ANALYTIC GEOMETRY AND THE CALCULUS

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PREFACE FOR THE TEACHER

Recent textbooks on the Calculus have generally three common characteristics: (a) the inclusion of analytic geometry as an integrated part of the course, (b) the use of vectors, and (c) the introduction of more rigor. There are, I believe, good reasons for (a) and (b), and since these are now in style there is no need to repeat here the arguments in their favor. As the title indicates, this book is a unified treatment of Analytic Geometry and the Calculus. Further, we introduce vectors as early as possible and use vectors whenever we can.

But item (c) is quite different. It seems to me that a really rigorous book on the Calculus would begin with the axioms of set theory, derive the Peano axioms, and then reproduce most of Landau's great *Foundations of Analysis*. After this preparation, the class (if there are any students left) would have no trouble following the proofs needed in the Calculus. Since a completely rigorous presentation of the Calculus is obviously neither practical nor desirable, the central question is which theorems should be proved and which should be left to the intuition of the student. It is my hope that this book comes somewhere near the correct answer for the majority of students.

I have attempted to encourage the proper attitude toward rigor by stating definitions and theorems clearly. In this connection, I contend that one should not try to state all of the hypotheses in a theorem, because the statement can become so long as to be incomprehensible to the average student. Certain hypotheses are tacitly understood by both author and reader and hence, for brevity, should be omitted. As an illustration of the problems involved, consider the following:

THEOREM *S*. *If* $y = f(u)$ *and* $u = g(x)$, *then the derivative of the composite function* $y = f(g(x))$ *is given by*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Here is a statement that is brief and simple, and the average student has a reasonable chance of understanding it. Now let us look at the same theorem when stated in a rigorous fashion.

THEOREM *R*. *Let f and g be two real-valued functions of a real variable and suppose that the range of g is a subset of the domain of f. Let h = fg be the composite function defined over the domain of g by setting $h(x) = f(g(x))$ for each x in the domain of g. If x_0 is an interior point of an interval contained in the domain of g, and g is a differentiable function at x_0, and if f is a differentiable function at $u_0 = g(x_0)$, then h is a differentiable function at x_0, and further the derivative is given by the formula*

$$h'(x_0) = f'(u_0)u'(x_0).$$

There is no doubt that R is the correct statement and S is full of gaps. However, the average student can learn and use *S*, but when *R* is presented he will either fall asleep or totally ignore it. It is just too complicated for him to master at this stage of his mathematics study. The presentation of *R* rather than *S* does real harm because it serves to repel many students who are originally attracted to mathematics and who might turn out to be capable technicians or teachers (if not creative thinkers) if they are given a reasonable chance to develop.

In this book we always give the short, simple, attractive, and perhaps erroneous statement in preference to the long, complicated, unattractive, but certainly correct version. For the very bright student (say one out of every twenty-five in the average university) who wants a rigorous treatment, it is a simple matter for the teacher to assign additional outside reading, such as Landau's *Foundations of Analysis*, his *Differential and Integral Calculus*, Hardy's *Pure Mathematics*, the two-volume work by Courant and McShane, *Differential and Integral Calculus*, Blachie and Don, Std. London, 1934, or the teacher's favorite book on Advanced Calculus.

The content of a course in Analytic Geometry and the Calculus, is reasonably standard, and a glance at the table of contents of this book will show that all of the essentials have been included, and these items do not require any further discussion. However, this book does have several minor innovations, which are listed here for convenience, together with a word or two of explanation.

1. The appendix contains short chapters on (a) mathematical induction, (b) inequalities, and (c) determinants. A student beginning the Calculus should know this material. It is included for the convenience and use of the poorly prepared student, but it is relegated to the appendix in order to avoid breaking into the natural sequence of ideas in the text proper.

2. Chapter 4 contains a detailed explanation of the summation notation and a list of problems devoted exclusively to this topic.

3. Centroids and moments of inertia have been postponed until Chapter 15. It has been my experience that students always have difficulty with this material, and the trouble may well disappear if we first allow them to develop more maturity. As a bonus for this postponement, we are able to amuse the student with a collection of weighted points that does not possess a center of gravity.

4. Chapter 11 contains some material on the computation of integrals by the method of undetermined coefficients.

5. In Chapter 13, on infinite series, we begin by stressing the computation and use of series, and postpone most of the theorems and proofs till the later part of the chapter. In this way the student can see the need for, and value of, the theorems before he tackles the proofs.

6. Chapter 16 includes a very brief introduction to the descriptive properties of point sets, together with a list of problems.

All topics are important in mathematics, but it must be admitted that some are more important than others. In order to devote enough space to a decent and detailed development of the essential ideas, and still hold this book to a reasonable length, it was necessary to omit certain items. The following topics were selected (with deep regrets and apologies) for exclusion: (a) equations of the angle bisectors for a pair of lines, (b) the radical axis of two circles, (c) average value of a function, (d) Kepler's laws of planetary motion, (e) contour lines as a method of graphing surfaces, and (f) line integrals and work.

Although the physical labor of organizing and writing this book was mine alone, it is obvious that any merit the book may have is ultimately due to the many persons who from every direction extended their helping hands. Just the list of all the mathematicians who have taught, guided, and inspired me is already too long for inclusion here, but the three who have had the deepest influence on my work are Otto Szasz, Paul Erdös, and Hans Rademacher. Unfortunately Otto Szasz is no longer with us, and Paul Erdös has been barred from entering this country for the last eight years, much to the detriment of our own young mathematical scholars. I must also acknowledge my great debt to my parents who understood and encouraged me in my efforts to study mathematics. Finally, special mention is due my wife, who while raising a family found the time and energy to type the entire manuscript of this book, and two other books as well.

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PREFACE FOR THE STUDENT

The Calculus is not an easy subject, and yet every year thousands of students manage to master it. You can do the same. The real difficulty is that the Calculus deals with *variables.* Because it is hard to give a precise definition of a variable this word is at present in ill-favor with some mathematicians. But the concept is easily understood by anyone who keeps his eyes and mind open while observing nature in action. For example, the distance of a car from some fixed reference point is a constant if the car is not moving, but the distance is a variable as soon as the car is in use. In the hands of a steady driver the speed will be constant, at least over short intervals of time, but in the vast majority of cases the speed is a variable. The height of a tree in a windstorm is a variable. The length of a steel bar changes as it is heated or cooled. The pressure of the gas in the cylinder of an automobile varies quite violently. The distance from the moon to Mars changes in a very complicated way.

The Calculus is a branch of mathematics that deals with variable quantities, and it is here that the student has his troubles. In trigonometry the student learns to solve triangles that are fixed (at least while he is solving them), or to prove identities (that are also fixed). In Algebra he learns how to solve a fixed equation for its fixed roots, or to find the value of a certain fixed determinant, etc. In the Calculus, however, we consider a variable quantity and ask such questions as "How fast is the quantity varying?", "What is the maximum value of the quantity?", "When is it increasing, and when is it decreasing?," etc.

As a result of this enlarged viewpoint, we can use the Calculus to solve problems that are very difficult or almost impossible by the means previously at our disposal. Probably the outstanding example is the computation of the area of the region bounded by a given curve. As long as the region is *fixed*, we are powerless to find the area, as all of the mathematicians prior to Newton and Leibniz (with the possible exception of Archimedes) will testify. But once we are willing to let one boundary of the region be variable, then the computation of the area becomes almost trivial. (See Chapter 4.)

The basic concept of the Calculus is expressed by the mysterious looking collection of symbols

$$\lim_{x \to a} y = b,$$

(read, the limit of y, as x approaches a, is b). Here y depends on x, and as x changes y also changes. Equation (1) tells us that as x gets closer to a, then y gets closer to b. How close is y to b? As close as we please. This means that if we select any small number, usually denoted by ϵ (the Greek letter epsilon) then we can make y within ϵ of b, in symbols

$$|y - b| < \epsilon, \tag{1}$$

by taking x sufficiently close to a.

To feel at home with (1) and its definition, you must have some knowledge of inequalities, and to assist you a brief treatment of this subject is given in Appendix 1. It is not necessary to master this material before starting the Calculus, but you should have some familiarity with the ideas and the symbols $<$ and $>$. You should also have some feeling for the relative magnitude of quantities. For example, it should be immediately obvious that if x is very large then $1/x$ is very small. This concept is expressed by the symbols

$$\lim_{x \to \infty} \frac{1}{x} = 0. \tag{2}$$

Here x is a variable that is growing without bound ($x \to \infty$), and (2) merely states that as x grows without bound $1/x$ gets closer and closer to zero. More examples and details will be found in §3 and 4 of Chapter 2.

Every author hopes that his book can be read by the student, and this author is no exception. The trouble is that the author, in writing his book, cannot raise his voice or pound on the table for emphasis. You must supply these stage effects for yourself while reading. The best that we can do is to put the important formulas in boxes, and to put the theorems and definitions in italics so that you can spot the essential items. It is a good idea for you to memorize all of the theorems, definitions, and boxed formulas. Memorization is not a substitute for learning, but it frequently happens that once an item is memorized, the subconscious mind will mull it over, and this will hasten eventual understanding. Naturally, learning and understanding are the ultimate goals, but memorization plays a very important role that is not properly recognized today. A child who memorizes Lincoln's Gettysburg address, when the words are meaningless will come to understand its meaning and appreciate its beauty far more quickly than the child who does not memorize it.

Most of the exercise lists contain more problems than can be done in one study session. You should be content to work a representative selection (probably the teacher will make a definite assignment) and reserve the rest to be used in review or in studying for examinations. Some problems are more difficult than others, either involving more complicated concepts or more extensive computation. We have marked such problems with stars(*) so that you can be on guard. The double star (**) means that the problem is even harder than the ones marked with a single star.

The Calculus was discovered almost simultaneously by Isaac Newton (1642–1727) in England and Gottfried Leibniz (1646–1716) in Germany. Parts of the Calculus were anticipated much earlier by Archimedes (287?–212 B.C.) in Syracuse and by Pierre Fermat (1601–1665) in France. As first presented, the material was difficult to understand, but during the second century of its life it was smoothed, polished, simplified, and extended by a host of geniuses. The leaders in this activity were Leonard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), and Augustin-Louis Cauchy (1789–1857). With such an array of mental giants contributing to the subject we should expect it to be rich, elegant, and beautiful—but not easy.

As you begin your study of the Calculus, I am tempted to say "Good luck" to you. But it is not luck, just good hard work that will see you safely through. The climb is difficult, but when you reach the peak and look back on the material covered, you will see a mathematical design of great beauty in the mountain "Calculus." There are of course still other mathematical mountains to climb that are even higher and more difficult, and from their tops the scenery is still more beautiful. But the Calculus seems to be the one that separates the men from the boys.

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