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André Unterberger

Alternative Pseudodifferential Analysis

With an Application to Modular Forms

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To François Treves

Preface

The subject of the present work is pseudodifferential analysis: the motivations lie in harmonic analysis and modular form theory. So far as the last two domains are concerned, nothing more than some minimal familiarity is needed: some knowledge of the metaplectic representation, and of the definition of holomorphic and nonholomorphic modular forms, will help. Even though the symbolic calculus introduced here is entirely new, and does not depend on any technical result concerning pseudodifferential operators, it would not be honest to claim that no previous acquaintance with that field is necessary: the analysis developed here is strikingly different from the usual one, some knowledge of which – in particular, its representation-theoretic aspects – is needed for comparison.

Modular form theory is a very appealing subject: some time ago already, we tried to approach it from an angle which, to us, was much more familiar, that of pseudodifferential analysis. It is possible to realize nonholomorphic modular forms as distributions in the plane [35, Sect. 18], the main benefit being that they can then be considered as symbols for a calculus of the usual species, to wit the Weyl calculus. Yes, there are difficulties on the way toward developing the symbolic calculus of associated operators, since distributions on $\mathbb{R}^2$ which correspond to modular forms, though beautiful objects from the point of view of arithmetic, are extremely singular. Still, one can survive these difficulties, as shown in [36].

Only the nonholomorphic modular form theory could be reached in this way. Needless to say, we tried to incorporate holomorphic modular form theory as well: this cannot work to a full extent, and the best one can do in this direction will be summed up in Sect. 5.2 of the present work. Then, in an independent piece of work [38], partly motivated by Physics, we introduced the "new" anaplectic analysis – like many new things, it is only a coherent rearrangement of old ones – and it turned out, to our unanticipated satisfaction, that this solved our old problem.

Only one-dimensional anaplectic analysis will concern us here – the higher-dimensional case is considerably harder – and, of course, we are not assuming that the reader has read, or borrowed, our book on the subject. It is our opinion that the version presented here, in Sects. 2.2 and 4.1, in which no proofs are given, will make easy reading. Though our main current interest in anaplectic analysis lies with Physics, it is clear, to us, that the approach to holomorphic modular form theory it leads to deserves to be explored further.

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Chapter 1
Introduction

The object of the present work is to introduce a new "pseudodifferential" analysis on the real line. This alternative analysis is endowed with just as many symmetries as the usual one, mostly of a different nature. It is a necessary counterpart of the celebrated Weyl calculus, so far as certain matters pertaining to representation theory or modular form theory are concerned. Whether it may be of any use, in the foreseeable future, in partial differential equations, the most important domain of applications of pseudodifferential analysis, is much more questionable, though we shall try to give some minimal hints regarding related possibilities.

Though it is not our intention to discuss at length the well-understood harmonic–analytic aspects of the Weyl calculus, recalling some of its basic definitions and properties may be of help to some readers: also, it will facilitate the comprehension of the main features of alternative pseudodifferential analysis, which parallel the corresponding ones from the Weyl calculus but violently contrast with these most of the time. Finally, most of our readers are probably not familiar with the relation of the Weyl calculus to modular form theory, an important topic toward our present purpose.

One-dimensional pseudodifferential analysis starts with a way of representing linear operators acting on functions of one variable by functions of two variables. If $h \in \mathcal{S}(\mathbb{R}^2)$, the operator $\mathrm{Op}(h)$, called the operator with Weyl symbol h [42], is the operator: $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined by the equation

$$(\mathrm{Op}(h)\,u)(x) = \int_{\mathbb{R}^2} h(\frac{x+y}{2}, \eta)\, u(y)\, e^{2i\pi(x-y)\eta} dy\, d\eta. \tag{1.1.1}$$

It is easily seen (writing its integral kernel) that, under the sole assumption that $h \in L^2(\mathbb{R}^2)$, one can define the operator $\mathrm{Op}(h)$ as a Hilbert–Schmidt operator in $L^2(\mathbb{R})$. It is important, especially in the arithmetic (automorphic) case, to use the elementary fact that one can still define $\mathrm{Op}(\mathfrak{S})$ as a linear operator from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ if $\mathfrak{S}$ is an arbitrary distribution in $\mathcal{S}'(\mathbb{R}^2)$.

As will be recalled in detail in Sect. 2.1, there is a unique (up to isomorphism) pair $(\widetilde{SL}(2,\mathbb{R}), \iota)$, where $\widetilde{SL}(2,\mathbb{R})$ is a connected Lie group and ι is a homomorphism from $\widetilde{SL}(2,\mathbb{R})$ onto $SL(2,\mathbb{R})$ with a kernel reducing to two elements. Moreover, there is a certain unitary representation $\text{Met}^{(1)}$, called the metaplectic representation, of $\widetilde{SL}(2,\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R})$, one of the main features of which is that, for some well-chosen $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$ lying above the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator $\text{Met}^{(1)}(\tilde{g})$ is $e^{-\frac{i\pi}{4}}$ times the Fourier transformation. Up to the multiplication by ± 1, any operator $\text{Met}^{(1)}(\tilde{g})$ with $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$ depends only on the image g of $\tilde{g}$ in $SL(2,\mathbb{R})$. All necessary details regarding this representation are to be found in Sect. 2.1. One then has the fundamental covariance formula

$$\text{Met}^{(1)}(\tilde{g})\,\text{Op}(\mathfrak{S})\,\text{Met}^{(1)}(\tilde{g})^{-1} = \text{Op}(\mathfrak{S} \circ g^{-1}) : \tag{1.1.2}$$

that the left-hand side makes sense depends on the fact that a transformation such as $\text{Met}^{(1)}(\tilde{g})$ is also well defined as an automorphism of $\mathcal{S}(\mathbb{R})$ or as an automorphism of $\mathcal{S}'(\mathbb{R})$.

There is another covariance formula, actually involving the Heisenberg representation, which can be summed up in the following terms: if one sets $(\tau_{y,\eta} u)(x) = u(x-y)\, e^{2i\pi(x-\frac{y}{2})\eta}$, one has

$$\tau_{y,\eta}\ \text{Op}(\mathfrak{S})\ \tau_{y,\eta}^{-1} = \text{Op}((x,\xi) \mapsto \mathfrak{S}(x-y,\, \xi-\eta)) \tag{1.1.3}$$

for every $(y,\eta) \in \mathbb{R}^2$.

These two covariance formulas lead to two quite different composition formulas. Given two symbols h_1, h_2 lying in $L^2(\mathbb{R}^2)$ (there is a considerable variety of other possibilities, which makes part of the tool kit of pseudodifferential analysis), the operator $\text{Op}(h_1)\,\text{Op}(h_2)$, as a Hilbert–Schmidt operator, can be written as $\text{Op}(h_1 \# h_2)$ for a unique symbol $h_1 \# h_2 \in L^2(\mathbb{R}^2)$. The sharp composition of symbols can be analyzed in combination with the decomposition of functions in $\mathbb{R}^2$ under the representation involved on the right-hand side of (1.1.2) or (1.1.3). In the second case, we are dealing with the representation of the commutative group $\mathbb{R}^2$ acting by translations on $L^2(\mathbb{R}^2)$: the differential operators on $\mathbb{R}^2$ which commute with this action are the differential operators with constant coefficients, the joint (generalized) eigenfunctions of which are the functions $X = (x,\xi) \mapsto e^{2i\pi\langle A,X\rangle}$ with $A \in \mathbb{R}^2$. Then, the formula we are looking for is simply

$$e^{2i\pi\langle A,X\rangle} \# e^{2i\pi\langle B,X\rangle} = e^{i\pi[A,B]}\, e^{2i\pi\langle A+B,X\rangle}, \tag{1.1.4}$$

where we have introduced the so-called symplectic form $[\ ,\ ]$ such that

$$[A,B] = -a\beta + b\alpha \qquad \text{if } A=(a,\alpha),\, B=(b,\beta). \tag{1.1.5}$$

When making the operator $\text{Op}(e^{2i\pi\langle A,X\rangle})$ explicit as $\tau_{a,\alpha}$, this formula is sometimes called the Weyl exponential version of Heisenberg's relation. More important to our purpose, when coupled with the decomposition of general symbols in $L^2(\mathbb{R}^2)$ into functions $X \mapsto e^{2i\pi\langle A,X\rangle}$ (this decomposition is nothing else than the standard Fourier transformation in $\mathbb{R}^2$), it leads to the quite well-known formula

$$(h_1 \# h_2)(X) = 4 \int_{\mathbb{R}^4} h_1(Y)\, h_2(Z)\, e^{-4i\pi[Y-X,Z-X]}\, dY\, dZ \tag{1.1.6}$$

or, taking advantage of the unitary group generated (Stone's theorem) by the operator $\mathcal{L}$ on $L^2(\mathbb{R}^4)$ such that

$$i\pi\mathcal{L} = (4i\pi)^{-1}\left(-\frac{\partial^2}{\partial y\,\partial\zeta}+\frac{\partial^2}{\partial z\,\partial\eta}\right) \quad \text{if } (Y;Z) = ((y,\eta);(z,\zeta)), \tag{1.1.7}$$

to the fully equivalent formula

$$(h_1\#h_2)(X) = \{e^{i\pi\mathcal{L}}(h_1(X+Y)\,h_2(X+Z))\}\,(Y=Z=0). \tag{1.1.8}$$

Expanding the exponential into a series, one obtains the so-called Moyal formula

$$(h_1\#h_2)(X)$$
$$\sim \sum_{n\geq 0}(4i\pi)^{-n}\sum_{j+k=n}\frac{(-1)^j}{j!k!}\left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial \xi}\right)^k h_1(x,\xi)\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial \xi}\right)^j h_2(x,\xi). \tag{1.1.9}$$

The right-hand side reduces to a finite sum and yields an exact formula, in the case when one is dealing with symbols of differential operators (of course, these are not Hilbert–Schmidt), since these are polynomial with respect to the second variable. That the formula is still correct in some suitable asymptotic sense, when h_1 and h_2 lie in appropriate (nonpolynomial) classes of symbols, can be proved to be true in a variety of cases, and is essential in applications of pseudodifferential analysis to partial differential equations.

The sharp composition of symbols reveals a quite different structure when emphasis is put on the first covariance formula (1.1.2). The algebra of differential operators on $\mathbb{R}^2$ which commute with the action of $SL(2,\mathbb{R})$ by linear changes of coordinates is generated by the single Euler operator

$$\mathcal{E} = (2i\pi)^{-1}\left(x\frac{\partial}{\partial x}+\xi\frac{\partial}{\partial \xi}+1\right), \tag{1.1.10}$$

the generalized eigenfunctions of which are just the functions on $\mathbb{R}^2\backslash\{0\}$ homogeneous of degrees $-1-i\lambda$, $\lambda\in\mathbb{R}$. Hence, there is an *exact* composition formula enabling one to expand $h_1\#h_2$, under the assumption that h_1 (resp. h_2) is homogeneous of degree $-1-i\lambda_1$ (resp. $-1-i\lambda_2$), as the integral with respect to $\lambda\in\mathbb{R}$ of a family of functions homogeneous of degrees $-1-i\lambda$. This formula can be found in [36, Theorem 17.1] and we shall not reproduce it here: let us only mention that it is more difficult to handle than any of the previous versions, since it involves the consideration of operators on the line with singular integral kernels. But the question is not which formula is more pleasant or easier to use: rather, what is the range of applicability of each.

For our present purpose, it is definitely the second (more complicated) formula which is important for comparison, and this will take us to our last topic discussed in connection with the Weyl calculus, to wit that of modular forms. Before coming to it, let us remind the reader that the so-called quasiregular representation of $SL(2,\mathbb{R})$ in the space $L^2_{\text{even}}(\mathbb{R}^2)$ is the one such that $(g.h)(x,\xi) = h(g^{-1}(x,\xi))$. It decomposes [10] as the direct integral of irreducible representations $\pi_{i\lambda}$, $\lambda \in \mathbb{R}$, the decomposition corresponding precisely to the decomposition of an even function $h \in L^2(\mathbb{R}^2)$ into its homogeneous components. Also, $\pi_{i\lambda}$ and $\pi_{-i\lambda}$ are unitarily equivalent, the intertwining operator being in this realization the symplectic Fourier transformation (the one with integral kernel $((x,\xi);(y,\eta)) \mapsto e^{2i\pi(x\eta - y\xi)}$, which commutes with all linear transformations of $\mathbb{R}^2$ associated with matrices in $SL(2,\mathbb{R})$, not only those in $SO(2)$). Each representation $\pi_{i\lambda}$ has another important realization in a Hilbert space of eigenfunctions, in the hyperbolic upper half-plane $\Pi \simeq SL(2,\mathbb{R})/SO(2)$, of the hyperbolic Laplacian Δ. Now, the Weyl calculus provides a simple characterization of even functions or tempered distributions on $\mathbb{R}^2$ by means of pairs of functions in Π, as follows: consider the first two (normalized) eigenfunctions u_i and u_i^1 of the harmonic oscillator $L = \pi\left(x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}\right)$ on the line: explicitly, $u_i(x) = 2^{\frac{1}{4}} e^{-\pi x^2}$ and $u_i^1(x) = 2^{\frac{5}{4}} \pi^{\frac{1}{2}} x e^{-\pi x^2}$. Given some $\tilde{g}$ lying above $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, the image of u_i or u_i^1 under the transformation $\mathrm{Met}(\tilde{g})$ only depends, up to the multiplication by some complex number of absolute value 1, on the point $z = \frac{ai+b}{ci+d}$. It is then possible, with some more or less arbitrary choice of the "phase factors," to define two families $(u_z)_{z\in\Pi}$ and $(u_z^1)_{z\in\Pi}$, coinciding with u_i and u_i^1 at $z = i$, each family being essentially permuted under any transformation in the image of the metaplectic representation. One can then characterize [36, p. 16] a symbol $\mathfrak{S} \in \mathcal{S}'_{\text{even}}(\mathbb{R}^2)$ by the pair (f_0, f_1) of functions on Π such that

$$\begin{aligned} f_0(z) &= (u_z \,|\, \mathrm{Op}(\mathfrak{S})\, u_z), \\ f_1(z) &= (u_z^1 \,|\, \mathrm{Op}(\mathfrak{S})\, u_z^1). \end{aligned} \tag{1.1.11}$$

Now, under each of these transfers, the quasiregular action of $SL(2,\mathbb{R})$ in $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$ transforms to the quasiregular action of this group on functions defined on Π: the difference is that, on Π, elements of G act by fractional-linear changes of the (complex) coordinate rather than linear ones. Besides, the operator $\pi^2\mathcal{E}^2$ transforms to $\Delta - \frac{1}{4}$, so that, from [10] again, the transfers under study intertwine the two classical realizations of the principal series $(\pi_{i\lambda})$.

This is especially interesting in the automorphic situation, i.e., when the distributions $\mathfrak{S}$ under study are invariant under the linear changes of coordinates in $\mathbb{R}^2$ associated with matrices in a given arithmetic group, say $SL(2,\mathbb{Z})$: then, the pair (f_0, f_1) consists of automorphic functions and can be identified [36, p. 30] with a pair of Cauchy data for the Lax–Phillips scattering theory relative to the automorphic wave equation [18]. That pairs of automorphic functions have to be considered reflects the fact that the concept of automorphic distribution in $\mathbb{R}^2$ is slightly more precise than that of automorphic function in Π: for instance, the two nonholomorphic Eisenstein series $E_{\frac{1\pm i\lambda}{2}}$ (cf. any book on nonholomorphic modular form theory) are proportional, whereas the two *Eisenstein distributions* they come from are only