ELEMENTARY DIFFERENTIAL EQUATIONS

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INTERNATIONAL TEXTBOOK COMPANY
Scranton, Pennsylvania

INTERNATIONAL TEXTBOOKS IN MATHEMATICS

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Preface

This text is intended for use in a first course in ordinary differential equations following elementary calculus and is designed to meet a variety of needs. Where a minimal coverage of techniques is the objective in a one-semester, three-hour course, Chapters 1 through 6 with all the starred material omitted should serve. At the other extreme, a serious and longer course, with some attention to fine points, may be based on the full contents of the book, including starred material. It is hoped that the book will be well suited for use in institutions where a course in differential equations is generally required, as is the case in some engineering schools.

Since there is an increasing tendency to pay attention to individual differences among students (e.g., through ability grouping by course sections or within course sections), some instructors may wish to have enrichment materials available for their more capable students. The starred subsections and accompanying starred exercises should in many cases help to fill this need.

Where possible we have attempted, even in the main portions of the text, to inject some of the modern spirit into the presentation, while at the same time respecting fully valid classical objectives centering about techniques and skills. For example, in the solution of $x\,dy + y\,dx = 0$, we deviate from the traditional practice of dividing by xy and subsequently pretending that the division had not occurred. Again, we have tried to stress what is obvious but usually avoided, that a function is not specified until its domain is given; accordingly, the search for a function (e.g., a solution of a differential equation) entails also the search for its domain.

Despite an apparent decrease in interest in operators, we have elected to include them, even with some emphasis, for four reasons: 1) they simplify the development of the theory of linear equations both conceptually and notationally; 2) they aid in the presentation of our point of view toward applications in Chapter 6; 3) many students are stimulated, and thus better motivated, by contact with the theory of operators, probably because of the high level of precision and clarity which characterizes it; and 4) the theory of operators can be especially profitable for students who have already been exposed, perhaps even in high school, to some abstract mathematics.

The Laplace transform has been included in such a way that it can be omitted entirely, touched upon briefly, or emphasized, according to the desires of the instructor. In Chapter 7 we have included a section on Fourier series primarily because the elements of this topic are needed early by engineering students but are not normally accessible in elementary texts. In this chapter and elsewhere we have freely used the summation notation. Our experience has shown that students at this level can become thoroughly familiar with it and skilled in its use; we believe that it can help students to develop good habits of expression.

The primary reason, in our opinion, for including applications in elementary mathematics books is motivation. Yet it is senseless to try to apply mathematics before having some mathematics to apply. Consequently, we feel that to intersperse applications throughout the book would have produced greater harm than benefit. We have therefore segregated them and placed them in two chapters, 3 and 6.

In selecting exercises, we have been guided by educational objectives and not by an urge to outdo other authors in respect to quantity. We have endeavored to meet the demands posed by individual instructors' tastes, while avoiding long lists of repetitious "busy-work" problems. Supplementary groups at the ends of Chapters 2 and 5 are included in order to give students an opportunity to develop judgment in the selection of an appropriate or "best" method from among many that are available. Similar groups in Chapters 3 and 6 may meet the desires of instructors who wish to stress applications.

Answers to odd-numbered exercises appear in Appendix 4; similar answers to even-numbered exercises are available separately on special order by faculty members. Where approximate numerical answers are given, they have been computed with the help of seven-place tables to ensure accuracy; use of four- or five-place tables may thus result in some discrepancies. A reasonably complete table of Laplace transforms and some miscellaneous reference materials which may be helpful to the student appear in Appendices 1, 2, 3.

We are grateful for the superb cooperation and understanding of International Textbook Company during all stages of work on the book—from the inception of the writing project, through preparation of the manuscript, and finally publication. We owe much to many colleagues who aided us directly or indirectly; outstanding among these is R. A. Struble, who was associated with our project for a time, and who contributed substantially to the development of our approach to linear equations and especially their applications. Many thanks are due also to Mrs. Herbert J. Curtis, who aided in reading proof; to Mrs. L. R. Wilcox, who typed the manuscript, read proof, and offered much valuable advice; and to our respective institutions, which furnished the proving ground for many of our ideas.

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June, 1961

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Chapter

1

Introduction

1–1. PRELIMINARIES

Much of mathematics is concerned with equations of one type or another. The reader is already familiar with numerical equations, such as are studied extensively in algebra. There we have a given function f of a numerical variable and are interested in those particular numbers x for which $f(x) = 0$. The statement $f(x) = 0$ is called an *equation*, and the numbers x, which when substituted into it make the statement true, are the *solutions*. Generally, an equation specifies, though indirectly, the collection of all its solutions; mathematical questions centering around the equation are directed toward learning everything possible about this collection of all solutions.

Example 1–1. If a, b are given real numbers, $ax + b = 0$ is an equation specifying a certain collection of (real) solutions. If $a \neq 0$, it is shown in algebra that this collection consists of exactly one number, and, in fact, that this number is $-b/a$.

Example 1–1 suggests that, while the task of "solving an equation," that is, finding in some specific form all solutions, is the most important one connected with an equation, it might be useful first to settle two preliminary questions:

I Is there any solution at all?

II If so, exactly how many solutions are there?

When the answer to Question I is *No*, one proceeds no further, and all other questions disappear. Knowledge of this fact may save considerable effort. When the answer to Question I is *Yes*, an answer to Question II gives some direction to the search for solutions.

Example 1–2. Consider $\sqrt{x} + 1 = 0$, where x is complex. Here no solution exists, because $\sqrt{x}$ is non-negative if x is real and non-negative, and is non-real otherwise; in no case can $\sqrt{x}$ be equal to -1.

Example 1–3. Consider $x^3 - x = 0$, where x is complex. The general theorems on polynomial equations tell us that a cubic equation has just three solutions. By inspection, or by factorization, we see that 0, 1, -1 are solutions. All solutions have thus been found.

Equations of the type to be treated in this book have, to some extent, been encountered in the calculus. If f is a given function of the real variable t defined, say, for $a \leqq t \leqq b$, a and b being given, one may consider the equation

$$\frac{d}{dt} y(t) = f(t). \tag{1–1}$$

Thus (1–1) is a statement about a function y of t which may or may not be true; the *solutions* of (1–1) are those functions y of t which make (1–1) true when substituted into it. It is understood that the equality in (1–1) means that y possesses a derivative for all values of t such that $a \leqq t \leqq b$, and that this derivative function is the same as the function $f(t)$.

The problem represented by (1–1) is a central one in calculus; any solution of (1–1) is called an *antiderivative* (sometimes *indefinite integral*) of $f(t)$ and is often denoted by $\int f(t)\,dt$. Questions I and II are of significance in connection with such equations as (1–1), as we shall see.

Example 1–4. Consider $\frac{d}{dt} y(t) = 3t^2$, where t is unrestricted, but real. A little knowledge of differentiation yields that the function t^3 is a solution. Thus $t^3 + c$ is also a solution, where c is an arbitrary constant (function). It is shown in the calculus that no further solutions exist (see page 5). Note that as soon as the solution t^3 is produced, Question I is answered in the affirmative; existence of infinitely many solutions gives an answer to Question II. Finally, all solutions are given by $y(t) = t^3 + c$.

Equations of the form (1–1) always have infinitely many solutions if one exists. Hence it is important to establish conditions for the existence of a solution. Under such conditions, Question II is of little value in its original form; a modified form will be given later.

The term *differential equations* will be used throughout this text to refer to equations of the form (1–1) and to certain generalizations of these equations; no attempt will be made to specify in a precise manner exactly what class or classes of equations will be so called.

Loosely speaking, differential equations "involve" derivatives (or differentials), and the "unknowns" are functions. As certain algebraic equations are classified according to "degree," so certain differential equations are classified according to "order." Thus (1–1) has *first order* because no derivatives (or differentials) of $y(t)$ of order greater than the first appear. More generally, the term nth *order* is applied to a differential equation in which derivatives (or differentials) of an "unknown" function of various orders may appear, those of all orders greater than n being absent and that of order n being certainly present. Thus,[1]

$$\frac{dy}{dt} = ty,$$

$$(t + y)\,dy = (t - y)\,dt,$$

$$\sin\left(\frac{d^5y}{dt} + y\right) = \ln\,(\sqrt{t} + y)$$

are examples of differential equations of order 1, 1, 5, respectively. (NOTE: We have written and shall frequently write y instead of $y(t)$ for simplicity.)

★Domains of Functions. Mean Value Theorem.[2] It has been indicated that throughout this book we are concerned with functions; sometimes they are given and sometimes they are to be determined. It is important to recognize precisely what information is required in the specification of a function.

Basically, a function is a correspondence f associating with each member of a set A of objects, called the *domain*, a unique object of a set B called the *range set*.[3] In our work, the domain is almost always a set of real numbers, and the range set also consists of numbers (sometimes real, sometimes complex). In order to specify fully a function f, it is necessary to tell what the sets A and B are and then what the rule of correspondence is.

In practice, we often tend to overlook the necessity of specifying the domain and range set, because most of the familiar functions have a "natural" or "maximal" domain. Thus, if we define a func-

[1]The notation $\ln x$ denotes $\log_e x$. The notation $\log x$ is reserved for $\log_{10} x$.

[2]For a less comprehensive treatment of the subject, the sections marked by ★ can be omitted without interrupting the continuity of the presentation.

[3]The term *range* is used to designate the set of those objects in B which actually correspond to objects in the domain. It may consist of all of B or only a part of B.

tion f by the requirement $f(t) = t^2$, it is natural to assume that the domain as well as the range set consists of all real numbers t. Again, if $f(t) = 1/t$, then the "natural" domain is the set of all non-zero real numbers; if $f(t) = \sqrt{t}$, then the domain consists of all non-negative real numbers.

This practice is generally acceptable, but it can on occasion lead to difficulties. The reader is cautioned that a critical study of differential equations is not possible without the devotion of careful attention to the domains of functions which arise.

In performing the task of presenting the solution functions of a differential equation, it is often more or less satisfactory to present each function by a formula or rule of correspondence. The tacit assumption might be that the domain is the "natural" domain, while the range set is the set of all real (or complex) numbers. Yet it must be remembered that when a function is unknown, its domain is unknown too, and part of the task of determining such a function is to find its domain. Thus, in Example 1-4, the equation $y'(t) = 3t^2$ carried with it a statement that the domain of $3t^2$ was to be taken as the set of all real numbers. The solution $y(t) = t^3 + c$ has the same set as its natural domain.

A rather curious anomaly occurs, however, in case of the differential equation $y' = 1/t$, where the domain of $1/t$ is the set of all real $t \neq 0$. Here integral calculus suggests that $y = \ln |t| + c$ gives all solutions. However, if the domain of $\ln |t| + c$ is taken again as all $t \neq 0$, it is clear that further solutions exist. For example, if we define $z(t)$ for $t \neq 0$ so that

$$z(t) = \begin{cases} \ln |t| + c_1 & \text{for } t > 0, \\ \ln |t| + c_2 & \text{for } t < 0, \end{cases}$$

where $c_1 \neq c_2$, then, for every $t \neq 0$, $z'(t) = 1/t$, so that $z(t)$ is a solution of the equation. Two principles are therefore suggested:

1. Admissible solutions of differential equations shall be functions whose domains are intervals, that is, sets of the form

   all real numbers t such that $a \leqq t \leqq b$,

   or

   all real numbers t such that $a \leqq t$,

   or

   all real numbers t such that $t \leqq b$,

   or

   all real numbers t.

2. Solutions of a differential equation should have, but need not have, maximal domains subject to the restrictions in principle 1. It is not required that their domains agree with those of functions appearing in the equation.

From these principles, it appears that $\ln|t| + c$ $(t \neq 0)$ would not be admissible as a solution of $y' = 1/t$ $(t \neq 0)$ because principle 1 is violated. However, by principle 2, $\ln t + c$ $(t > 0)$ [or $\ln(-t) + c$ $(t < 0)$] would be admissible, even though its domain is less than that of $1/t$. Despite the ambiguity, we shall follow custom and write $\ln|t| + c$ to designate succinctly the two classes of solutions.

In Example 1–4, it was suggested that *all* solutions of the differential equation $y' = f(t)$ are given by $y = Y(t) + c$, where $Y(t)$ is a particular solution satisfying the equation. Before proving this result, we state the following theorem.

Mean Value Theorem. If $\phi(t)$ is continuous for $t_1 \leqq t \leqq t_2$ and if $\phi'(t)$ exists for $t_1 < t < t_2$, then there exists t_0 with $t_1 < t_0 < t_2$ such that

$$\phi(t_2) - \phi(t_1) = (t_2 - t_1)\phi'(t_0).$$

A proof of this theorem is to be found in most calculus books.

Let $Y(t)$ be a particular solution of $y' = f(t)$ and let $y(t)$ be any solution; it is assumed that all three functions are defined for all t such that $a \leqq t \leqq b$. Define $\phi(t) = y(t) - Y(t)$, so that

$$\phi'(t) = y'(t) - Y'(t) = f(t) - f(t) = 0$$

for $a \leqq t \leqq b$. Let t be given, and apply the mean value theorem with $t_1 = a$, $t_2 = t$; there exists t_0 between a and t such that

$$\phi(t) - \phi(a) = (t - a)\phi'(t_0) = 0.$$

If c is defined as $\phi(a)$, we have

$$y(t) - Y(t) = \phi(t) = \phi(a) = c,$$

so that $y(t) = Y(t) + c$ for all t such that $a \leqq t \leqq b$.

EXERCISES

In each of exercises 1 to 10, find answers to Questions I and II for the given numerical equation. Where solutions exist, find all of them. Use the real number system.

1. $x^3 - 2x^2 - 8x = 0.$
2. $x^3 + x - 2 = 0.$
3. $\dfrac{1}{x^2} - 1 = 0.$
4. $\dfrac{1}{x - 1} = \dfrac{2}{x^2 - 1}.$
5. $x = \sqrt{5x - 6}.$
6. $x = \sqrt{x + 2}.$
7. $2\sqrt{-x} + x = 1.$
8. $\sec x = \dfrac{1}{2}.$
9. $\dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}.$
10. $\dfrac{1 - \cos x}{2} = \sin x.$

For the equation in each of exercises 11 to 14, answer Questions I and II.

11. $\tan x = x.$
12. $\sin x = x.$
13. $e^x = b$ (b a given real number).
14. $e^x + e^{-x} = 1.$

Each of exercises 15 to 21 contains an equation for an unknown real function $y(t)$ of the real variable t. For each equation find one solution; then write all solutions.

15. $\dfrac{dy}{dt} = \dfrac{1}{t^2}.$
16. $y' = \cos^2 t.$
17. $y' = \dfrac{1}{t^3 - 1}.$
18. $y' = te^t.$
19. $y' = \dfrac{1}{\sqrt{t^2 + 2t}}.$
20. $y' = t \ln t.$
21. $y' = \dfrac{t^2 + 1}{t(t - 2)}.$

In each of exercises 22 to 25, determine the order of the differential equation.

22. $\dfrac{d^3y}{dt^3} - \dfrac{dy}{dt} = 0.$

23. $(y^2 + 1)\,dy - (t^2 + y^2)\,dt = 0.$

24. $\dfrac{d^4y}{dt^4} + 3y\dfrac{dy}{dt} = t^2\left(1 + \dfrac{1}{t^2}\dfrac{d^4y}{dt^4}\right).$

25. $y + \sin^2\dfrac{d^2y}{dt^2} + \cos^2\dfrac{d^2y}{dt^2} = \dfrac{dy}{dt}.$

In each of exercises 26 to 29, y is an unknown function and the equation is assumed to be true for all real numbers u and v. Show that in each case infinitely many solutions exist.[4]

★26. $y(u + v) = y(u) + y(v).$

★27. $y(u + v) = y(u) \cdot y(v).$

★28. $y(u + 2\pi) = y(u).$

★29. $y^2\left(u + \dfrac{\pi}{2}\right) + y^2(u) = 1.$

★30. In exercise 19, determine the "natural" domain of each solution.

★31. In each of exercises 18, 20, and 21, specify appropriate domains for the function in the right side and the solutions found.

1–2. FAMILIES OF SOLUTIONS

Let us examine some possibilities relative to the collection of all solutions of a differential equation. It will be recalled that the equation[5]

$$y'(t) = 3t^2 \tag{1–2}$$

of Example 1–4 has an infinitude of solutions

$$y(t) = t^3 + c. \tag{1–3}$$

Now consider the second-order equation

$$y''(t) = 6t; \tag{1–4}$$

[4]Exercises marked by ★ are more difficult than the rest; they are intended primarily for students who are studying the entire text, including the sections similarly identified.

[5]As usual in the calculus, $y', y'', \cdots$ are alternative notations for $\dfrac{dy}{dt}, \dfrac{d^2y}{dt^2}, \cdots$.