 370–374, 424
- Increasing, function, 93–96
  - sequence, 416

- Increment in two variables, 553–554  
 Indefinite integral, 129  
 Independent, constants, 130, 631  
     variable, 35  
 Indeterminate forms, 358–370  
 Induction, 63, 150, 153, 672–678  
 Inequalities, 589–591, 667–671  
 Infinite series, 397–453, *see* series  
 Infinity, 49  
     order of, 74 (prob. 10)  
 Inflection point, 110  
 Initial conditions, 131, 134 (probs. 13–21),  
     659 (probs. 20–25)  
 Integral, 89, 103  
     indefinite, 129  
     definite, 140, 144, 161–162  
     improper, 370–374  
 Integral curve, 631  
 Integrand, 129  
 Integrating factor, 643 (probs. 10–15)  
 Integration, 89, 103  
     limits of, 140  
     of series, 413, 444  
 Integration formulas, collection 282–283  
     exponential function, 261  
     hyperbolic functions, 270  
     logarithm function, 257  
     trigonometric functions, 231, 235, 258  
     table, 719–728  
 Integration methods, 280–309  
     algebraic substitution, 283–285  
     hyperbolic substitution, 292–293 (probs.  
         18–23)  
     partial fractions, 293–301  
     parts, 301–304  
     trigonometric integrand, 285–289, 304–  
         307  
     trigonometric substitutions, 289–292  
     undetermined coefficients, 307–309  
 Intercept, 18, 19  
     equation of a line, 20 (prob. 11)  
 Interior point, 552  
 Intersection, of two curves, 38–40  
     curve of two surfaces, 577 (probs. 19–28)  
 Interval, 34, 93  
 Invariance of, center of gravity, 513 (probs.  
     7, 8)  
     curvature, 346 (probs. 17, 18)  
 Invariant quantities, 224, 227 (probs.  
     12–16), 343, 346 (probs. 17, 18)  
 Inverse functions, 81–85, 238–249, 251,  
     266–268  
 Inverse relation, 238  
 Inversions in a permutation, 694, 695  
     (prob. 5)  
 Involute of a circle, 328 (prob. 16), 335  
 Iterated integrals, 591–592, 615–620  
     inversion of order, 599  
 Knopp, K., 447  
 Krypton, 662 (prob. 3)  
 Landau, Edmund, 101  
 Laplace's equation, 544 (probs. 15, 16),  
     642 (prob. 9)  
     in polar coordinates, 564 (probs. 18, 19)  
 Laws, of exponents, 250–251  
     of logarithms, 252, 253 (probs. 12, 13),  
         259 (prob. 39)  
 Left-hand derivative, 58 (prob. 9)  
 Leibniz, Gottfried (1646–1716), 129  
 Lemniscate, 393 (prob. 14), 396 (prob. 7)  
 Length of a curve, 182, 183, 494  
     as a parameter, 332–334, 348–351, 495  
     in polar coordinates, 390–393  
 Length of a vector, 313  
 Level, curves, 572  
     surfaces, 574  
 L'Hospital's rule, 214 (prob. 22), 361, 363,  
     364  
 Light, 216 (probs. 7, 8), 291 (prob. 6)  
 Limacon, 396 (prob. 8)  
 Limit, 44–52, 679–682  
     evaluation by L'Hospital's rule, 358–370  
     of a sequence, 415  
     of a vector function, 330, 337 (probs.  
         14–17)  
 Line, 9, 10, *see* equation of a line  
     motion on, 86–91  
 Linear differential equation, first order,  
     643–645  
     higher order, 645–654  
     with constant coefficients, 645–654  
 Lines, parallel, 21  
     perpendicular, 21, *see* normal, and  
     orthogonal  
 Locus, 10  
 Logarithmic differentiation, 256, 262  
 Logarithmic function, definition, 251, 275  
     derivative, 254, 255  
     series for, 411 (probs. 4, 5), 440 (prob.  
         3), 446 (probs. 1, 2, 3, 10)